HIGHER-ORDER INDECOMPOSABLE ISOLS

BY
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0. **Introduction.** An effective analogue of the theory of cardinal numbers was created about ten years ago by J. C. E. Dekker. (See [1] and Dekker-Myhill [4].) In it the only sets considered are subsets of the natural numbers and the only functions considered are 1–1 partial recursive functions. In the classical theory a cardinal number may be considered to be an equivalence class containing all sets for which there is a 1–1 function mapping the set onto a given set. The Dekker analogue of a cardinal number is an equivalence class on the set of all subsets of E ($E=\{0, 1, 2, \dots\}$) containing all sets α for which there is a 1–1 partial recursive function f such that the domain of f includes α and the image of α under f is a given subset of E. These equivalence classes are called recursive equivalence types, or RETs. The collection of all RETs is denoted $\mathbf{\Omega}$. The RET to which a set, α, belongs is denoted $\langle\alpha\rangle$.

Addition is defined on the RETs in the following manner. If $\alpha\subseteq E$ and $\beta\subseteq E$, α and β are called recursively separated if there exist disjoint recursively enumerable (RE) sets ω and θ such that $\alpha\subseteq\omega$ and $\beta\subseteq\theta$. Let A and B be RETs. The sum of A and B is defined to be the RET represented by $\alpha\cup\beta$ where $A=\langle\alpha\rangle$, $B=\langle\beta\rangle$, and α and β are recursively separated.

We define $A\leqq B$ for RETs A and B if there is an RET C such that $A+C=B$. The $\leqq$ relation is a partial ordering of the RETs. If $A\leqq B$ we say A is a predecessor of B.

An RET is called an isol if it satisfies the additive cancellation law. The collection of all isols is denoted $\mathbf{\Lambda}$. Thus $A\in\mathbf{\Lambda}$ if and only if for all RETs B and C, $A+B=A+C$ implies $B=C$. A subset, α, of E is called isolated if there is no 1–1 partial recursive function f whose domain includes α and such that $f(\alpha)\subsetneqq\alpha$. $\langle\alpha\rangle\in\mathbf{\Lambda}$ if and only if α is isolated. A subset of E contains no infinite RE subset if and only if it is isolated. Thus $\mathbf{\Lambda}$ is the collection of equivalence classes of sets which do not have infinite RE subsets. $\mathbf{\Lambda}$ is closed under addition and predecessor. The assertions of this paragraph are proved in [4].

Dekker-Myhill [4, p. 114] define an ideal in $\mathbf{\Lambda}$ as a subsystem of $\mathbf{\Lambda}$ closed under addition and predecessor. A sequence of ideals will be defined here in order to discuss the results of this paper. Variables X, Y, Z, V, W range over $\mathbf{\Lambda}$.

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DEFINITION 0.1. $I_0=\{X : X \text{ is finite}\}$.

For ordinals α, $\alpha>0$, the definition is continued by induction.

$$P_\alpha = \Big\{X : X = Y+Z \rightarrow Y \in \bigcup_{\alpha'<\alpha} I_{\alpha'} \vee Z \in \bigcup_{\alpha'<\alpha} I_{\alpha'}\Big\}.$$

Elements of $P_\alpha - \bigcup_{\alpha'<\alpha} I_{\alpha'}$ will be called α-order indecomposables; they are indecomposable with respect to $\bigcup_{\alpha'<\alpha} I_{\alpha'}$.

$$S_\alpha = \Big\{X : X = Y+Z \,\&\, Z \notin \bigcup_{\alpha'<\alpha} I_{\alpha'} \rightarrow (\exists V)(\exists W)\Big[Z = V+W \,\&\, V \notin \bigcup_{\alpha'<\alpha} I_{\alpha'} \,\&\, W \notin \bigcup_{\alpha'<\alpha} I_{\alpha'}\Big]\Big\}.$$

Elements of $S_\alpha - \bigcup_{\alpha'<\alpha} I_{\alpha'}$ will be called α-order highly decomposable isols; they are highly decomposable with respect to $\bigcup_{\alpha'<\alpha} I_{\alpha'}$. I_α is the ideal generated by $P_\alpha \cup S_\alpha$.

DEFINITION 0.2. The following two definitions are made for each positive ordinal α.

(a) $X=_\alpha Y \leftrightarrow (\exists V)(\exists W)[X+V=Y+W \,\&\, V \in \bigcup_{\alpha'<\alpha} I_{\alpha'} \,\&\, W \in \bigcup_{\alpha'<\alpha} I_{\alpha'}]$.

(b) $X \leqq_\alpha Y \leftrightarrow (\exists Z)[X+Z=_\alpha Y]$.

The main results of this paper are existence theorems for $P_\alpha - \bigcup_{\alpha'<\alpha} I_{\alpha'}$ and $S_\alpha - \bigcup_{\alpha'<\alpha} I_{\alpha'}$ for countable ordinals α. Let c denote the power of the continuum. Theorem 3.8 asserts that for each positive countable ordinal α there exists a set of c α-order indecomposables which are pairwise α-incomparable. Theorem 3.15 asserts that for each positive countable ordinal α there exists a set of c α-order highly decomposable isols which are pairwise α-incomparable. Theorem 4.1 asserts that for each positive countable ordinal α there exists an α-order highly decomposable isol which is multiple-free with respect to $=_\alpha$: that is, there is a Z such that for no $n>1$ is there an X satisfying $nX=_\alpha Z$. In view of Theorem 1.4, which asserts that every isol is in $\bigcup_{\alpha'<\omega_1} I_{\alpha'}$ where ω_1 is the first uncountable ordinal, the first two existence results are as strong as possible.

The existence of c mutually incomparable first-order indecomposables and of c first-order highly decomposable isols has been known at least since 1958. (See Dekker [2, T1] and Dekker-Myhill [4, pp. 112–113].) The existence of c mutually incomparable first-order highly decomposable isols is an observation of Nerode although the proof given below is not his.

1. Basic properties of the ideals I_α.

The following result will be used frequently. It is a consequence of Theorems 15(l) and 19 of Dekker-Myhill [4, pp. 80–81].

The refinement property. If $A_i \in \Lambda$ for $i=1,\ldots,n$ and $B_j \in \Lambda$ for $j=1,\ldots,p$ and $A_1+\cdots+A_n=B_1+\cdots+B_p$, there exists a matrix $(C_{i,j})_{i=1}^{n}{}_{j=1}^{p}$ of isols such that $A_i=C_{i,1}+\cdots+C_{i,p}$ for $i=1,\ldots,n$ and $B_j=C_{1,j}+\cdots+C_{n,j}$ for $j=1,\ldots,p$.

The relations satisfied by the elements $C_{i,j}$ will often be indicated in the following way:

$$\begin{array}{ccccccc} C_{1,1} + & C_{2,1} + & \cdots & + C_{n,1} & = & B_1 \\ + & + & & + & & \\ C_{1,2} + & C_{2,2} + & \cdots & + C_{n,2} & = & B_2 \\ + & + & & + & & \\ \vdots & \vdots & & \vdots & & \vdots \\ + & + & & + & & \\ C_{1,p} + & C_{2,p} + & \cdots & + C_{n,p} & = & B_p \\ \| & \| & \cdots & \| & & \\ A_1 & A_2 & \cdots & A_n & & \end{array}$$

LEMMA 1.1. $\bigcup_{\alpha'<\alpha} I_{\alpha'} = P_\alpha \cap S_\alpha$.

Proof. $\subseteq$. Let $X \in I_{\alpha'}$ for some $\alpha' < \alpha$. If $X = Y+Z$, then $Y \in I_{\alpha'}$ and $Z \in I_{\alpha'}$ since $I_{\alpha'}$ is an ideal. Therefore, $X \in P_\alpha$. Further, $X \in S_\alpha$ since the antecedent of the defining condition of S_α is never satisfied.

$\supseteq$. Let $X \in S_\alpha - \bigcup_{\alpha'<\alpha} I_{\alpha'}$. Since $X = 0 + X$, $X \notin \bigcup_{\alpha'<\alpha} I_{\alpha'}$, and $X \in S_\alpha$, there exist isols V and W such that $X = V + W$, $V \notin \bigcup_{\alpha'<\alpha} I_{\alpha'}$, and $W \notin \bigcup_{\alpha'<\alpha} I_{\alpha'}$. This shows $X \notin P_\alpha$.

Notational remark. The equality of Lemma 1.1 may be used to shorten some expressions. E.g., $P_\alpha - \bigcup_{\alpha'<\alpha} I_{\alpha'} = P_\alpha - S_\alpha$, and the latter form may be used when the former is intended.

LEMMA 1.2. $\alpha_1 < \alpha_2 \rightarrow I_{\alpha_1} \subseteq I_{\alpha_2}$. $\bigcup_{\alpha'<\alpha+1} I_{\alpha'} = I_\alpha$.

Proof. $I_{\alpha_1} \subseteq \bigcup_{\alpha'<\alpha_2} I_{\alpha'} \subseteq P_{\alpha_2} \cap S_{\alpha_2} \subseteq I_{\alpha_2}$.

LEMMA 1.3. *$\bigcup_{\alpha'<\alpha} I_{\alpha'}$ is an ideal.*

Proof. Clear from Lemma 1.2.

THEOREM 1.4. $X \in \Lambda \rightarrow X \in \bigcup_{\alpha'<\omega_1} I_{\alpha'}$.

Proof. Let $X \in \Lambda$. Define

$$\gamma = \text{l.u.b.}\, \{\alpha : \alpha < \omega_1 \;\&\; (\exists Y)[Y \leqq X \;\&\; Y \in P_\alpha - S_\alpha]\} + 1.$$

Note that if α is an element of the set occurring in the definition, $\alpha < \gamma$. Also, $\gamma < \omega_1$ since there are only countably many predecessors of X. We will show $X \in S_\gamma$, hence $X \in I_\gamma$. Suppose $X = Y + Z$ and $Z \notin P_\gamma \cap S_\gamma$. If $Z = V + W$ implies $V \in P_\gamma \cap S_\gamma$ or $W \in P_\gamma \cap S_\gamma$, then $Z \in P_\gamma - S_\gamma$ and

$$\gamma \in \{\alpha : \alpha < \omega_1 \;\&\; (\exists Y)(Y \leqq X \;\&\; Y \in P_\alpha - S_\alpha)\}.$$

We conclude $\gamma < \gamma$. Hence the implication fails, and there are V and W such that $Z = V + W$ and $V \notin P_\gamma \cap S_\gamma$ and $W \notin P_\gamma \cap S_\gamma$. Therefore $X \in S_\gamma$.

LEMMA 1.5. *P_α is closed under predecessor. I.e.,* $Y \leqq X$ & $X \in P_\alpha \to Y \in P_\alpha$.

Proof. $X = Y + Z$ for some Z. Let $Y = V + W$. Then $X = V + (W + Z)$. Since $X \in P_\alpha$, $V \in P_\alpha \cap S_\alpha$, in which case there is nothing to prove, or $(W+Z) \in P_\alpha \cap S_\alpha$. In the latter case, since $P_\alpha \cap S_\alpha$ is an ideal (Lemma 1.3), $W \in P_\alpha \cap S_\alpha$. Therefore $Y \in P_\alpha$.

LEMMA 1.6. *S_α is an ideal.*

Proof. First we prove that S_α is closed under predecessor. Let $X_1 \leqq X$ and $X \in S_\alpha$. Then there is an X_2 such that $X_1 + X_2 = X$. Now suppose $X_1 = Y + Z$ and $Z \notin P_\alpha \cap S_\alpha$. Then $X = (X_2 + Y) + Z$ and, since $X \in S_\alpha$ and $Z \notin P_\alpha \cap S_\alpha$, Z has the desired decomposition. Therefore, $X_1 \in S_\alpha$.

Secondly, suppose $X_1 \in S_\alpha$, $X_2 \in S_\alpha$, and $X_1 + X_2 = Y + Z$, and $Z \notin P_\alpha \cap S_\alpha$. By the refinement property, there exist isols Y_1, Y_2, Z_1, Z_2 such that

$$\begin{array}{ccccc} Y_1 & + & Z_1 & = & X_1 \\ + & & + & & \\ Y_2 & + & Z_2 & = & X_2 \\ \| & & \| & & \\ Y & & Z & & \end{array}$$

Since $P_\alpha \cap S_\alpha$ is an ideal, $Z_1 \notin P_\alpha \cap S_\alpha$ or $Z_2 \notin P_\alpha \cap S_\alpha$. $Z_i \notin P_\alpha \cap S_\alpha$ implies the existence of V and W such that $Z_i = V + W$, $V \notin P_\alpha \cap S_\alpha$, and $W \notin P_\alpha \cap S_\alpha$, since $Z_i \leqq X_i$ and $X_i \in S_\alpha$. Then $Z = (Z_{3-i} + V) + W$ is a decomposition of the desired form. We may conclude that $(X_1 + X_2) \in S_\alpha$.

LEMMA 1.7. *$X \in I_\alpha$ if and only if there is a finite set of isols $X_1, \ldots, X_n, Z$ such that each $X_i \in P_\alpha$, $Z \in S_\alpha$ and $X = \sum_{i=1}^{n} X_i + Z$.*

Proof. The ideal generated by any set is the collection of all predecessors of finite sums of elements of the set. Thus I_α is the collection of all isols X for which there are isols $Y_1, \ldots, Y_n, Z_1, \ldots, Z_m$ such that each $Y_i \in P_\alpha$, each $Z_i \in S_\alpha$, and

$$X \leqq Y_1 + \cdots + Y_n + Z_1 + \cdots + Z_m.$$

Since S_α is an ideal, $Z_1 + \cdots + Z_m = Z_0 \in S_\alpha$. By the refinement property, $X \leqq Y_1 + \cdots + Y_n + Z_0$ implies the existence of isols $X_1, \ldots, X_n, Z$ such that $X = X_1 + \cdots + X_n + Z$, $Z \leqq Z_0$, and each $X_i \leqq Y_i$. Since both S_α and P_α are closed under predecessors, $Z \in S_\alpha$ and each $X_i \in P_\alpha$. Hence, if $X \in I_\alpha$, X satisfies the conditions specified in the lemma (i.e., $\to$ is proved.). The converse is clear.

THEOREM 1.8. *If $X \in I_\alpha$, then there is a finite n such that any linear decomposition of X includes at most n isols of $P_\alpha - S_\alpha$. More precisely, if $\sum_{i=1}^{n+k} Y_i = X$, then there are at most n i's for which $Y_i \in P_\alpha - S_\alpha$.*

Proof. Let $X \in I_\alpha$. By Lemma 1.7, there are isols $X_1, \ldots, X_n, Z$ such that $X = X_1 + \cdots + X_n + Z$, $Z \in S_\alpha$, and each $X_i \in P_\alpha$. We claim that this n satisfies the conditions of the theorem. To simplify notation, we prove an example in which $n=2$ and $k=2$. Suppose $X = X_1 + X_2 + Z$, each $X_i \in P_\alpha$ and $Z \in S_\alpha$. Suppose further that $X = Y_1 + Y_2 + Y_3 + Y_4$. We must prove that there are at most two i's such that $Y_i \in P_\alpha - S_\alpha$. Equivalently we must show that there are at least two i's such that $Y_i \notin P_\alpha - S_\alpha$. By the refinement property, there exist isols $Y_{i,j}$ such that

$$\begin{array}{ccccccccc}
Y_{1,1} & + & Y_{1,2} & + & Y_{1,3} & + & Y_{1,4} & = & X_1 \\
+ & & + & & + & & + & & \\
Y_{2,1} & + & Y_{2,2} & + & Y_{2,3} & + & Y_{2,4} & = & X_2 \\
+ & & + & & + & & + & & \\
Y_{3,1} & + & Y_{3,2} & + & Y_{3,3} & + & Y_{3,4} & = & Z \\
\parallel & & \parallel & & \parallel & & \parallel & & \\
Y_1 & & Y_2 & & Y_3 & & Y_4 & &
\end{array}$$

Since $X_i \in P_\alpha$, there is at most one $j=j(i)$ such that $Y_{i,j(i)} \notin P_\alpha \cap S_\alpha$. Thus $K = \{1, 2, 3, 4\} - \{j(1), j(2)\}$ has at least two elements. It suffices, to prove the theorem, to prove that $k \in K$ implies $Y_k \notin P_\alpha - S_\alpha$. Let $k \in K$. Then $Y_{i,k} \in S_\alpha$ for $i \leqq 2$ since $Y_{i,k} \in P_\alpha \cap S_\alpha \subseteq S_\alpha$. $Y_{3,k} \in S_\alpha$ since $Y_{3,k} \leqq Z$ and $Z \in S_\alpha$. Therefore, $Y_k = Y_{1,k} + Y_{2,k} + Y_{3,k}$ is an element of S_α (Lemma 1.6), and so $Y_k \notin P_\alpha - S_\alpha$.

Corollary of Definition 0.2(a). *$=_\alpha$ is an equivalence relation. Addition is well defined on the $=_\alpha$ equivalence classes.*

Proof. Reflexivity and symmetry are clear. If $(V_1 + W_1 + V_2 + W_2) \in P_\alpha \cap S_\alpha$ and $X_1 + V_1 = Y + W_1$ and $Y + V_2 = X_2 + W_2$, then $X_1 + V_1 + V_2 = X_2 + W_1 + W_2$ so $=_\alpha$ is transitive. Again, if $(V_1 + W_1 + V_2 + W_2) \in P_\alpha \cap S_\alpha$, $X_1 + V_1 = Y_1 + W_1$, and $X_2 + V_2 = Y_2 + W_2$, then

$$X_1 + X_2 + V_1 + V_2 = Y_1 + Y_2 + W_1 + W_2,$$

so addition is well defined on $=_\alpha$ equivalence classes.

Corollary of Definition 0.2(b). *$\leqq_\alpha$ is a partial order.*

Proof. We have only to show that $\leqq_\alpha$ is antisymmetric. To do this we note that $X + Y =_\alpha X + Z \rightarrow Y =_\alpha Z$. For if $V \in P_\alpha \cap S_\alpha$ and $W \in P_\alpha \cap S_\alpha$ and $X + Y + V = X + Z + W$, then, since $X \in \Lambda$, $Y + V = Z + W$. To prove the corollary, suppose $X_1 \leqq_\alpha X_2$ and $X_2 \leqq_\alpha X_1$. Then there exist Z_1, Z_2 such that $X_1 + Z_1 =_\alpha X_2$ and $X_1 =_\alpha X_2 + Z_2$. Therefore, $X_1 + Z_1 + Z_2 =_\alpha X_2 + Z_2 =_\alpha X_1$. Thus $Z_1 + Z_2 =_\alpha 0$. I.e., $(Z_1 + Z_2) \in P_\alpha \cap S_\alpha$. So $Z_1 \in P_\alpha \cap S_\alpha$ and $X_1 =_\alpha X_2$.

Lemma 1.9. $X + Z =_\alpha Y \;\&\; Y \in P_\alpha \rightarrow X \in P_\alpha \cap S_\alpha \vee Z \in P_\alpha \cap S_\alpha$.

Proof. Suppose $X+Z=_\alpha Y$, $X \notin P_\alpha \cap S_\alpha$, and $Z \notin P_\alpha \cap S_\alpha$. There exist V and W, $(V+W) \in P_\alpha \cap S_\alpha$, such that $X+Z+V=Y+W$. Applying the refinement property, there exist X_i, V_i, Z_i such that

$$\begin{array}{l} X_1+Z_1+V_1 = Y \\ +\quad+\quad+ \\ X_2+Z_2+V_2 = W \\ \|\quad\;\|\quad\;\| \\ X\quad Z\quad V \end{array}$$

Since $X_2 \leqq W \in P_\alpha \cap S_\alpha$, $X_2 \in P_\alpha \cap S_\alpha$. Since $X \notin P_\alpha \cap S_\alpha$, $X_1 \notin P_\alpha \cap S_\alpha$. Similarly $Z_1 \notin P_\alpha \cap S_\alpha$. Therefore $(Z_1+V_1) \notin P_\alpha \cap S_\alpha$. Thus $Y=X_1+(Z_1+V_1)$ is not an element of P_α.

COROLLARY. $X \notin P_\alpha \cap S_\alpha$ & $Y \in P_\alpha$ & $X \leqq_\alpha Y \rightarrow X=_\alpha Y$.

Proof. $X+Z=_\alpha Y$ for some Z. By the lemma, $Z=_\alpha 0$. Thus $X=_\alpha Y$.

LEMMA 1.10. *If $X \in P_\alpha - S_\alpha$ and each $Y_i \in P_\alpha$ and $X \leqq_\alpha \sum_{i=1}^n Y_i$, then there is an i for which $X=_\alpha Y_i$.*

Proof. To simplify notation, we will prove the special case of $n=2$. There exist Z, V, W such that $(V+W) \in P_\alpha \cap S_\alpha$ and $X+Z+V=Y_1+Y_2+W$. Applying the refinement property, there exist X_i, Z_i, V_i satisfying

$$\begin{array}{l} X_1+Z_1+V_1 = Y_1 \\ +\quad+\quad+ \\ X_2+Z_2+V_2 = Y_2 \\ +\quad+\quad+ \\ X_3+Z_3+V_3 = W \\ \|\quad\;\|\quad\;\| \\ X\quad Z\quad V \end{array}$$

Since $X_3 \leqq W \in P_\alpha \cap S_\alpha$, $X_3 \in P_\alpha \cap S_\alpha$. Since $X \in P_\alpha - S_\alpha$, there is exactly one i, $i=1$ or $i=2$, for which $X_i \notin P_\alpha \cap S_\alpha$. Thus $X_i =_\alpha X$. By Lemma 1.9, since $Y_i \in P_\alpha$, $(Z_i+V_i) \in P_\alpha \cap S_\alpha$. Thus $X_i =_\alpha Y_i$ and so $X=_\alpha Y_i$.

LEMMA 1.11. *If $X \leqq \sum_{i=1}^n Y_i$ and, for each i, $Y_i \in P_\alpha - S_\alpha$, then either there is a Z such that $Z \leqq X$ and $Z \in P_\alpha - S_\alpha$, or $X \in P_\alpha \cap S_\alpha$.*

Proof. Suppose $X+W=\sum_{i=1}^n Y_i$. By the refinement property, there exist isols X_i, W_i such that for $i \leqq n$, $X_i+W_i=Y_i$ and such that $X=\sum_{i=1}^n X_i$. By Lemma 1.5, for each i, $X_i \in P_\alpha$. If, for each i, $X_i \in S_\alpha$, then $X=\sum_{i=1}^n X_i \in P_\alpha \cap S_\alpha$ by Lemmas 1.1 and 1.3. Otherwise there is an i for which $X_i \in P_\alpha - S_\alpha$ and, of course, $X_i \leqq X$.

2. Two constructions related to the sequence of ideals.

Notation. If α is a countable ordinal and $\alpha > 1$, we will say that $\alpha_i \uparrow \alpha$ only if one of the following two conditions is satisfied: α is a successor ordinal and, for all i, $0 \leqq i < \omega$, $\alpha_i = \alpha - 1$; or α is a limit ordinal, α is the least upper bound of $\{\alpha_i\}_{i=0}^{\infty}$, and for all i, $0 \leqq i < \omega$, $\alpha_i < \alpha_{i+1}$.

If $\beta \subseteq E$, $\beta = \beta_1 \cup \beta_2$, and there is a pair of disjoint RE sets, (ω, θ), such that $\beta_1 \subseteq \omega$ and $\beta_2 \subseteq \theta$, then we may write $\beta = \beta_1 + \beta_2$ or $\beta = \beta_1 + \beta_2$ since (ω, θ). We call β the *sum* of β_1 and β_2 in this case. More generally, if $n \geqq 2$, $1 \leqq i < j \leqq n$ implies $\theta_i \cap \theta_j = \varnothing$, and $\beta_i \subseteq \theta_i \subseteq E$ and θ_i is RE for $i = 1, \ldots, n$, we may write

$$\bigcup_{i=1}^{n} \beta_i = \sum_{i=1}^{n} \beta_i$$

or

$$\bigcup_{i=1}^{n} \beta_i = \sum_{i=1}^{n} \beta_i \quad \text{since } \{\theta_i\}_{i=1}^{n}.$$

If $\delta \subseteq \beta$ and $\beta = \delta + (\beta - \delta)$, we may write $\delta \prec \beta$ and call δ a *predecessor* of β.

$\{(\chi_f, \psi_f)\}_{f=0}^{\infty}$ is a sequence of pairs of disjoint RE subsets of $E \times E$ containing all such pairs and such that

$$(\chi_0, \psi_0) = (\{2x : x \in E\} \times E, \{2x+1 : x \in E\} \times E).$$

If $\delta \subseteq E$, we abbreviate $\delta \times \{i\}$ by δ^i. Finally, if ω is an RE subset of $E \times E$, then ω_{ev} and ω_{od} are disjoint RE sets such that $\omega = \omega_{\text{ev}} \cup \omega_{\text{od}}$ and whenever $\omega \cap E^i$ is infinite both $\omega_{\text{ev}} \cap E^i$ and $\omega_{\text{od}} \cap E^i$ are infinite.

In the remainder of this section we assume that $\alpha_i \uparrow \alpha$, $\alpha > 1$, and for each i, $Z_i \in P_{\alpha_i} - S_{\alpha_i}$. We will describe and discuss two constructions from the sequence $\{Z_i\}_{i=0}^{\infty}$. The first is designed to construct representatives of isols in $P_\alpha - S_\alpha$, the second representatives of isols in $S_\alpha - P_\alpha$. Any isol which has a representative resulting from the first construction, and only such an isol, will be, by definition, an element of $P[\{Z_i\}]$. $S[\{Z_i\}]$ is defined in the same manner, using the second construction instead of the first.

An observation which will be used in Construction I is that given any countable sequence of subsets of E, say $\{\xi_f\}_{f=0}^{\infty}$, there is an infinite subset of E, say ξ, such that for each f, $\xi \cap \xi_f$ is finite or $\xi \cap \bar{\xi}_f$ is finite. Dekker-Myhill [4, p. 102] give a proof for the special case in which $\xi_f = \omega_f$ is the fth set in an enumeration of all RE sets. However their proof shows that the conclusion is valid for any countable sequence of sets.

Construction I. For $i \in E$, let ζ_i be a representative of Z_i. We first construct a set δ_i such that (i) $\delta_i \prec \zeta_i$, $\langle \zeta_i - \delta_i \rangle \in P_{\alpha_i} \cap S_{\alpha_i}$, and (ii) if $f \leqq i$ and $\delta_i^i \subseteq \chi_f \cup \psi_f$, then $\delta_i^i \cap \chi_f = \varnothing$ or $\delta_i^i \cap \psi_f = \varnothing$. Let $\zeta_{i,0} = \zeta_i$. Construct $\zeta_{i,n+1}$ from $\zeta_{i,n}$ in such a way that (iii) $\zeta_{i,n+1} \prec \zeta_{i,n}$, and (iv) $\langle \zeta_{i,n} - \zeta_{i,n+1} \rangle \in P_{\alpha_i} \cap S_{\alpha_i}$ according to the following instructions. Initially, for $f \leqq i$, say f is unsatisfied for i. At step n, $n \geqq 0$, let f be such that $0 \leqq f < i+1$ and $n \equiv f \pmod{(i+1)}$. If f has been satisfied for i at a previous step or if $\neg((\zeta_{i,n})^{\iota} \subseteq \chi_f \cup \psi_f)$, let $\zeta_{i,n+1} = \zeta_{i,n}$ and go to step $n+1$. If f has not been satisfied for i and $(\zeta_{i,n})^{\iota} \subseteq \chi_f \cup \psi_f$, say f is satisfied for i and define $\zeta_{i,n+1}$ according to the following:

$$\begin{aligned}(\zeta_{i,n+1})^{\iota} &= (\zeta_{i,n})^{\iota} \cap \chi_f \quad \text{if} \quad ((\zeta_{i,n})^{\iota} \cap \psi_f) \in P_{\alpha_i} \cap S_{\alpha_i} \\ &= (\zeta_{i,n})^{\iota} \cap \psi_f \quad \text{if} \quad ((\zeta_{i,n})^{\iota} \cap \chi_f) \in P_{\alpha_i} \cap S_{\alpha_i}.\end{aligned}$$

Exactly one of the two conditions must occur since

$$(\zeta_{i,n})^{\iota} = ((\zeta_{i,n})^{\iota} \cap \chi_f) + ((\zeta_{i,n})^{\iota} \cap \psi_f)$$

and $\langle(\zeta_{i,n})^{\iota}\rangle \in P_{\alpha_i} - S_{\alpha_i}$. Properties (iii) and (iv) are clearly satisfied. Now go to step $n+1$. Observe that if f is satisfied for i at step n, then $\chi_f \cap (\zeta_{i,n'})^{\iota} = \varnothing$ or $\psi_f \cap (\zeta_{i,n'})^{\iota} = \varnothing$ for all $n' > n$ by the construction and (iii). There are at most $i+1$ steps at which some f becomes satisfied for i. Let n be strictly larger than any n' such that at step n' some f becomes satisfied for i. Define $\delta_i = \zeta_{i,n}$. δ_i does satisfy property (i) for $\zeta_i - \delta_i = \sum_{j=0}^{n-1} (\zeta_{i,j} - \zeta_{i,j+1})$, each $\langle \zeta_{i,j} - \zeta_{i,j+1} \rangle \in P_{\alpha_i} \cap S_{\alpha_i}$, and $P_{\alpha_i} \cap S_{\alpha_i}$ is an ideal. Property (ii) is satisfied by δ_i since $\delta_i^{\iota} \subseteq \chi_f \cup \psi_f$ and $f \leqq i$ implies that f is satisfied for i, hence that property (ii) holds.

Define $\rho_f = \{i : i \geqq f \;\&\; \delta_i^{\iota} \subseteq \chi_f \cup \psi_f\}$, $\xi_f = \rho_f \cap \{i : \chi_f \cap \delta_i^{\iota} \neq \varnothing\}$. Since $i \in \rho_f$ implies $\delta_i^{\iota} \cap \chi_f = \varnothing$ or $\delta_i^{\iota} \cap \psi_f = \varnothing$ (property (ii)), observe $\xi_f \subseteq \rho_f \cap \{i : \psi_f \cap \delta_i^{\iota} = \varnothing\}$. Thus (v) $\rho_f \cap \{i : \psi_f \cap \delta_i^{\iota} \neq \varnothing\} \subseteq \rho_f \cap \bar{\xi}_f$.

Let ξ be a set indecomposable with respect to $\{\xi_f\}_{f=0}^{\infty}$. By this we mean that ξ is an infinite set such that for each f, $\xi \cap \xi_f$ is finite or $\xi \cap \bar{\xi}_f$ is finite. Define $\beta = \bigcup_{i \in \xi} \delta_i^{\iota}$. β is the result of this construction: that is, $\langle\beta\rangle$ represents a typical element of $P[\{Z_i\}]$. (We will see later that $\langle\beta\rangle \in \mathbf{\Lambda}$.)

LEMMA 2.1. *If $\langle\beta\rangle = V + W$, there is a finite set ρ such that either $V \leqq \sum_{i \in \rho} Z_i$ or $W \leqq \sum_{i \in \rho} Z_i$.*

Proof. Since $\langle\beta\rangle = V + W$, there is a pair of disjoint RE subsets of $E \times E$, (χ_f, ψ_f) for some f, such that $\beta = (\chi_f \cap \beta) + (\psi_f \cap \beta)$, $\langle \chi_f \cap \beta \rangle = V$, and $\langle \psi_f \cap \beta \rangle = W$. Since $\beta \subseteq \chi_f \cup \psi_f$, $\xi \subseteq \{i : i < f\} \cup \rho_f$. By the definition of ξ, $\xi \cap \xi_f$ or $\xi \cap \bar{\xi}_f$ is finite.

Suppose first that $\xi \cap \xi_f$ is finite. We see that

$$\begin{aligned}\chi_f \cap \beta &= \bigcup_{i < f \& i \in \xi} (\chi_f \cap \delta_i^{\iota}) \cup \bigcup_{i \in \xi \cap \xi_f} (\chi_f \cap \delta_i^{\iota}) \\ &= \sum_{i < f \& i \in \xi} (\chi_f \cap \delta_i^{\iota}) + \sum_{i \in \xi \cap \xi_f} (\chi_f \cap \delta_i^{\iota}) \quad \text{since} \quad \{E^{\iota}\}_{i < f, i \in \xi \cap \xi_f} \\ &\prec \sum_{i < f \& i \in \xi} \delta_i^{\iota} + \sum_{i \in \xi \cap \xi_f} \delta_i^{\iota} \quad \text{since } (\chi_f, \psi_f) \\ &\prec \sum_{i < f \& i \in \xi} \zeta_i^{\iota} + \sum_{i \in \xi \cap \xi_f} \zeta_i^{\iota} \quad \text{for } \delta_i^{\iota} \prec \zeta_i^{\iota}.\end{aligned}$$

Thus $V = \langle \chi_f \cap \beta \rangle \leqq \sum_{i < f} Z_i + \sum_{i \in \xi \cap \xi_f} Z_i$.

If $\xi \cap \xi_f$ is finite, then

$$\xi \cap \{i : \psi_f \cap \delta_i^i \neq \varnothing\} \subseteq \{i : i < f\} \cup (\xi \cap \rho_f \cap \{i : \psi_f \cap \delta_i^i \neq \varnothing\})$$
$$\subseteq \{i : i < f\} \cup (\xi \cap \rho_f \cap \xi_f) \quad \text{by property (v).}$$

Since $\xi \cap \xi_f$ is finite, essentially the same argument as before shows that

$$W \leqq \sum_{i \in \xi \cap \xi_f} Z_i.$$

COROLLARY. $\langle\beta\rangle \in \mathbf{\Lambda}$.

Proof. If $\langle\beta\rangle \in \mathbf{\Omega}-\mathbf{\Lambda}$, then there exist V and W such that $\langle\beta\rangle = V+W$, $V \in \mathbf{\Omega}-\mathbf{\Lambda}$, and $W \in \mathbf{\Omega}-\mathbf{\Lambda}$. But the lemma shows that if $\langle\beta\rangle = V+W$, then there is a finite set ρ for which $V \leqq \sum_{i\in\rho} Z_i$ or $W \leqq \sum_{i\in\rho} Z_i$. Since each Z_i is in $\mathbf{\Lambda}$, $\sum_{i\in\rho} Z_i \in \mathbf{\Lambda}$ and, therefore, $V \in \mathbf{\Lambda}$ or $W \in \mathbf{\Lambda}$.

LEMMA 2.2. *For any finite n, there exist isols $X_1, \dots, X_n$, Y, $Y_1, \dots, Y_n$ such that $\langle\beta\rangle = X_1 + \cdots + X_n + Y$ and, for each i, there is a j such that $j > n$, $X_i + Y_i = Z_j$, and $Y_i \in P_{\alpha_j} \cap S_{\alpha_j}$. In particular, therefore, $X_i \in P_{\alpha_j} - S_{\alpha_j}$.*

Proof. The second statement follows from the first since $Z_j \in P_{\alpha_j} - S_{\alpha_j}$. Let $j_1, \dots, j_n$ be distinct elements of ξ, each $j_i > n$. Then

$$\beta = \delta_{j_1}^{j_1} + \cdots + \delta_{j_n}^{j_n} + \bigcup_{i \in \xi - \{j_1, \dots, j_n\}} \delta_i^i.$$

By (i), $\zeta_j^j = \delta_j^j + (\zeta_j^j - \delta_j^j)$ and $\langle \zeta_j^j - \delta_j^j\rangle \in P_{\alpha_j} \cap S_{\alpha_j}$. Thus, to prove the lemma, we may let $X_i = \langle \delta_{j_i}^{j_i}\rangle$, $Y_i = \langle \zeta_{j_i}^{j_i} - \delta_{j_i}^{j_i}\rangle$, $Y = \langle \bigcup_{i \in \xi - \{j_1, \dots, j_n\}} \delta_i^i\rangle$, and, for each i the required j is j_i.

CONSTRUCTION II. We first construct, for each $i \in E$, a set δ_i which is a subset of E^i and satisfies the following two conditions. First, there is a finite set of isols, $R_1, \dots, R_n$, $S_1, \dots, S_n$, R such that $\langle\delta_i\rangle = R_1 + \cdots + R_n + R$, R is finite, and, for each j, $R_j + S_j = Z_i$ and $S_j \in P_{\alpha_i} \cap S_{\alpha_i}$. Secondly, if $f \leqq i$ and $\delta_i \subseteq \chi_f \cup \psi_f$, then either $\delta_i \cap \chi_f = \varnothing$ ($\delta_i \cap \psi_f = \varnothing$) or $\delta_i \cap \chi_{f,\mathrm{ev}}$ ($\delta_i \cap \psi_{f,\mathrm{ev}}$) and $\delta_i \cap \chi_{f,\mathrm{od}}$ ($\delta_i \cap \psi_{f,\mathrm{od}}$) each contain a predecessor in $P_{\alpha_i} - S_{\alpha_i}$.

Let

$$\varepsilon_i = \bigcup_{\{f : f \leqq i \,\&\, \chi_f \cap E^i \text{ is finite}\}} (\chi_f \cap E^i) \cup \bigcup_{\{f : f \leqq i \,\&\, \psi_f \cap E^i \text{ is finite}\}} (\psi_f \cap E^i).$$

The construction will force δ_i to be disjoint from ε_i. For $f \leqq i$, we will say that f is excluded from i whenever $\neg(E^i - \varepsilon_i \subseteq \chi_f \cup \psi_f)$. If f is excluded from i, let x_f be an element of $(E^i - \varepsilon_i) \cap [(\chi_f \cup \psi_f)]^-$. Let $\mu_i = \{x_f : f \text{ is excluded from } i\}$. The construction will force μ_i to be a subset of δ_i. Define $F_i = E^i - (\varepsilon_i \cup \mu_i)$. Since F_i differs from E^i by a finite set, F_i is an infinite recursive set. We observe that (i) if

$f \leqq i$ then $F_i \subseteq \chi_f \cup \psi_f$ unless f is excluded from i: that is, if $f \leqq i$ and $\neg(F_i \subseteq \chi_f \cup \psi_f)$, then f is excluded from i.

We next define a sequence of sets $\delta_{i,n}$ for $n=0, 1, \ldots, 8(i+1)$ inductively on n. Inductively we show, as the sequence is defined, that for each n: (a) $\delta_{i,n} \subseteq F_i$; (b) $\delta_{i,n}$ is a finite sum of sets $\lambda_1, \ldots, \lambda_m$ such that for each j there are sets ν_j and ζ_j satisfying $\lambda_j + \nu_j = \zeta_j$, $\langle \nu_j \rangle \in P_{\alpha_i} \cap S_{\alpha_i}$, and $\langle \zeta_j \rangle = Z_i$; and (c) if (ω, θ) is a disjoint pair of RE sets covering F_i and $\omega \cap \delta_{i,n-1}$ has a predecessor in $P_{\alpha_i} - S_{\alpha_i}$, then $\omega \cap \delta_{i,n}$ also has a predecessor in $P_{\alpha_i} - S_{\alpha_i}$. We remark that (ii) if f is not excluded from i and $f \leqq i$, then (χ_f, ψ_f) is a pair of sets satisfying the first condition of (c) as are $(\chi_{f,\mathrm{ev}}, \chi_{f,\mathrm{od}} \cup \psi_f)$, $(\chi_{f,\mathrm{od}}, \chi_{f,\mathrm{ev}} \cup \psi_f)$, $(\psi_{f,\mathrm{ev}}, \psi_{f,\mathrm{od}} \cup \chi_f)$, and $(\psi_{f,\mathrm{od}}, \psi_{f,\mathrm{ev}} \cup \chi_f)$.

Let $\delta_{i,0} = \varnothing$. Assume $\delta_{i,8f}$ defined. *Step* $8f+1$. If f is excluded from i, let

$$\delta_{i,8f+1} = \delta_{i,8f+2} = \cdots = \delta_{i,8f+8} = \delta_{i,8f}$$

and go to step $8(f+1)+1$. If f is not excluded from i, go to step $(8f+1)$(A). *Step* $(8f+1)$(A). If $\chi_f \cap E^i$ is finite, let

$$\delta_{i,8f+1} = \delta_{i,8f+2} = \delta_{i,8f+3} = \delta_{i,8f+4} = \delta_{i,8f}$$

and go to step $8f+5$. If $\chi_f \cap E^i$ is infinite, go to step $(8f+1)$(B). *Step* $(8f+1)$(B). If $\chi_{f,\mathrm{ev}} \cap \delta_{i,8f}$ has a predecessor in $P_{\alpha_i} - S_{\alpha_i}$, let $\delta_{i,8f+1} = \delta_{i,8f+2} = \delta_{i,8f}$ and go to step $8f+3$. If $\chi_{f,\mathrm{ev}} \cap \delta_{i,8f}$ has no predecessor in $P_{\alpha_i} - S_{\alpha_i}$, go to step $(8f+1)$(C).

Step $(8f+1)$(C). Let $\delta_{i,8f+1} = \delta_{i,8f} - (\delta_{i,8f} \cap \chi_{f,\mathrm{ev}})$. Go to step $8f+2$. Since $\delta_{i,8f} \cap \chi_{f,\mathrm{ev}} \prec \delta_{i,8f}$, the required separating sets being $(\chi_{f,\mathrm{ev}}, \chi_{f,\mathrm{od}} \cup \psi_f)$, and since $\delta_{i,8f} \cap \chi_{f,\mathrm{ev}}$ has no predecessor in $P_{\alpha_i} - S_{\alpha_i}$, we may apply (b) and Lemma 1.11 to conclude $\langle \delta_{i,8f} \cap \chi_{f,\mathrm{ev}} \rangle \in P_{\alpha_i} \cap S_{\alpha_i}$. We now verify conditions (a)–(c). (a) is clear. To demonstrate (b), suppose $\delta_{i,8f} = \lambda_1 + \cdots + \lambda_m$, $\lambda_j + \nu_j = \zeta_j$, $\langle \nu_j \rangle \in P_{\alpha_i} \cap S_{\alpha_i}$, and $\langle \zeta_j \rangle = Z_i$. Then $\delta_{i,8f+1} = \sum_{j=1}^{m} (\lambda_j - (\lambda_j \cap \chi_{f,\mathrm{ev}}))$. Furthermore, for each j,

$$\lambda_j = (\lambda_j - (\lambda_j \cap \chi_{f,\mathrm{ev}})) + (\lambda_j \cap \chi_{f,\mathrm{ev}}) \quad \text{since } (\chi_{f,\mathrm{od}} \cup \psi_f, \chi_{f,\mathrm{ev}}).$$

Therefore:

$$(\lambda_j - (\lambda_j \cap \chi_{f,\mathrm{ev}})) + ((\lambda_j \cap \chi_{f,\mathrm{ev}}) + \nu_j) = \zeta_j.$$

To complete the proof of (b) we have only to show that $\langle (\lambda_j \cap \chi_{f,\mathrm{ev}}) + \nu_j \rangle \in P_{\alpha_i} \cap S_{\alpha_i}$. Since $P_{\alpha_i} \cap S_{\alpha_i}$ is an ideal and $\langle \nu_j \rangle \in P_{\alpha_i} \cap S_{\alpha_i}$, it suffices to show $\langle \lambda_j \cap \chi_{f,\mathrm{ev}} \rangle \in P_{\alpha_i} \cap S_{\alpha_i}$. Since $\langle \delta_{i,8f} \cap \chi_{f,\mathrm{ev}} \rangle \in P_{\alpha_i} \cap S_{\alpha_i}$, it suffices to show

$$\lambda_j \cap \chi_{f,\mathrm{ev}} \prec \delta_{i,8f} \cap \chi_{f,\mathrm{ev}}.$$

Now

$$\begin{aligned} \delta_{i,8f} \cap \chi_{f,\mathrm{ev}} &= \Big(\lambda_j + \bigcup_{k \neq j} \lambda_k\Big) \cap \chi_{f,\mathrm{ev}} \\ &= (\lambda_j \cap \chi_{f,\mathrm{ev}}) + \Big(\Big(\bigcup_{k \neq j} \lambda_k\Big) \cap \chi_{f,\mathrm{ev}}\Big) \end{aligned}$$

and thus $\lambda_j \cap \chi_{f,\mathrm{ev}} \prec \delta_{i,8f} \cap \chi_{f,\mathrm{ev}}$. To demonstrate (c) suppose that (ω, θ) is a pair of disjoint RE sets covering F_i, that,

(iii) $\omega \cap \delta_{i,8f} = (\beta_1 \cap \delta_{i,8f}) + (\beta_2 \cap \delta_{i,8f})$ since (β_1, β_2),

and that $\langle \beta_1 \cap \delta_{i,8f} \rangle \in P_{\alpha_i} - S_{\alpha_i}$. It follows that

$$\omega \cap \delta_{i,8f+1} = (\beta_1 \cap \delta_{i,8f+1}) + (\beta_2 \cap \delta_{i,8f+1}) \quad \text{since } (\beta_1, \beta_2),$$

and because $\beta_1 \cup \beta_2 \supseteq \omega \cap \delta_{i,8f} \supseteq \omega \cap \delta_{i,8f+1}$. To demonstrate (c) it suffices to show that $\langle \beta_1 \cap \delta_{i,8f+1} \rangle \in P_{\alpha_i} - S_{\alpha_i}$. Now

$$\beta_1 \cap \delta_{i,8f} = (\beta_1 \cap \delta_{i,8f+1}) + (\beta_1 \cap \chi_{f,\mathrm{ev}} \cap \delta_{i,8f})$$

since $(\chi_{f,\mathrm{od}} \cup \psi_f, \chi_{f,\mathrm{ev}})$ and because $\delta_{i,8f} \subseteq \delta_{i,8f+1} \cup (\delta_{i,8f} \cap \chi_{f,\mathrm{ev}})$. Since

$$\langle \beta_1 \cap \delta_{i,8f} \rangle \in P_{\alpha_i} - S_{\alpha_i}, \langle \beta_1 \cap \delta_{i,8f+1} \rangle \in P_{\alpha_i}$$

by Lemma 1.5, and in order to prove $\langle \beta_1 \cap \delta_{i,8f+1} \rangle \notin S_{\alpha_i}$ it suffices to prove $\langle \beta_1 \cap \chi_{f,\mathrm{ev}} \cap \delta_{i,8f} \rangle \in P_{\alpha_i} \cap S_{\alpha_i}$ by Lemma 1.6. Now

$$\chi_{f,\mathrm{ev}} \cap \delta_{i,8f} = (\beta_1 \cap \omega \cap \chi_{f,\mathrm{ev}} \cap \delta_{i,8f}) + ((\beta_2 \cup \theta) \cap \chi_{f,\mathrm{ev}} \cap \delta_{i,8f})$$

since $(\beta_1 \cap \omega, \beta_2 \cup \theta)$ and because $(\beta_1 \cap \omega) \cup (\beta_2 \cup \theta) \supseteq \delta_{i,8f} \supseteq \chi_{f,\mathrm{ev}} \cap \delta_{i,8f}$. (To see the first inclusion we may argue as follows:

$$\begin{aligned}
&((\beta_1 \cap \omega) \cup (\beta_2 \cup \theta)) \cap \delta_{i,8f} \\
&\qquad = ((\beta_1 \cup \beta_2 \cup \theta) \cap (\omega \cup \beta_2 \cup \theta)) \cap \delta_{i,8f} \\
&\qquad = (\beta_1 \cup \beta_2 \cup \theta) \cap \delta_{i,8f} \quad \text{because } \omega \cup \theta \supseteq F_i \supseteq \delta_{i,8f} \\
&\qquad = ((\beta_1 \cup \beta_2) \cap \delta_{i,8f}) \cup (\theta \cap \delta_{i,8f}) \\
&\qquad = (\omega \cap \delta_{i,8f}) \cup (\theta \cap \delta_{i,8f}) \quad \text{by (iii)} \\
&\qquad = (\omega \cup \theta) \cap \delta_{i,8f} = \delta_{i,8f}.)
\end{aligned}$$

$$\beta_1 \cap \omega \cap \delta_{i,8f} = \beta_1 \cap \delta_{i,8f} \quad \text{by (iii) because } \beta_1 \cap \beta_2 = \varnothing.)$$

Therefore

$$\beta_1 \cap \omega \cap \chi_{f,\mathrm{ev}} \cap \delta_{i,8f} = \beta_1 \cap \chi_{f,\mathrm{ev}} \cap \delta_{i,8f} \prec \chi_{f,\mathrm{ev}} \cap \delta_{i,8f}.$$

Since $\langle \chi_{f,\mathrm{ev}} \cap \delta_{i,8f} \rangle \in P_{\alpha_i} \cap S_{\alpha_i}$, we have $\langle \beta_1 \cap \chi_{f,\mathrm{ev}} \cap \delta_{i,8f} \rangle \in P_{\alpha_i} \cap S_{\alpha_i}$ and (c) is proved.

Step $8f+2$. Let λ be a subset of $\chi_{f,\mathrm{ev}} \cap F_i$ such that $\langle \lambda \rangle = Z_i$. Such a λ exists since $\chi_{f,\mathrm{ev}} \cap F_i$ is an infinite RE set. Let $\delta_{i,8f+2} = \delta_{i,8f+1} \cup \lambda$. Go to step $8f+3$. Note that $\delta_{i,8f+2} = \delta_{i,8f+1} + \lambda$ since $(\chi_{f,\mathrm{od}} \cup \psi_f, \chi_{f,\mathrm{ev}})$. (a) holds since $\delta_{i,8f+1} \subseteq F_i$ and $\lambda \subseteq F_i$. (b) is immediate. To verify (c), suppose that (ω, θ) is a disjoint pair of RE sets covering F_i and that

$$\omega \cap \delta_{i,8f+1} = (\beta_1 \cap \delta_{i,8f+1}) + (\beta_2 \cap \delta_{i,8f+1}) \quad \text{since } (\beta_1, \beta_2)$$

where $\langle\beta_1 \cap \delta_{i,8f+1}\rangle \in P_{\alpha_i} - S_{\alpha_i}$. Then

$$\begin{aligned}\omega \cap \delta_{i,8f+2} &= \omega \cap (\delta_{i,8f+1} \cup \lambda) \\ &= (\omega \cap \delta_{i,8f+1}) \cup (\omega \cap \lambda) \\ &= (\beta_1 \cap \delta_{i,8f+1}) \cup (\beta_2 \cap \delta_{i,8f+1}) \cup (\omega \cap \lambda) \\ &= (\beta_1 \cap \delta_{i,8f+1}) + ((\beta_2 \cap \delta_{i,8f+1}) \cup (\omega \cap \lambda))\end{aligned}$$

since $(\beta_1 \cap (\chi_{f,\mathrm{od}} \cup \psi_f), \beta_2 \cup \chi_{f,\mathrm{ev}})$. This verifies (c).

Steps $8f+3$ and $8f+4$ are described by making two changes in the descriptions of the steps labelled $(8f+1)$(B) through $8f+2$. Each occurrence of $8f$ in the descriptions is replaced by $8f+2$, and $\chi_{f,\mathrm{ev}}$ and $\chi_{f,\mathrm{od}}$ are interchanged. With these changes, step $8f+3$ is the step labelled step $(8f+3)$(B).

Steps $8f+5$ through $8f+8$ are described by making similar changes in the descriptions of steps $(8f+1)$(A) through $8f+4$. More precisely, each occurrence of $8f$ is replaced by $8f+4$, and χ_f and ψ_f are to be interchanged throughout the descriptions of steps $(8f+1)$(A) through $8f+4$; in particular, $\chi_{f,\mathrm{ev}}$ is to be replaced with $\psi_{f,\mathrm{ev}}$ *etc.* With these changes, step $8f+5$ is the step labelled step $(8f+5)$(A).

Finally, set $\delta_i = \delta_{i,8(i+1)} \cup \mu_i = \delta_{i,8(i+1)} + \mu_i$ since $(E^i - \mu_i, \mu_i)$ and because $\delta_{i,8(i+1)} \subseteq F_i$ by (a). We observe that δ_i has the following properties.

(iv) There is a finite collection of sets $\lambda_1, \ldots, \lambda_m, \nu_1, \ldots, \nu_m, \rho$ such that $m \geqq 1$ and $\delta_i = \lambda_1 + \cdots + \lambda_m + \rho$, ρ is finite, and, for each j, $\langle\lambda_j + \nu_j\rangle = Z_i$ and $\langle\nu_j\rangle \in P_{\alpha_i} \cap S_{\alpha_i}$. This follows from (b) for $\delta_{i,8(i+1)}$, the finiteness of μ_i, and the observation that $\delta_{i,2} \neq \varnothing$ since 0 is not excluded from i.

(v) If $f \leqq i$ and $\chi_f \cap E^i$ $(\psi_f \cap E^i)$ is finite, then $\chi_f \cap \delta_i = \varnothing$ $(\psi_f \cap \delta_i = \varnothing)$. This is proved by noting that $\delta_{i,8(i+1)} \subseteq F_i$, hence $\delta_i \cap \varepsilon_i = \varnothing$.

(vi) If $f \leqq i$ and f is excluded from i, then $\neg(\delta_i \subseteq \chi_f \cup \psi_f)$. To see this recall $x_f \in \mu_i \subseteq \delta_i$ and $x_f \notin \chi_f \cup \psi_f$.

(vii) If $f \leqq i$ and f is not excluded from i and $\chi_f \cap E^i$ $(\psi_f \cap E^i)$ is infinite, then $\chi_{f,\mathrm{ev}} \cap \delta_i$ and $\chi_{f,\mathrm{od}} \cap \delta_i$ $(\psi_{f,\mathrm{ev}} \cap \delta_i$ and $\psi_{f,\mathrm{od}} \cap \delta_i)$ each contain a predecessor in $P_{\alpha_i} - S_{\alpha_i}$. Under the assumed conditions

$$\chi_{f,\mathrm{ev}} \cap \delta_{i,8f+2} \text{ and } \chi_{f,\mathrm{od}} \cap \delta_{i,8f+4}\ (\psi_{f,\mathrm{ev}} \cap \delta_{i,8f+6} \text{ and } \psi_{f,\mathrm{od}} \cap \delta_{i,8f+8})$$

each contain a predecessor in $P_{\alpha_i} - S_{\alpha_i}$. It follows by property (c) that $\delta_{i,8(i+1)}$, and hence δ_i, also satisfy the conclusion.

Let ξ be an infinite isolated subset of E and let $\beta = \bigcup_{i \in \xi} \delta_i$. β is the result of this construction: that is, $\langle\beta\rangle$ represents a typical element of $S[\{Z_i\}]$. We note that since ξ is isolated and each δ_i is isolated, β is isolated so that $\langle\beta\rangle \in \mathbf{\Lambda}$.

LEMMA 2.3. *For any finite n, there exist isols $X_1, \ldots, X_n, Y_1, \ldots, Y_n, Y$ such that $\langle\beta\rangle = X_1 + \cdots + X_n + Y$ and for each i there is a j which is greater than n and such that $X_i + Y_i = Z_j$ and $Y_i \in P_{\alpha_j} \cap S_{\alpha_j}$.*

Proof. The proof is immediate from (iv) since ξ is infinite.

LEMMA 2.4. *If $Z \leqq \langle\beta\rangle$ and $Z \notin P_\alpha \cap S_\alpha$, then there exist isols V and W such that $Z = V + W$, $V \notin P_\alpha \cap S_\alpha$, and $W \notin P_\alpha \cap S_\alpha$.*

Proof. If $Z \leqq \langle\beta\rangle$, then there exists a pair of disjoint RE subsets of $E \times E$, say (χ_f, ψ_f), such that $Z = \langle \chi_f \cap \beta\rangle$ and $\beta \subseteq \chi_f \cup \psi_f$. Let $\sigma = \{i : i \geqq f \ \& \ \chi_f \cap \delta_i \neq \varnothing\}$. First we show that σ is not finite. Suppose that σ is finite. Then

$$\begin{aligned} \chi_f \cap \beta &= \bigcup_{i \in \xi \& i < f} (\chi_f \cap \delta_i) \cup \bigcup_{i \in \sigma} (\chi_f \cap \delta_i) \\ &= \sum_{i \in \xi \& i < f} (\chi_f \cap \delta_i) + \sum_{i \in \sigma} (\chi_f \cap \delta_i) \\ &\prec \sum_{i \in \xi \& i < f} \delta_i + \sum_{i \in \sigma} \delta_i. \end{aligned}$$

We apply (iv) to see that each $\langle\delta_i\rangle \in I_{\alpha_i}$ since $\langle\delta_i\rangle$ is in the ideal generated by P_{α_i}. Thus each $\langle\delta_i\rangle \in P_\alpha \cap S_\alpha$ by Lemma 1.1 since each $\alpha_i < \alpha$. By Lemmas 1.1 and 1.3, any predecessor of a finite sum of elements of $P_\alpha \cap S_\alpha$ is an element of $P_\alpha \cap S_\alpha$. This, together with the above inequality gives $\langle \chi_f \cap \beta\rangle \in P_\alpha \cap S_\alpha$ contradicting the assumption that $Z \notin P_\alpha \cap S_\alpha$.

Since $\chi_f \cap \beta = (\chi_{f,\mathrm{ev}} \cap \beta) + (\chi_{f,\mathrm{od}} \cap \beta)$ it suffices, in order to prove the lemma, to prove $\langle \chi_{f,\mathrm{ev}} \cap \beta\rangle \notin P_\alpha \cap S_\alpha$ and $\langle \chi_{f,\mathrm{od}} \cap \beta\rangle \notin P_\alpha \cap S_\alpha$. If $i \in \xi$ and $f \leqq i$, then f is not excluded from i by (vi) since $\delta_i \subseteq \bigcup_{i \in \xi} \delta_i \subseteq \chi_f \cup \psi_f$. Furthermore, if $i \in \sigma$, then $\chi_f \cap E^i$ is infinite by (v). Therefore, the hypotheses of (vii) are satisfied whenever $i \in \sigma$ so that we may conclude that if $i \in \sigma$ then both $\chi_{f,\mathrm{ev}} \cap \delta_i$ and $\chi_{f,\mathrm{od}} \cap \delta_i$ contain a predecessor in $P_{\alpha_i} - S_{\alpha_i}$. If $\alpha = (\alpha - 1) + 1$ is a successor ordinal we use the fact that σ is infinite and Theorem 1.8 to conclude that $\langle \chi_{f,\mathrm{ev}} \cap \beta\rangle \notin I_{\alpha-1} = P_\alpha \cap S_\alpha$ and $\langle \chi_{f,\mathrm{od}} \cap \beta\rangle \notin P_\alpha \cap S_\alpha$. If α is a limit ordinal, then we argue by contradiction. Suppose $\langle \chi_{f,\mathrm{ev}} \cap \beta\rangle \in P_\alpha \cap S_\alpha = \bigcup_{\alpha' < \alpha} I_{\alpha'}$. Then for some $\alpha' < \alpha$, $\langle \chi_{f,\mathrm{ev}} \cap \beta\rangle \in I_{\alpha'}$. Since σ is unbounded and $\alpha_i \uparrow \alpha$, let $j \in \sigma$ be such that $\alpha_j > \alpha'$. By (vii) $\langle \chi_{f,\mathrm{ev}} \cap \beta\rangle$ contains a predecessor in $P_{\alpha_j} - S_{\alpha_j}$ and thus $\langle \chi_{f,\mathrm{ev}} \cap \beta\rangle \notin I_{\alpha'}$. This contradiction shows $\langle \chi_{f,\mathrm{ev}} \cap \beta\rangle \notin P_\alpha \cap S_\alpha$. The same argument shows $\langle \chi_{f,\mathrm{od}} \cap \beta\rangle \notin P_\alpha \cap S_\alpha$.

LEMMA 2.5. *If $Z \leqq \langle\beta\rangle$ and $Z \in P_\alpha \cap S_\alpha$, then there is a finite collection of isols $X_1, \ldots, X_n, V_1, \ldots, V_n, Y$ and indices $j_1, \ldots, j_n$ such that $Z \leqq \sum_{i=1}^n X_i + Y$, Y is finite, and, for each i, $X_i + V_i = Z_{j_i}$.*

Proof. Since $Z \leqq \langle\beta\rangle$, there is a disjoint pair of RE subsets of $E \times E$, say (χ_f, ψ_f), such that $Z = \langle \chi_f \cap \beta\rangle$ and $\beta \subseteq \chi_f \cup \psi_f$. If $\{i : \chi_f \cap \delta_i \neq \varnothing\}$ were infinite, we could conclude that $Z \notin P_\alpha \cap S_\alpha$ by using (vii) and Theorem 1.8 as in the previous proof. Thus $\{i : \chi_f \cap \delta_i \neq \varnothing\} = \rho$ is finite. By (iv) there exist sets $\lambda_{i,j}$, $\nu_{i,j}$, ρ_i such that $\delta_i = \lambda_{i,1} + \cdots + \lambda_{i,m_i} + \rho_i$, $\langle \lambda_{i,j} + \nu_{i,j}\rangle = Z_i$ and ρ_i is finite. Therefore

$$\begin{aligned} Z &\leqq \sum_{i \in \rho} \left(\sum_{j=1}^{m_i} \langle \lambda_{i,j}\rangle + \langle \rho_i \rangle \right) \\ &= \sum_{i \in \rho} \left(\sum_{j=1}^{m_i} \langle \lambda_{i,j}\rangle \right) + \sum_{i \in \rho} \langle \rho_i \rangle. \end{aligned}$$

3. Some existence theorems.

LEMMA 3.1. *If* $\alpha_i \uparrow \alpha$, $Z_i \in P_{\alpha_i} - S_{\alpha_i}$ *for each i, and* $Z \in P[\{Z_i\}]$, *then* $Z \in P_\alpha - S_\alpha$.

Proof. $Z \in P_\alpha$ follows immediately from Lemma 2.1 since $Z_i \in P_{\alpha_i} \subseteq P_\alpha$. If α is a successor ordinal, then since Z satisfies Lemma 2.2, it cannot satisfy Theorem 1.8 for $\alpha - 1$ and thus $Z \notin I_{\alpha-1} = P_\alpha \cap S_\alpha$. If α is a limit ordinal, we see that $Z \notin P_\alpha \cap S_\alpha$ using Lemma 2.1 and an argument similar to that used in Lemma 2.4 to show that $\langle \chi_{f,ev} \cap \beta \rangle \notin P_\alpha \cap S_\alpha$ in case α is a limit ordinal.

THEOREM 3.2. *For each ordinal* α, $0 < \alpha < \omega_1$, $P_\alpha - S_\alpha$ *is not empty.*

Proof. The proof is an induction on α. For $\alpha = 1$, the theorem is implied by Theorem 43(b) of Dekker-Myhill [4, p. 102]. Assume now that $\alpha > 1$. Let $\{\alpha_i\}_{i=0}^\infty$ be a sequence of ordinals such that $\alpha_i \uparrow \alpha$. By the inductive hypothesis there exists a sequence of isols, $\{Z_i\}_{i=0}^\infty$, such that, for each i, $Z_i \in P_{\alpha_i} - S_{\alpha_i}$. Let $Z \in P[\{Z_i\}]$. By the preceding lemma, $Z \in P_\alpha - S_\alpha$.

The following lemmas lead to a strengthening of this result. Theorem 3.8 asserts that $P_\alpha - S_\alpha$ is not only nonempty, but also contains a set of c isols any two of which are α-incomparable.

LEMMA 3.3. *If* $\alpha_i \uparrow \alpha$, $Z_i \in P_{\alpha_i} - S_{\alpha_i}$ *for each i,* $B \in P[\{Z_i\}]$, $Z \leqq B$, *and* $Z \in P_\alpha \cap S_\alpha$, *then there is a finite set* ρ *such that* $Z \leqq \sum_{i \in \rho} Z_i$.

Proof. Let $\{Z_i\}$, Z, and B satisfy the hypotheses of the lemma. Let Y be such that $Z + Y = B$. By Lemma 2.1, there is a finite set ρ such that $Z \leqq \sum_{i \in \rho} Z_i$ or $Y \leqq \sum_{i \in \rho} Z_i$. If the latter then $Y \in P_\alpha \cap S_\alpha$ since $\sum_{i \in \rho} Z_i \in P_\alpha \cap S_\alpha$. In this case $B = Z + Y \in P_\alpha \cap S_\alpha$ by Lemmas 1.1 and 1.3 since $Z \in P_\alpha \cap S_\alpha$ by assumption. This contradicts Lemma 3.1. Therefore $Z \leqq \sum_{i \in \rho} Z_i$.

DEFINITION. For $X \in P_1 - I_0$ and $1 \leqq \alpha < \omega_1$ we define $P_\alpha[X]$ by induction on α. $P_1[X] = \{X\}$. For $\alpha > 1$,

$$P_\alpha[X] = \bigcup_{\alpha_i \uparrow \alpha \,\&\, Z_i \in P_{\alpha_i}[X]} P[\{Z_i\}].$$

For the next three lemmas we assume that $X \in P_1 - I_0$.

LEMMA 3.4. *If* $X_\alpha \in P_\alpha[X]$, *then* $X_\alpha \in P_\alpha - S_\alpha$.

Proof. The proof is by induction on α. For $\alpha = 1$, the lemma is clear. The inductive step is proved using Lemma 3.1. If $X_\alpha \in P_\alpha[X]$, then there is a sequence of isols $\{Z_i\}_{i=0}^\infty$ and there is a sequence of ordinals $\{\alpha_i\}_{i=0}^\infty$ such that $\alpha_i \uparrow \alpha$, $X_\alpha \in P[\{Z_i\}]$, and $Z_i \in P_{\alpha_i}[X]$ for each i. By the inductive hypothesis, $Z_i \in P_{\alpha_i} - S_{\alpha_i}$. This shows that the hypotheses of Lemma 3.1 are satisfied for X_α so that we may conclude that $X_\alpha \in P_\alpha - S_\alpha$.

LEMMA 3.5. *If* $Y \leqq X_\alpha$, $X_\alpha \in P_\alpha[X]$, *and* $Y \in P_1 - I_0$, *then* $Y =_1 X$.

Proof. The proof is by induction on α. For $\alpha=1$, we have that $Y+Z=X$ and $Y\in P_1-I_0$ implies $Z\in I_0$ and $Y=_1 X$. Now assume $X_\alpha\in P[\{Z_i\}]$ where $\alpha_i\uparrow\alpha$ and $Z_i\in P_{\alpha_i}[X]$ for each i. Suppose $Y\leqq X_\alpha$ and $Y\in P_1-I_0$. Since $P_1\subseteq P_\alpha\cap S_\alpha$ we may apply Lemma 3.3 to prove the existence of a finite n such that $Y\leqq\sum_{i=1}^n Z_i$. By the refinement property there exist isols $Y_1,\ldots,Y_n$ such that $Y_i\leqq Z_i$ for $i=1,\ldots,n$ and $Y=\sum_{i=1}^n Y_i$. Since $Y\in P_1-I_0$ there is exactly one i' such that $Y_{i'}\notin I_0$. Therefore $Y=_1 Y_{i'}$. Since $Y_{i'}\leqq Z_{i'}\in P_{\alpha_{i'}}[X]$, we can apply the inductive hypothesis to conclude that $Y_{i'}=_1 X$. Therefore $Y=_1 X$.

LEMMA 3.6. *If $X_\alpha\in P_\alpha[X]$ and $Z=_\alpha X_\alpha$, then there is a Y such that $Y=_1 X$ and $Y\leqq Z$.*

Proof. Let X_α and Z be isols satisfying the hypothesis of the lemma. There exist isols V and W in $P_\alpha\cap S_\alpha$ such that $Z+V=X_\alpha+W$. Applying the refinement property, there exist isols Z_1, V_1, Z_2, V_2 such that

$$\begin{array}{ccccc} Z_1 & + & V_1 & = & X_\alpha \\ + & & + & & \\ Z_2 & + & V_2 & = & W \\ \| & & \| & & \\ Z & & V & & \end{array}$$

Since $V\in P_\alpha\cap S_\alpha$, $V_1\in P_\alpha\cap S_\alpha$. Since $X_\alpha\notin P_\alpha\cap S_\alpha$ by Lemma 3.4, $Z_1\notin P_\alpha\cap S_\alpha$. If there were no Y such that $Y\in P_1-I_0$ and $Y\leqq Z_1$, then we would have $Z_1\in S_1$ and thus $Z_1\in P_\alpha\cap S_\alpha$. Therefore there is a Y such that $Y\in P_1-I_0$ and $Y\leqq Z_1$. Since $Z_1\leqq Z$ and $Z_1\leqq X_\alpha$, $Y\leqq Z$ and $Y\leqq X_\alpha$. By Lemma 3.5, $Y=_1 X$.

LEMMA 3.7. *If $X\in P_1-I_0$, $U\in P_1-I_0$, and $\neg(X=_1 U)$, and if $X_\alpha\in P_\alpha[X]$ and $U_\alpha\in P_\alpha[U]$, then $\neg(X_\alpha=_\alpha U_\alpha)$.*

Proof. Let X, U, X_α, U_α satisfy the hypothesis and suppose $X_\alpha=_\alpha U_\alpha$. Then there exist isols V and W in $P_\alpha\cap S_\alpha$ such that $X_\alpha+V=U_\alpha+W$. Applying the refinement property, there exist isols $X_{\alpha,1}, X_{\alpha,2}, V_1, V_2$ such that

$$\begin{array}{ccccc} X_{\alpha,1} & + & V_1 & = & U_\alpha \\ + & & + & & \\ X_{\alpha,2} & + & V_2 & = & W \\ \| & & \| & & \\ X_\alpha & & V & & \end{array}$$

Since $X_{\alpha,2}\leqq W$ and $W\in P_\alpha\cap S_\alpha$, $X_{\alpha,2}\in P_\alpha\cap S_\alpha$. Therefore $X_{\alpha,1}=_\alpha X_\alpha$. By Lemma 3.6 there is an isol Y such that $Y\in P_1-I_0$, $Y\leqq X_{\alpha,1}$, and $Y=_1 X$. But $Y\leqq X_{\alpha,1}\leqq U_\alpha$ implies $Y\leqq U_\alpha$. Since $Y\in P_1-I_0$, we may apply Lemma 3.5 to see that $Y=_1 U$. $Y=_1 X$ and $Y=_1 U$ implies $X=_1 U$, contradicting the hypothesis.

THEOREM 3.8. *For any ordinal α, $0<\alpha<\omega_1$, there exists a set of c isols in $P_\alpha - S_\alpha$, any two of which are α-incomparable.*

Proof. For $\alpha=1$, this theorem is Corollary 1 of Theorem 44 of Dekker-Myhill [4, p. 103]. Suppose $1<\alpha<\omega_1$. Let Θ be a set of c indecomposable mutually incomparable isols: i.e., Θ is a set satisfying the conclusion of the theorem for $\alpha=1$. For each $X\in\Theta$, let X_α be an element in $P_\alpha[X]$. $\{X_\alpha : X\in\Theta\}$ is a set satisfying the conclusion of the theorem for α. For each $X\in\Theta$, $X_\alpha\in P_\alpha - S_\alpha$ by Lemma 3.4. If X and Y are elements of Θ and $X\neq Y$, then $\neg(X=_1 Y)$ since $X=_1 Y$ implies $X\leqq Y$ or $Y\leqq X$. By Lemma 3.7 it follows that $\neg(X_\alpha=_\alpha Y_\alpha)$. Finally, the corollary to Lemma 1.9 shows that X_α and Y_α are α-incomparable.

We now consider the second construction of the preceding section and derive existence theorems for $S_\alpha - P_\alpha$.

LEMMA 3.9. *If $\alpha_i \uparrow \alpha$, $Z_i\in P_{\alpha_i} - S_{\alpha_i}$ for each i, and $Z\in S[\{Z_i\}]$, then $Z\in S_\alpha - P_\alpha$.*

Proof. $Z\in S_\alpha$ follows immediately from Lemma 2.4. The proof that $Z\notin P_\alpha \cap S_\alpha$ is exactly the same as the proof of the corresponding assertion in Lemma 3.1 using Lemma 2.3 in place of Lemma 2.2.

THEOREM 3.10. *For each ordinal α, $0<\alpha<\omega_1$, $S_\alpha - P_\alpha$ is not empty.*

Proof. For $\alpha=1$, the theorem is implied by Theorem 49* of Dekker-Myhill [4, p. 112]. For $\alpha>1$, the theorem follows from the preceding lemma and Theorem 3.2.

The following definition and lemmas lead to Theorem 3.15, which is a strengthening of Theorem 3.10.

DEFINITION. For $X\in P_1 - I_0$ and $1<\alpha<\omega_1$ we define

$$S_\alpha[X] = \bigcup_{\alpha_i\uparrow\alpha \& Z_i\in P_{\alpha_i}[X]} S[\{Z_i\}].$$

For the next three lemmas we assume that $X\in P_1 - I_0$.

LEMMA 3.11. *If $X_\alpha\in S_\alpha[X]$, then $X_\alpha\in S_\alpha - P_\alpha$.*

Proof. The proof is a direct application of Lemmas 3.4 and 3.9.

LEMMA 3.12. *If $Y\leqq X_\alpha$, $X_\alpha\in S_\alpha[X]$, and $Y\in P_1 - I_0$, then $Y=_1 X$.*

Proof. Let Y and X_α satisfy the hypothesis of the lemma. Assume $X_\alpha\in S[\{Z_i\}]$ where $\alpha_i\uparrow\alpha$ and $Z_i\in P_{\alpha_i}[X]$ for each i. Since $Y\in P_\alpha\cap S_\alpha$, by Lemma 2.5 there exists a finite isol N and a finite sequence of indices $j_1,\ldots,j_n$ (possibly with repetitions) such that $Y\leqq \sum_{i=1}^n Z_{j_i}+N$. Applying the refinement property, we may decompose Y into a sum, $Y=\sum_{i=0}^n Y_i$ such that $Y_0\leqq N$ and $Y_i\leqq Z_{j_i}$ for $i>0$. Since $Y\in P_1 - I_0$, there is exactly one i' such that $Y_{i'}\notin I_0$. $i'\neq 0$ since $Y_0\leqq N$ and $N\in I_0$. By Lemma 3.5, since $Y_{i'}\leqq Z_{j_{i'}}\in P_\beta[X]$ where $\beta=\alpha_{j_{i'}}$, $Y_{i'}=_1 X$. Since $Y=_1 Y_{i'}$, $Y=_1 X$.

LEMMA 3.13. *If* $X_\alpha \in S_\alpha[X]$ *and* $Z =_\alpha X_\alpha$, *then there is a* Y *such that* $Y =_1 X$ *and* $Y \leqq Z$.

Proof. The proof is the same as the proof of Lemma 3.6 using Lemmas 3.11 and 3.12 in place of Lemmas 3.4 and 3.5.

LEMMA 3.14. *If* $X \in P_1 - I_0$, $U \in P_1 - I_0$, *and* $\neg(X =_1 U)$, *and if* $X_\alpha \in S_\alpha[X]$ *and* $U_\alpha \in S_\alpha[U]$, *then* $\neg(X_\alpha \leqq_\alpha U_\alpha)$.

Proof. Let X, U, X_α, U_α satisfy the hypotheses of the lemma and suppose $X_\alpha \leqq_\alpha U_\alpha$. Then there exist isols Z, V, W such that $V \in P_\alpha \cap S_\alpha$, $W \in P_\alpha \cap S_\alpha$, and $X_\alpha + Z + V = U_\alpha + W$. By the refinement property there exist isols $X_{\alpha,1}$, $X_{\alpha,2}$, Z_1, Z_2, V_1, V_2 such that

$$\begin{array}{ccccc} X_{\alpha,1} + & Z_1 + & V_1 & = & U_\alpha \\ + & + & + & & \\ X_{\alpha,2} + & Z_2 + & V_2 & = & W \\ \| & \| & \| & & \\ X_\alpha & Z & V & & \end{array}$$

Since $X_{\alpha,2} \leqq W$ and $W \in P_\alpha \cap S_\alpha$, $X_{\alpha,2} \in P_\alpha \cap S_\alpha$. Therefore $X_{\alpha,1} =_\alpha X_\alpha$ and, by Lemma 3.13, there is an isol Y such that $Y \leqq X_{\alpha,1}$ and $Y =_1 X$. Since $X_{\alpha,1} \leqq U_\alpha$, $Y \leqq U_\alpha$. Since $Y \in P_1 - I_0$ and $Y \leqq U_\alpha$, $Y =_1 U$ by Lemma 3.12. Therefore $X =_1 U$ contradicting the hypothesis.

THEOREM 3.15. *For any ordinal* α, $0 < \alpha < \omega_1$, *there exists a set of* c *isols in* $S_\alpha - P_\alpha$ *any two of which are* α*-incomparable.*

Proof. For $\alpha = 1$, the result may be proved in the following way. If δ is a retraceable set and $\langle\delta\rangle \leqq \langle\beta\rangle$, then δ is Turing-reducible to β. By Sacks [6, Chapter 2], there exists a set of c mutually incomparable Turing degrees. By Dekker-Myhill [3, pp. 364–365], each Turing degree except the lowest can be represented by an isolated retraceable set. Since each infinite isolated retraceable set represents an element of $S_1 - I_0$ (see Dekker-Myhill [4, Theorem 49*]), the theorem follows for $\alpha = 1$.

For $\alpha > 1$, the proof is similar to that of Theorem 3.8. Let Θ be a set of c indecomposable isols which are pairwise incomparable. For each $X \in \Theta$ let $X_\alpha \in S_\alpha[X]$. $\{X_\alpha : X \in \Theta\}$ is a set satisfying the conclusion of the theorem for α. For each $X \in \Theta$, $X_\alpha \in S_\alpha - P_\alpha$ by Lemma 3.11. If X and U are distinct elements of Θ, then $\neg(X =_1 U)$ since $X =_1 U$ implies $X \leqq U$ or $U \leqq X$. By Lemma 3.14, X_α and U_α are α-incomparable.

4. Multiple-free highly decomposable isols.

THEOREM 4.1. *Let* α *be a positive countable ordinal. There exists a* Z *such that* $Z \in S_\alpha - P_\alpha$ *and for no natural number* n, $n > 1$, *does there exist an* X *satisfying* $nX =_\alpha Z$.

Let $\alpha_i \uparrow \alpha$. Let $\{A_{i,j}\}$ be pairwise incomparable elements of $P_1 - I_0$ for $i=0, 1, 2, \ldots$ and $j=0, 1, 2, \ldots$.

MODIFIED CONSTRUCTION II. Modify Construction II of §2 in the following way. If, at the jth step in the construction of δ_i a representative of Z_i is added to $\delta_{i,j-1}$, instead add a representative of an element of $P_{\alpha_i}[A_{i,j}]$. The result of the modified construction will be called a typical element of $S_\alpha[\{A_{i,j}\}]$.

For each i there exist sets ρ_i and $\lambda_{i,j}$ and $\nu_{i,j}$ as j ranges over an appropriate nonempty subset of $\{1, \ldots, 8(i+1)\}$ such that $\delta_i = \sum_j \lambda_{i,j} + \rho_i$, ρ_i is finite, and, for each j, $\langle \lambda_{i,j} + \nu_{i,j} \rangle \in P_{\alpha_i}[A_{i,j}]$ and $\langle \nu_{i,j} \rangle \in P_{\alpha_i} \cap S_{\alpha_i}$. The proof is the same as that of (iv) in Construction II. Let $Z_{i,j} = \langle \lambda_{i,j} \rangle$. By Lemma 3.5, if $A \leqq Z_{i,j}$ and $A \in P_1 - I_0$, then $A =_1 A_{i,j}$. By Lemma 3.6, there is an A such that $A \in P_1 - I_0$ and $A \leqq Z_{i,j}$.

(4.1) If $Z \in S_\alpha[\{A_{i,j}\}]$, $Y \leqq Z$, and $Y \in P_\alpha \cap S_\alpha$, then there is a finite isol X and a finite range of indices (i,j) such that $Y \leqq \sum Z_{i,j} + X$.

The proof is the same as the proof of Lemma 2.5.

(4.2) If $Z \in S_\alpha[A_{i,j}]$, then $Z \in S_\alpha - P_\alpha$.

(4.2) is established by the same chain of reasoning that proves Lemma 3.9.

Until the end of the proof of Lemma 4.3 let $Z \in S_\alpha[\{A_{i,j}\}]$ and let the isols $Z_{i,j}$ be related to Z in the way specified above.

LEMMA 4.2. *If α is a countable successor ordinal greater than* 1, *then for any $n > 1$ there is no X such that $nX =_\alpha Z$.*

Proof. In order to avoid notational complexity it will only be shown that 2 does not divide Z with respect to $=_\alpha$. The general proof is essentially the same. Since $\alpha = (\alpha-1)+1$, each $Z_{i,j} \in P_{\alpha-1} - S_{\alpha-1}$. Suppose $2X =_\alpha Z$. Let V and W be such that $V \in P_\alpha \cap S_\alpha$, $W \in P_\alpha \cap S_\alpha$, and $2X + V = Z + W$. By Lemmas 1.2 and 1.8, there exist $V_0', \ldots, V_n'$, $W_0', \ldots, W_m'$ such that $V = \sum_{i=0}^n V_i'$, $W = \sum_{i=0}^m W_i'$, $V_0' \in S_{\alpha-1}$, $W_0' \in S_{\alpha-1}$, $V_i' \in P_{\alpha-1} - S_{\alpha-1}$ for $i = 1, \ldots, n$, and $W_i' \in P_{\alpha-1} - S_{\alpha-1}$ for $i = 1, \ldots, m$. Since there exist infinitely many $Z_{i,j}$ and the $Z_{i,j}$ are pairwise $(\alpha-1)$-incomparable by the proof of Theorem 3.8, there exists a $Z_{i,j}$ which is $(\alpha-1)$-incomparable with each V_i' and W_i' for $i > 0$. Let $Z_{i,j}$ have this property. Let $Z = Z_{i,j} + Z'$.

(i) if $A \leqq Z'$ and $A \in P_1 - I_0$, then $\neg(A =_1 A_{i,j})$. To see this suppose A satisfies the antecedent. Then $A + Z_{i,j} \leqq Z' + Z_{i,j} = Z$ and $A + Z_{i,j} \in I_{\alpha-1} = P_\alpha \cap S_\alpha$. By (4.1) there exist a finite set of pairs (i', j') and a finite isol Y for which

$$A + Z_{i,j} \leqq \sum Z_{i',j'} + Y.$$

$Z_{i,j}$ must be one of the $Z_{i',j'}$ by Lemma 1.10 so that $A \leqq \sum Z_{i',j'} + Y$ where the summation is over the same range of (i', j') as before except for (i, j). By the refinement property, there is an A' such that $A' =_1 A$ and $A' \leqq Z_{i',j'}$ for some (i', j'). Therefore $A' =_1 A_{i',j'}$ and thus $A =_1 A_{i',j'}$. Since $A_{i',j'}$ is incomparable with $A_{i,j}$, $\neg(A =_1 A_{i,j})$.

Apply the refinement property to the equation $2X+V=Z_{i,j}+Z'+W$ obtaining

$$
\begin{array}{ccccccl}
X_{1,1} & + & X_{1,2} & + & V_1 & = & Z_{i,j} \\
+ & & + & & + & & \\
X_{2,1} & + & X_{2,2} & + & V_2 & = & Z' \\
+ & & + & & + & & \\
X_{3,1} & + & X_{3,2} & + & V_3 & = & W \\
\| & & \| & & \| & & \\
X & & X & & V & &
\end{array} \tag{4.3}
$$

Since $Z_{i,j} \in P_{\alpha-1} - S_{\alpha-1}$, exactly one of the isols $X_{1,1}$, $X_{1,2}$, V_1 is $(\alpha-1)$-equal to $Z_{i,j}$. Since $Z_{i,j}$ is $(\alpha-1)$-incomparable with every predecessor of V in $P_{\alpha-1}-S_{\alpha-1}$, $X_{1,1} =_{\alpha-1} Z_{i,j}$ or $X_{1,2} =_{\alpha-1} Z_{i,j}$. Assume the former with no loss of generality.

Apply the refinement property to the equation

$$X_{1,2}+X_{2,2}+X_{3,2} = X_{1,1}+X_{2,1}+X_{3,1}$$

obtaining

$$
\begin{array}{ccccccl}
X_{1,2,1} & + & X_{2,2,1} & + & X_{3,2,1} & = & X_{1,1} \\
+ & & + & & + & & \\
X_{1,2,2} & + & X_{2,2,2} & + & X_{3,2,2} & = & X_{2,1} \\
+ & & + & & + & & \\
X_{1,2,3} & + & X_{2,2,3} & + & X_{3,2,3} & = & X_{3,1} \\
\| & & \| & & \| & & \\
X_{1,2} & & X_{2,2} & & X_{3,2} & &
\end{array} \tag{4.4}
$$

Again, exactly one of $X_{1,2,1}$, $X_{2,2,1}$, $X_{3,2,1}$ is $(\alpha-1)$-equal to $X_{1,1}$ since $X_{1,1} \in P_{\alpha-1}-S_{\alpha-1}$. $X_{1,2,1} \leqq X_{1,2} \in P_{\alpha-1} \cap S_{\alpha-1}$ so $\neg(X_{1,2,1} =_{(\alpha-1)} X_{1,1})$. $X_{3,2,1} \leqq X_{3,2} \leqq W$ and any predecessor of W in $P_{\alpha-1}-S_{\alpha-1}$ is $(\alpha-1)$-incomparable with $Z_{i,j}$. Since $X_{1,1} =_{(\alpha-1)} Z_{i,j}$, $\neg(X_{3,2,1} =_{(\alpha-1)} X_{1,1})$. Therefore $X_{2,2,1} =_{(\alpha-1)} X_{1,1} =_{(\alpha-1)} Z_{i,j}$. By Lemma 3.6, $X_{2,2,1}$ contains a predecessor A such that $A =_1 A_{i,j}$. But $X_{2,2,1} \leqq X_{2,2} \leqq Z'$ and thus $A \leqq Z'$. This contradicts (i) and proves the lemma.

LEMMA 4.3. *If α is a positive countable limit ordinal, then for any $n>1$ there is no X such that $nX =_\alpha Z$.*

Proof. Again it will only be shown that 2 does not divide Z with respect to $=_\alpha$. Suppose $2X =_\alpha Z$. Let V and W be such that $V \in P_\alpha \cap S_\alpha$, $W \in P_\alpha \cap S_\alpha$, and $2X+V=Z+W$. Then $V+W \in \bigcup_{\alpha'<\alpha} I_{\alpha'}$. Therefore $V+W \in I_{\alpha'}$ for some $\alpha' < \alpha$. Since Z satisfies an appropriately modified version of Lemma 2.3, there is an α'' and there is a pair (i,j) such that $\alpha' < \alpha''$ and $Z_{i,j} \in P_{\alpha''} - S_{\alpha''}$. Let α'' and (i,j) satisfy these properties. Let $Z = Z_{i,j} + Z'$. As in the proof of (i) for Lemma 4.2, it may be shown that (ii) if $A \leqq Z'$ and $A \in P_1 - I_0$, then $\neg(A =_1 A_{i,j})$.

As in the proof of Lemma 4.2, apply the refinement property to the equation $2X+V=Z_{i,j}+Z'+W$ obtaining the equations (4.3). Since $Z_{i,j} \in P_{\alpha''} - S_{\alpha''}$, exactly one of $X_{1,1}$, $X_{1,2}$, V_1 is α''-equal to $Z_{i,j}$. Since $V_1 \leqq V$, $V \in I_{\alpha'}$, and $I_{\alpha'} \subseteq P_{\alpha''} \cap S_{\alpha''}$, $\neg(V_1 =_{\alpha''} Z_{i,j})$. Assume without loss of generality that $X_{1,1} =_{\alpha''} Z_{i,j}$.

As in the proof of Lemma 4.2, reapply the refinement property to obtain (4.4). Exactly one of $X_{1,2,1}$, $X_{2,2,1}$, $X_{3,2,1}$ is α''-equal to $X_{1,1}$ since $X_{1,1} \in P_{\alpha''} - S_{\alpha''}$. $\neg(X_{1,2,1} =_{\alpha''} X_{1,1})$ since $X_{1,2,1} \leqq X_{1,2} \in P_{\alpha''} \cap S_{\alpha''}$. $\neg(X_{3,2,1} =_{\alpha''} X_{1,1})$ since $X_{3,2,1} \leqq X_{3,2} \leqq W$ and $W \in P_{\alpha''} \cap S_{\alpha''}$. Therefore $X_{2,2,1} =_{\alpha''} X_{1,1} =_{\alpha''} Z_{i,j}$. By Lemma 3.6, $X_{2,2,1}$ has a predecessor A such that $A =_1 A_{i,j}$. Since $X_{2,2,1} \leqq X_{2,2} \leqq Z'$, $A \leqq Z'$. This contradicts (ii) and proves the lemma.

LEMMA 4.4. *There exists an isol Z such that $Z \in S_1 - P_1$ and for any $n > 1$ there is no X such that $nX =_1 Z$.*

Proof. Define $\times^n \delta$ to be the n-fold Cartesian product of δ with itself for $\delta \subseteq E$ and $n > 1$. Let $\{\omega_i\}_{i=0}^{\infty}$ be the set of all infinite RE subsets of E. For fixed n, $n > 1$, let $\{f_{n,i}\}_{i=0}^{\infty}$ be the set of all 1–1 partial recursive functions whose domain is a subset of $\times^n E$ and includes $\times^n \delta$ for some infinite δ. If β and δ are subsets of E, define $\beta =_1 \delta$ if and only if there exist finite sets ρ and μ such that $\beta \cup \rho = \delta \cup \mu$.

To prove the lemma it suffices to prove the existence of a set satisfying (4.5), (4.6), and (4.7).

(4.5) For each pair (n, i) of natural numbers with $n > 1$, there is no set δ such that $f_{n,i}(\times^n \delta) =_1 \beta$.

(4.6) For each i, $\neg(\omega_i \subseteq \beta)$.

(4.7) For each i, $\omega_i \cap \beta$ is infinite.

(4.6) implies $\langle\beta\rangle \in \Lambda$, (4.7) implies $\langle\beta\rangle \in S_1 - P_1$, and (4.5) implies that $\langle\beta\rangle$ is multiple-free with respect to $=_1$.

(4.5) may be replaced by (4.8).

(4.8) For each pair (n, i) of natural numbers with $n > 1$ there are infinitely many pairs of natural numbers (x, y) satisfying (iii) $f_{n,i}(x, y, x, \ldots, x) \in \beta$ and $f_{n,i}(y, x, x, \ldots, x) \notin \beta$. To see that (4.8) implies (4.5) suppose $f_{n,i}(\times^n \delta) =_1 \beta$ for some β satisfying (4.8). Let ρ and μ be finite sets such that $f_{n,i}(\times^n \delta) \cup \rho = \beta \cup \mu$. Of the infinitely many pairs (x, y) satisfying (iii), $f_{n,i}(x, y, x, \ldots, x) \in \rho$ for only finitely many. Hence there are infinitely many pairs (x, y) satisfying (iii) such that $\{x, y\} \subseteq \delta$. Of these infinitely many pairs $f_{n,i}(y, x, x, \ldots, x) \in \mu$ for only finitely many. Therefore there are infinitely many pairs (x, y) satisfying (iii) and such that $f_{n,i}(y, x, x, \ldots, x) \in \beta$. This contradiction shows that it suffices to prove the existence of a set, β, satisfying (4.6), (4.7), and (4.8).

Let j be a function mapping E onto $\{(n, i) : n > 1\}$ in such a way that the inverse image of (n, i) is infinite for each (n, i) with $n > 1$. Let m be a function mapping E onto E in such a way that the inverse image of i is infinite for each $i \in E$. To construct β define a sequence of pairs of sets (β_k, δ_k) by induction on k. Let $\beta_0 = \delta_0 = \varnothing$. Assume inductively that $\beta_{k-1} \cap \delta_{k-1} = \varnothing$, $\beta_{k-1} \cup \delta_{k-1}$ is finite, and, for all $k' < k-1$, $\beta_{k'} \subseteq \beta_{k'+1}$ and $\delta_{k'} \subseteq \delta_{k'+1}$. If $k = 3l+1$, let $\beta_k = \beta_{k-1}$, $\delta_k = \delta_{k-1} \cup \{z\}$ where $z \in \omega_l \cap \bar{\beta}_{k-1}$. If $k = 3l+2$, let $\delta_k = \delta_{k-1}$, $\beta_k = \beta_{k-1} \cup \{z\}$ where

$$z \in \omega_{m(l)} \cap [(\beta_{k-1} \cup \delta_{k-1})]^{-}.$$

If $k=3l$ and $l>0$, let $\beta_k=\beta_{k-1}\cup\{f_{n,i}(x, y, x, \ldots, x)\}$ and let

$$\delta_k = \delta_{k-1}\cup\{f_{n,i}(y, x, x, \ldots, x)\}$$

where $(n, i)=j(l)$ and $\{f_{n,i}(x, y, x, \ldots, x), f_{n,i}(y, x, x, \ldots, x)\}\cap(\beta_{k-1}\cup\delta_{k-1})=\varnothing$. Note that in each of the three cases the inductive assumption remains satisfied. Define $\beta=\bigcup_{k=0}^{\infty}\beta_k$.

$\beta\cap(\bigcup_{k=0}^{\infty}\delta_k)=\varnothing$ since $\beta_k\cap\delta_k=\varnothing$ and $\beta_k\subseteq\beta_{k+1}$ and $\delta_k\subseteq\delta_{k+1}$ for all k. The definition of δ_{3l+1} shows that β satisfies (4.6) for i. The definition of β_{3l+2} for the infinitely many l such that $m(l)=i$ shows that β satisfies (4.7) for i. Finally the definitions of β_{3l} and δ_{3l} for the infinitely many positive l such that $j(l)=(n, i)$ show that β satisfies (4.8) for any (n, i) with $n>1$.

Proof of Theorem 4.1. For $\alpha=1$, apply Lemma 4.4. For $\alpha>1$, Theorem 3.8, (4.2), and either Lemma 4.2 if α is a successor ordinal or Lemma 4.3 if α is a limit ordinal imply the result.

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