REDUCED TEICHMÜLLER SPACES[1]

BY
CLIFFORD J. EARLE

Introduction. This paper falls naturally into three parts. In the first, §§1–5, we define and study the reduced Teichmüller space $T^{\#}(\Gamma)$ of a normalized Fuchsian group Γ. For a group Γ of the first kind, $T^{\#}(\Gamma)$ coincides with the Teichmüller space $T(\Gamma)$, and our results are for the most part restatements of results of Bers [6], [8]. For a group Γ of the second kind, $T^{\#}(\Gamma)$ coincides with the space $S(\Gamma)$ which we studied in [10], and our results continue the research begun in that paper. In the second part of the paper, §§6–7, we study the finite dimensional spaces $T^{\#}(\Gamma)$. We show in particular that $T^{\#}(\Gamma)$ has finite dimension if and only if Γ is finitely generated. In the final section, we study the reduced Teichmüller space of a Riemann surface.

The debt which this paper owes to the work of Ahlfors and Bers is obvious. The author is especially grateful to Professor Ahlfors for introducing him to the study of Teichmüller spaces.

1. **The reduced Teichmüller space.** We denote by G the group of all Möbius transformations which leave the upper half plane U invariant. G is a Lie group, canonically isomorphic to $SL(2, \mathbf{R})$ modulo its center. Let Γ be a discrete subgroup of G. The *limit set* $L(\Gamma)$ is the set of accumulation points of the orbit Γi. $L(\Gamma)$ is a subset of the extended real axis. We say that Γ is a *normalized Fuchsian group* if $0, 1, \infty \in L(\Gamma)$. The letters Γ, Γ' will always denote normalized Fuchsian groups. Γ is of the *first kind* if $L(\Gamma)$ is the extended real axis. Otherwise, $L(\Gamma)$ is a perfect nowhere dense subset of $\mathbf{R} \cup \{\infty\}$, and Γ is of the *second kind.*

We denote by Σ^* the group of all quasi-conformal self-mappings of U which leave the points $0, 1, \infty$ fixed. The mapping w in Σ^* is said to be *compatible* with the group Γ if $w \circ A \circ w^{-1}$ is conformal for each A in Γ. In this case, w *induces* the isomorphism $A \mapsto w \circ A \circ w^{-1}$ of Γ onto the normalized Fuchsian group $w \circ \Gamma \circ w^{-1}$. We call an isomorphism θ of Γ onto Γ' a *qc isomorphism* if some w in Σ^* induces θ. The set of *qc* isomorphisms with the domain Γ is the *reduced Teichmüller space* $T^{\#}(\Gamma)$.

Following Bers [6], we call w_1 and w_2 in Σ^* *equivalent* if $w_1(x) = w_2(x)$ for all real x. The *Teichmüller space* $T(\Gamma)$ is the set of all equivalence classes of mappings in Σ^* compatible with Γ. It is important to compare $T(\Gamma)$ and $T^{\#}(\Gamma)$.

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LEMMA 1. *Let w, w_0 in Σ^* induce θ, θ_0 in $T^{\#}(\Gamma)$ respectively. Then $\theta=\theta_0$ if and only if $w(x)=w_0(x)$ for all x in $L(\Gamma)$.*

Proof. We may assume that w_0 is the identity map. (Otherwise, replace w, θ, Γ by $w \circ w_0^{-1}$, $\theta \circ \theta_0^{-1}$, $\theta_0(\Gamma)$.) Suppose $w(x)=x$ for all x in $L(\Gamma)$. Then for all A in Γ and x in $L(\Gamma)$, $Ax=w(Ax)=\theta(A)x$. Since $L(\Gamma)$ contains more than two points, θ is the identity. Conversely, assume θ is the identity. Let x be the attractive fixed point of the hyperbolic transformation A in Γ. As $n\to\infty$, $A^n(w(i))\to x$. But $A^n(w(i))=w(A^n(i))\to w(x)$, so $w(x)=x$. Since the attractive fixed points are dense in $L(\Gamma)$ and w is continuous, the lemma is proved.

COROLLARY. *There is a canonical surjection f: $T(\Gamma)\to T^{\#}(\Gamma)$. f is injective if and only if Γ is of the first kind.*

Proof. If w induces θ in $T^{\#}(\Gamma)$, we put $f([w])=\theta$, where $[w]$ is the equivalence class of w. By Lemma 1, f is well defined for all Γ and injective for Γ of the first kind. Clearly, f is always surjective. Finally, if Γ is of the second kind, one can easily construct a map w in Σ^* which is compatible with Γ and induces the identity automorphism of Γ but does not leave all points of the real axis fixed.

2. **Beltrami differentials.** Every quasiconformal mapping w has distributional derivatives

$$w_z = \tfrac{1}{2}(w_x - iw_y), \qquad w_{\bar{z}} = \tfrac{1}{2}(w_x + iw_y)$$

which are locally L_2 functions and satisfy Beltrami's equation

$$w_{\bar{z}} = \mu w_z \tag{2.1}$$

where $\mu(z)$ belongs to the open unit ball in L_∞. Conversely, each μ in the open unit ball of $L_\infty(U)$ determines a unique mapping w_μ in Σ^* which satisfies (2.1).

μ in $L_\infty(U)$ is called a *Beltrami differential* of Γ if it satisfies

$$\mu(Az)A'(z)^*/A'(z) = \mu(z) \qquad \text{for all } A \text{ in } \Gamma. \tag{2.2}$$

The Beltrami differentials of Γ form a closed subspace of $L_\infty(U)$. We denote its open unit ball by $M(\Gamma)$. An easy calculation shows that w_μ is compatible with Γ if and only if $\mu \in M(\Gamma)$.

For μ in $M(\Gamma)$, let θ_μ be the *qc* isomorphism in $T^{\#}(\Gamma)$ induced by w_μ. *We endow $T^{\#}(\Gamma)$ with the quotient topology associated with the surjective map $\mu \mapsto \theta_\mu$.* The same topology is induced by the *Teichmüller metric* on $T^{\#}(\Gamma)$, which is defined as follows. For any *qc* isomorphism θ, let $K(\theta)$ be the least number K such that a K-quasiconformal mapping induces θ. If $\theta_1, \theta_2 \in T^{\#}(\Gamma)$, the Teichmüller distance $d(\theta_1, \theta_2)=\log K(\theta_2 \circ \theta_1^{-1})$. It is easy to prove that $T^{\#}(\Gamma)$ with the Teichmüller metric is a complete metric space.

3. **Quadratic differentials.** Let Ω $(=\Omega(\Gamma))$ be that component of the complement of $L(\Gamma)$ which contains the lower half plane U^*. Ω is either U^* or the complement of $L(\Gamma)$, according as Γ is of the first or second kind. In either case Ω is

invariant under Γ and carries a Poincaré metric $ds=\lambda(z)|dz|$ with curvature -4 which satisfies

$$\lambda(Az)|A'(z)| = \lambda(z) \qquad \text{for all } A \text{ in } \Gamma. \tag{3.1}$$

The Banach space $B^{\#}(\Gamma)$ of *bounded quadratic differentials* is the set of functions φ holomorphic in Ω, real at real points of Ω, and satisfying

$$\varphi(Az)A'(z)^2 = \varphi(z) \qquad \text{for all } A \text{ in } \Gamma \tag{3.2}$$

and

$$\|\varphi\| = \sup |\varphi(z)|\lambda(z)^{-2} < \infty. \tag{3.3}$$

For Γ of the first kind, $B^{\#}(\Gamma)$ is a complex Banach space under the usual operations. If Γ is of the second kind, only multiplication by real scalars is possible, and $B^{\#}(\Gamma)$ is a real Banach space.

THEOREM 1. *There is a canonical homeomorphism j which maps $T^{\#}(\Gamma)$ onto an open bounded domain in $B^{\#}(\Gamma)$. Each φ in $B^{\#}(\Gamma)$ with $\|\varphi\|<2$ is of the form $j(\theta_\mu)$ where $\mu(z)=-\frac{1}{2}\varphi(\bar{z})\lambda(\bar{z})^{-2}$.*

REMARK. $B^{\#}(\Gamma)$ carries a natural analytic structure, complex or real according as Γ is of the first or second kind. This structure, together with the mapping j, induces a *canonical analytic structure* on $T^{\#}(\Gamma)$.

Proof. For Γ of the first kind the theorem is a restatement of results of Bers ([8], Theorem 6) and Ahlfors-Weill ([4]; see also [2, Chapter VI C]). If $\mu \in M(\Gamma)$, let w^μ be the unique quasiconformal self-mapping of the plane which leaves $0, 1, \infty$ fixed, satisfies (2.1) in U, and is conformal in $U^*=\Omega$. We recall that $j(\theta_\mu)=\{w^\mu, z\}$, the Schwarzian derivative of w^μ.

Now suppose Γ is of the second kind. Let $\pi\colon U \to \Omega$ represent U as the universal covering surface of Ω. Let Γ^π be the group of B in G such that $\pi \circ B = A \circ \pi$ for some A in Γ. Γ^π is a Fuchsian group of the first kind. Let $j^\pi\colon T^{\#}(\Gamma^\pi) \to B^{\#}(\Gamma^\pi)$ be the canonical map. The map $j\colon T^{\#}(\Gamma) \to B^{\#}(\Gamma)$ will be defined with the help of j^π and two auxiliary mappings.

Extend each μ in $M(\Gamma)$ to Ω by $\mu(\bar{z})=\mu(z)^*$ and put $(\mu\cdot\pi)(z)=\mu(\pi(z))\pi'(z)^*/\pi'(z)$. $\mu \mapsto \mu\cdot\pi$ is an isometric mapping of $M(\Gamma)$ into $M(\Gamma^\pi)$. For φ in $B^{\#}(\Gamma)$, set $(\varphi\times\pi)(\bar{z})^* = \varphi(\pi(z))\pi'(z)^2$. $\varphi \mapsto \varphi\times\pi$ is an isometry of $B^{\#}(\Gamma)$ onto a real subspace $B'(\Gamma^\pi)$ of $B^{\#}(\Gamma^\pi)$. According to [10, Theorems 3 and 4], the map $\theta_\mu \mapsto j^\pi(\theta_{\mu\cdot\pi})$ is a bijection from $T^{\#}(\Gamma)$ to an open subset of $B'(\Gamma^\pi)$. We may therefore define $j(\theta_\mu)$ by the formula $j(\theta_\mu)\times\pi = j^\pi(\theta_{\mu\cdot\pi})$. j maps $T^{\#}(\Gamma)$ onto a bounded open domain in $B^{\#}(\Gamma)$. In [10], we made certain technical assumptions on the mapping π in order to prove that the image of $T^{\#}(\Gamma)$ is open in $B'(\Gamma^\pi)$. These assumptions are unnecessary. This follows from the fact, which we are about to prove, that j does not depend on the choice of π.

Suppose π is replaced by $\rho=\pi\circ A$ where $A\in G$. By the composition laws for Schwarzian derivatives,

$$j^{\rho}(\theta_{\mu\cdot\rho}) = \{w^{\mu\cdot\rho}, z\} = \{w^{\mu\cdot\pi}\circ A, z\} = \{w^{\mu\cdot\pi}, Az\}A'(z)^2.$$

Clearly $\varphi\times\rho=((\varphi\times\pi)\circ A)(A')^2$ for all φ in $B^{\#}(\Gamma)$. We have proved that j is independent of the choice of π.

Suppose now that $\varphi\in B^{\#}(\Gamma)$ and $\|\varphi\|<2$. From (3.1), (3.2), and (3.3) we find that $\mu(z)=-\frac{1}{2}\varphi(\bar z)\lambda(z)^{-2}$ belongs to $M(\Gamma)$. From the identity

$$\lambda(\pi(z))|\pi'(z)| = |z-\bar z|^{-1}$$

we obtain $(\mu\cdot\pi)(z)=-\frac{1}{2}|z-\bar z|^2(\varphi\times\pi)(\bar z)$, so that $j(\theta_\mu)=\varphi$.

We must still prove that j is a homeomorphism onto its image; in other words, that the map $\mu\mapsto j(\theta_\mu)$ is open and continuous. The continuity is clear, for the maps $\mu\mapsto\mu\cdot\pi$ and $\varphi\mapsto\varphi\times\pi$ are isometries, and j^{π} is continuous. Moreover, we have proved that each neighborhood of zero in $M(\Gamma)$ covers an open subset of $B^{\#}(\Gamma)$. Hence j^{-1} is continuous at the origin. We shall prove that j^{-1} is continuous at every point after we state Theorem 2, on which our proof depends. Apart from this step, the proof of Theorem 1 is complete.

Remark. By the definition of j for Γ of the second kind, the map $\theta_\mu\mapsto\theta_{\mu\cdot\pi}$ is a real analytic embedding of $T^{\#}(\Gamma)$ in $T^{\#}(\Gamma^{\pi})$.

4. **Allowable maps.** We shall define a class of allowable maps between reduced Teichmüller spaces. First we define two classes of primitive allowable maps. The first consists of all maps $\alpha: T^{\#}(\Gamma)\to T^{\#}(\Gamma_0)$ of the form $\alpha(\theta)=\theta\circ\theta_0$ where $\theta_0:\Gamma_0\to\Gamma$ belongs to $T^{\#}(\Gamma_0)$. The second class contains all maps $\alpha: T^{\#}(\Gamma)\to T^{\#}(\Gamma_0)$ such that $\Gamma=A\circ\Gamma_0\circ A^{-1}$ for some A in G and $\alpha(\theta_\mu)=\theta_{\mu\cdot A}$ for all μ in $M(\Gamma)$, where $(\mu\cdot A)(z)=\mu(Az)A'(z)^*/A'(z)$. An *allowable map* is the composite of finitely many primitive allowable maps.

Clearly, every allowable map $\alpha: T^{\#}(\Gamma)\to T^{\#}(\Gamma_0)$ preserves Teichmüller distances. Moreover, α is analytic. If Γ is of the first kind this follows from a theorem of Bers [8, Theorem 8]. If Γ is of the second kind and α is of the form $\theta\mapsto\theta\circ\theta_0$, the proof is given in [10, Theorem 5]. For Γ of the second kind and α of the form $\theta_\mu\mapsto\theta_{\mu\cdot A}$, the proof is an easy calculation. Indeed, let $j: T^{\#}(\Gamma)\to B^{\#}(\Gamma)$ and $j_0: T^{\#}(\Gamma_0)\to B^{\#}(\Gamma_0)$ be the canonical maps and let the map $\varphi\mapsto\varphi\times A$ of $B^{\#}(\Gamma)$ onto $B^{\#}(\Gamma_0)$ be defined by $(\varphi\times A)(z)=\varphi(Az)A'(z)^2$. Then $j_0(\theta_{\mu\cdot A})=j(\theta_\mu)\times A$, and α is analytic. Since the inverse of every allowable map is allowable we have proved

Theorem 2. *Every allowable map $\alpha: T^{\#}(\Gamma)\to T^{\#}(\Gamma_0)$ is an analytic diffeomorphism which is an isometry with respect to the Teichmüller metric.*

We shall now finish the proof of Theorem 1. Let Γ_0 be a group of the second kind, and let $j_0: T^{\#}(\Gamma_0)\to B^{\#}(\Gamma_0)$ be the canonical map. We shall prove that j_0^{-1} is continuous. Consider any $\theta_0:\Gamma_0\to\Gamma$ in $T^{\#}(\Gamma_0)$, and let $\alpha: T^{\#}(\Gamma)\to T^{\#}(\Gamma_0)$

be the allowable map $\alpha(\theta)=\theta\circ\theta_0$. α is a homeomorphism, for it preserves Teichmüller distances. Let $j\colon T^{\#}(\Gamma)\to B^{\#}(\Gamma)$ be the canonical map, and let $\beta=j_0\circ\alpha\circ j^{-1}$. By Theorem 2, β is a diffeomorphism. Moreover, we know that j^{-1} is continuous at the origin. Therefore $j_0^{-1}=\alpha\circ j^{-1}\circ\beta^{-1}$ is continuous at $\beta(0)=j_0^{-1}(\theta_0)$. Since we may choose θ_0 arbitrarily, the proof is complete.

5. **Smooth isomorphisms.** We call the isomorphism θ in $T^{\#}(\Gamma)$ *smooth* if there is a real analytic quasiconformal diffeomorphism w which induces θ. Equivalently, $\theta=\theta_\mu$ for some real analytic μ in $M(\Gamma)$. The following fact, well known to students of Teichmüller theory, has, to the author's knowledge, never been published.

THEOREM 3. *Every θ in $T^{\#}(\Gamma)$ is smooth.*

Proof. Let $S(\Gamma)$ be the set of smooth θ in $T^{\#}(\Gamma)$. By Theorem 1, $S(\Gamma)$ is a neighborhood of the origin. (The *origin* of $T^{\#}(\Gamma)$ is the identity self-mapping of Γ.) we assert that $S(\Gamma)$ is open and closed in $T^{\#}(\Gamma)$. If $\theta\colon\Gamma\to\Gamma'$ is smooth, the allowable map $\theta'\mapsto\theta'\circ\theta$ carries $S(\Gamma')$ into $S(\Gamma)$ and maps $S(\Gamma')$ onto a neighborhood of θ. Hence $S(\Gamma)$ is open. Next, let the sequence $\{\theta_n\}$ in $S(\Gamma)$ converge to $\theta\colon\Gamma\to\Gamma'$ in $T^{\#}(\Gamma)$. For large n, $\theta_n\circ\theta^{-1}\in S(\Gamma')$. Hence $\theta=(\theta_n\circ\theta^{-1})^{-1}\circ\theta_n$ is smooth, and $S(\Gamma)$ is closed. Since $T^{\#}(\Gamma)$ is connected, $S(\Gamma)=T^{\#}(\Gamma)$, and the theorem is proved.

6. **Finite dimensional spaces.** We denote by $\Omega'(\Gamma)$ the set of points in $\Omega(\Gamma)$ which are not left fixed by any element of Γ except the identity. The orbit space $\Gamma\backslash\Omega'(\Gamma)$ has a unique conformal structure such that the natural projection of $\Omega'(\Gamma)$ on $\Gamma\backslash\Omega'(\Gamma)$ is holomorphic.

The Riemann surface S is said to be *finite* if there are a closed surface S_0 and a conformal map $f\colon S\to S_0$ such that $S_0-f(S)$ is a finite point set.

THEOREM 4. *The following statements are equivalent*:

(a) *$T^{\#}(\Gamma)$ has finite dimension.*

(b) *$\Gamma\backslash\Omega'(\Gamma)$ is a finite Riemann surface.*

(c) *Γ is a finitely generated group.*

Proof. By Theorem 1, $T^{\#}(\Gamma)$ is an analytic manifold whose dimension equals the dimension of $B^{\#}(\Gamma)$. Therefore $T^{\#}(\Gamma)$ and $B^{\#}(\Gamma)$ are interchangeable in (a). Let $A(\Gamma)$ be the complex Banach space of functions φ holomorphic in Ω, satisfying (3.2), and with norm

$$\iint_{\Gamma\backslash\Omega}|\varphi(z)|\,dx\,dy<\infty.$$

It is well known [7, p. 213] that $\Gamma\backslash\Omega'(\Gamma)$ is finite if and only if $A(\Gamma)$ has finite dimension. Let $B(\Gamma)$ be the complex Banach space of all functions $\varphi=\varphi_1+i\varphi_2$ where $\varphi_1,\varphi_2\in B^{\#}(\Gamma)$, normed by (3.3). (If Γ is of the first kind, of course, $B(\Gamma)=B^{\#}(\Gamma)$.) According to a theorem of Ahlfors [1, Theorem 3], $B(\Gamma)$ is canonically isomorphic to the conjugate space of $A(\Gamma)$. Indeed, each bounded linear functional on $A(\Gamma)$ can be written uniquely in the form (φ, ψ), ψ in $B(\Gamma)$, where

$$(\varphi, \psi) = \iint_{\Gamma\backslash\Omega} \varphi(z)\psi(z)^*\lambda(z)^{-2}\,dx\,dy. \tag{6.1}$$

(This result for Γ of the first kind is due to Bers, [7, Theorem 1].) We conclude that (a) and (b) are equivalent.

It is well known and easy to prove that (b) implies (c). To obtain the reverse implication and complete the proof of the theorem, we shall prove that (c) implies (a). Let $\{A_1, \ldots, A_n\}$ be a set of generators for Γ. The map $P\colon \theta \mapsto (\theta(A_1), \ldots, \theta(A_n))$ is a one-to-one map of $T^{\#}(\Gamma)$ into G^n. It is clear, for instance from [3, Theorem 9], that P is continuous. But the Lie group G has finite dimension three. Hence, by invariance of domain, $T^{\#}(\Gamma)$ is finite dimensional. The theorem is proved.

Remark. The equivalence of (b) and (c) is proved in the manuscript of Nielsen and Fenchel [11]. Other proofs have been given by Ahlfors [1], Bers [7], Heins [12], and Marden [14]. The above proof is a rather simple application of the theory of moduli.

7. **P is an immersion.** In the proof of Theorem 4 we introduced a continuous one-to-one map $P\colon T^{\#}(\Gamma) \to G^n$. We shall now prove that P is regular at every point of $T^{\#}(\Gamma)$. (Recall that P is regular at θ if and only if P is real analytic at θ and its differential at θ is a one-to-one map.) We formulate this result as

Theorem 5. *Let $\{A_1, \ldots, A_n\}$ be a set of generators for Γ. The map*

$$P\colon \theta \mapsto (\theta(A_1), \ldots, \theta(A_n))$$

of $T^{\#}(\Gamma)$ into G^n is a real analytic one-to-one immersion.

Proof. We have already seen that P is one-to-one. We shall concentrate now on proving that P is regular at the origin, or, equivalently, $P \circ j^{-1}$ is regular at the origin in $B^{\#}(\Gamma)$.

By [3, Theorem 11 and its corollary], the map $\mu \mapsto \theta_\mu(A)$ from $M(\Gamma)$ into G is real analytic for each A in Γ. In addition,

$$\dot{A}[\mu](z) = \lim_{t\to 0} \frac{\theta_{t\mu}(A)(z) - z}{t} \qquad (t \text{ real})$$

exists for all z in U and satisfies

$$\dot{A}[\mu] = f[\mu] \circ A - A' f[\mu],$$

where

$$f[\mu](\zeta) = -\frac{1}{\pi}\iint_U (\mu(z)R(z, \zeta) + \mu(z)^* R(\bar{z}, \zeta))\,dx\,dy,$$

and $R(z, \zeta) = \zeta(\zeta - 1)/z(z-1)(z-\zeta)$. (See [2, Chapter VI D].)

We call the Beltrami differential μ *stationary* if $f[\mu] \circ A = A'f[\mu]$ for all A in Γ. From [1, Lemma 9], we find that μ is stationary if and only if

$$\iint_{\Gamma\backslash\Omega} \mu(z)\varphi(z)\, dx\, dy = 0 \qquad \text{for all } \varphi \text{ in } A(\Gamma). \tag{7.1}$$

Here μ is extended to Ω by putting $\mu(\bar{z}) = \mu(z)^*$, and $A(\Gamma)$ is as in the proof of Theorem 4.

Now for ψ in the open unit ball of $B^\#(\Gamma)$, we have

$$(P \circ j^{-1})(\psi) = (\theta_\mu(A_1), \ldots, \theta_\mu(A_n)),$$

where

$$\mu(z) = -\tfrac{1}{2}\psi(\bar{z})\lambda(\bar{z})^{-2}. \tag{7.2}$$

Since the map $\mu \mapsto \theta_\mu(A)$ is real analytic, $P \circ j^{-1}$ is real analytic at the origin. Moreover, the differential of $P \circ j^{-1}$ at the origin is a one-to-one mapping. Otherwise there would exist a nonzero μ of the form (7.2) which was stationary. From (7.1) we would obtain

$$\iint_{\Gamma\backslash\Omega} \varphi(z)\psi(z)^*\lambda(z)^{-2}\, dx\, dy = 0 \qquad \text{for all } \varphi \text{ in } A(\Gamma).$$

But this says that the inner product (6.1) vanishes for all φ in $A(\Gamma)$, which is impossible unless $\psi = 0$. We conclude that P is regular at the origin.

To complete the proof of the theorem, we again use allowable maps. Let $\theta\colon \Gamma \to \Gamma'$ belong to $T^\#(\Gamma)$, and let $\alpha\colon T^\#(\Gamma') \to T^\#(\Gamma)$ be the allowable map $\alpha(\theta') = \theta' \circ \theta$. The generators $\theta(A_1), \ldots, \theta(A_n)$ of Γ' determine the map $P' = P \circ \alpha$ of $T^\#(\Gamma')$ into G^n. By what we have already proved, P' is regular at the origin of $T^\#(\Gamma')$. Since α is an analytic diffeomorphism, P is regular at θ. The theorem is proved.

Suppose that the generators $A_1, \ldots, A_n$ form a "standard sequence of generators," in the terminology of Keen [13]. Then the image of $T^\#(\Gamma)$ under P is diffeomorphic to a cell (see [13, especially Theorem 6]). Moreover, one can show that in this case P is an embedding. We conclude that *$T^\#(\Gamma)$ is diffeomorphic to a cell whenever Γ is finitely generated.* These matters will be discussed in a forthcoming paper.

8. **Spaces of Riemann surfaces.** In this section, Γ will always be a normalized Fuchsian group which contains no elliptic transformations. We consider those Riemann surfaces which are conformally equivalent to $\Gamma\backslash U$ for some Γ. Such a surface is called a *type* I (*type* II) surface if the group Γ is of the first (second) kind. Every type II surface S has a *double* Σ of type I. Indeed, the double of the type II surface $\Gamma\backslash U$ is $\Gamma\backslash\Omega$. (The terminology type I and type II was introduced by Rodlitz [15].)

Let f_1 and f_2 be quasiconformal mappings of the Riemann surface S on the Riemann surfaces S_1 and S_2 respectively. We call f_1 and f_2 *equivalent* if there is a conformal map h of S_1 on S_2 such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity map. The *reduced Teichmüller space* $T^{\#}(S)$ is the set of equivalence classes. The following lemma, if Γ is of the first kind, is equivalent to a result of Bers [6, §5.4, Proposition A(1)]. It is essentially equivalent to the lemma of Ahlfors in [2, Chapter VI A].

LEMMA 2. *Assume that* Γ *contains no elliptic transformations. There is a canonical one-to-one map* ρ *of* $T^{\#}(\Gamma)$ *on* $T^{\#}(\Gamma\backslash U)$.

Proof. Let $\pi: U \to \Gamma\backslash U$ be the natural projection map. Each μ in $M(\Gamma)$ determines a Fuchsian group $\Gamma_\mu = w_\mu \circ \Gamma \circ (w_\mu)^{-1}$, a Riemann surface $\Gamma_\mu\backslash U$, and a quasiconformal mapping $f_\mu: \Gamma\backslash U \to \Gamma_\mu\backslash U$ such that $f_\mu \circ \pi \circ (w_\mu)^{-1}$ is the natural projection of U on $\Gamma_\mu\backslash U$. We put $\rho(\theta_\mu) = [f_\mu]$, the equivalence class of f_μ. By [5, p. 99], ρ is a well-defined map of $T^{\#}(\Gamma)$ into $T^{\#}(\Gamma\backslash U)$.

If f is any quasiconformal mapping with domain $\Gamma\backslash U$, there is a unique μ in $M(\Gamma)$ such that $f \circ \pi \circ (w_\mu)^{-1}$ is holomorphic, and f is equivalent to f_μ. Therefore ρ is surjective. To verify that ρ is one-to-one, suppose that $\rho(\theta_\mu) = \rho(\theta_\nu)$. According to [5, p. 99] there is a Möbius transformation $B: U \to U$ such that $\theta_\mu(A) \circ B = B \circ \theta_\nu(A)$ for all A in Γ. By the reasoning used in the proof of Lemma 1, $B \circ w_\nu = w_\mu$ on $L(\Gamma)$. Hence B is the identity (it leaves 0, 1, ∞ fixed), and $\theta_\mu = \theta_\nu$. The lemma is proved.

REMARK. This lemma makes clear the significance of Theorem 3. If μ in $M(\Gamma)$ is real analytic, then the quasiconformal map $f_\mu: \Gamma\backslash U \to \Gamma_\mu\backslash U$ used in Lemma 2 to define $\rho(\theta_\mu)$ is a real analytic diffeomorphism. Since each θ in $T^{\#}(\Gamma)$ is smooth, each equivalence class of maps in $T^{\#}(\Gamma\backslash U)$ contains a real analytic diffeomorphism.

Each quasiconformal map $f_0: S_0 \to S$ induces an *allowable map* $[f] \mapsto [f \circ f_0]$ of $T^{\#}(S)$ on $T^{\#}(S_0)$. We also call $\alpha: T^{\#}(S) \to T^{\#}(\Gamma)$ an allowable map if $\alpha = \rho^{-1} \circ \beta$ where $\beta: T^{\#}(S) \to T^{\#}(\Gamma\backslash U)$ is allowable and $\rho: T^{\#}(\Gamma) \to T^{\#}(\Gamma\backslash U)$ is the canonical map of Lemma 2. Since S is conformally equivalent to some $\Gamma\backslash U$, there always exists an allowable map of $T^{\#}(S)$ on some $T^{\#}(\Gamma)$. Recall that in §4 the class of allowable maps $\alpha: T^{\#}(\Gamma) \to T^{\#}(\Gamma')$ was defined. We omit the easy proof of

LEMMA 3. *Let* $\alpha: T^{\#}(S) \to T^{\#}(\Gamma)$ *be an allowable map. Then* $\alpha': T^{\#}(S) \to T^{\#}(\Gamma')$ *is allowable if and only if* $\alpha' \circ \alpha^{-1}: T^{\#}(\Gamma) \to T^{\#}(\Gamma')$ *is allowable.*

According to Theorem 2, every allowable map $\alpha: T^{\#}(\Gamma) \to T^{\#}(\Gamma')$ preserves the analytic and metric structures. Therefore, Lemma 3 allows us to define the analytic and metric structures of $T^{\#}(S)$ as the structures induced on $T^{\#}(S)$ by any allowable map $\alpha: T^{\#}(S) \to T^{\#}(\Gamma)$.

THEOREM 6. (a) *If* S *is a type* I *surface*, $T^{\#}(S)$ *is a complex analytic manifold.*

(b) *If S is a type* II *surface*, $T^{\#}(S)$ *is a real analytic manifold. If* Σ *is the double of* S, *there is a canonical real analytic imbedding of* $T^{\#}(S)$ *in* $T^{\#}(\Sigma)$.

(c) $T^{\#}(S)$ *has finite dimension if and only if S has a finitely generated fundamental group.*

Proof. Let $\alpha: T^{\#}(S) \to T^{\#}(\Gamma)$ be an allowable map. If S is a type I surface, Γ is a group of the first kind and (a) follows from Theorem 1. Similarly, if S is a type II surface, Γ is of the second kind, and Theorem 1 implies that $T^{\#}(S)$ is a real analytic manifold.

To complete the proof of (b), let $\pi: U \to \Omega(\Gamma)$ be a holomorphic covering map and let Γ^{π} be the Fuchsian group of B in G such that $\pi \circ B = A \circ \pi$ for some A in Γ. Then π determines a conformal map h of $\Gamma^{\pi}\backslash U$ on $\Gamma\backslash\Omega$, and this map determines a natural extension of the allowable map $\alpha: T^{\#}(S) \to T^{\#}(\Gamma)$ to an allowable map $\beta: T^{\#}(\Sigma) \to T^{\#}(\Gamma^{\pi})$. Indeed, α is determined by a quasiconformal map f of S on $\Gamma\backslash U$. f extends by reflection to a quasiconformal map f of Σ on $\Gamma\backslash\Omega$, and $h^{-1} \circ f: \Sigma \to \Gamma^{\pi}\backslash U$ determines β.

The natural map s of $T^{\#}(S)$ into $T^{\#}(\Sigma)$ is defined as follows. If $f: S \to S'$ is quasiconformal, then S' is a type II surface and f extends by reflection to a quasiconformal mapping $f: \Sigma \to \Sigma'$. We put $s([f]) = [f]$. This map is well defined by [9, Theorem 2]. Moreover, if α, β, Γ, and Γ^{π} are as above, then $\beta \circ s \circ \alpha^{-1}$ is simply the map $\theta_{\mu} \mapsto \theta_{\mu\cdot\pi}$ of $T^{\#}(\Gamma)$ into $T^{\#}(\Gamma^{\pi})$ which was defined during the proof of Theorem 1. It was remarked at the end of §3 that this map is a real analytic imbedding. Hence, so is the natural map s.

It remains to prove (c). It is well known that S has a finitely generated fundamental group if and only if either S is a finite (type I) surface in the sense of §6, or else S is a type II surface and Σ is finite. Thus, (c) is an immediate consequence of Theorem 4, and Theorem 6 is proved.

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INSTITUTE FOR ADVANCED STUDY,
PRINCETON, NEW JERSEY