MAXIMAL R-SETS, GRASSMANN SPACES, AND STIEFEL SPACES OF A HILBERT SPACE

BY

ERHARD LUFT

1. **Introduction.** The ordinary Fredholm theory of Hilbert spaces was generalized in [4] and [5] to apply to a wider range of problems in analysis. The ideal of compact linear maps, which plays a fundamental role in the ordinary theory, was replaced by a certain class of C^*-algebras called R-algebras in [4]. This generalization is based on the Rellich criterion for compact linear maps of a Hilbert space, which states that a linear map is compact if and only if for every $\varepsilon > 0$ there exists a closed linear subspace of finite codimension such that the norm of the restriction of the linear map to this subspace becomes smaller than ε. The R-algebras in [4] are defined by substituting for the collection of closed linear subspaces of finite codimension in this criterion a collection of closed linear subspaces which is an R-set.

The R-sets are subsets of the metric space of all closed linear subspaces of a Hilbert space, which satisfy a condition related to the metric. For the explicit definition we refer to Chapter 4. Each R-set is contained in a uniquely determined smallest maximal R-set. The algebraic objects attached to an R-set depend on this maximal R-set only, and its structure is of crucial importance.

The path components of a maximal R-set are simply Grassmann manifolds for a finite-dimensional Hilbert space. In the following we present a general theory of Grassmann and Stiefel manifolds associated with a maximal R-set of a Hilbert space. This theory contains many important features of the ordinary theory of Grassmann and Stiefel manifolds of a finite-dimensional Hilbert space. Our constructions and arguments are independent of the dimension, and the ordinary theory for finite-dimensional Hilbert spaces and its direct generalization to Hilbert spaces of arbitrary dimension appear as special cases.

We show that the path components of a maximal R-set are Banach manifolds and homogeneous spaces determined by the action of the group of continuous isomorphisms of the Hilbert space onto itself which leave the maximal R-set invariant. Furthermore, there is a natural fibre bundle structure.

Several other Banach manifolds are associated in a canonical way with a path component of a maximal R-set. They are homogeneous spaces with a fibre bundle structure and direct analogues of the ordinary Stiefel manifolds.

Finally, we investigate the maximal R-set of all closed linear subspaces of a

Hilbert space. This special case gives directly the ordinary theory of Grassmann and Stiefel spaces for arbitrary not necessarily finite-dimensional Hilbert spaces. The result of [9], that the general linear group of an infinite-dimensional Hilbert space is contractible, implies the Grassmann and Stiefel spaces of an infinite-dimensional Hilbert space are universal classifying spaces for the general linear group of certain Hilbert spaces.

We tried to make this manuscript as self-contained as possible. We avoided any spectral theory and elaborated a purely "geometric" approach. Further applications are planned in subsequent publications.

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2. **Preliminaries.** H denotes in the following a real, complex, or quaternionic Hilbert space, which also can be finite dimensional. If $x, y \in H$, then (x, y) denotes the inner product of x and y, and $\|x\| = (x, x)^{1/2}$ the length of x. If H and K are two Hilbert spaces, then $\mathfrak{L}(H, K)$ is the linear space of continuous ($=$ bounded) linear maps ($=$ operators) from H to K with the norm

$$\|\alpha\| = \sup \{\|\alpha(x)\|; \ x \in H \text{ and } \|x\| = 1\} \quad \text{for each } \alpha \in \mathfrak{L}(H).$$

We denote $\mathfrak{L}(H, H)$ by $\mathfrak{L}(H)$. $\mathbf{1} \in \mathfrak{L}(H)$ is the identity map. The general linear group $GL(H)$ of the Hilbert space H is the topological subspace of $\mathfrak{L}(H)$ consisting of the invertible elements of $\mathfrak{L}(H)$. It is an open subset of $\mathfrak{L}(H)$. The mapping Inv: $GL(H) \to GL(H)$ which assigns to any $\gamma \in GL(H)$ its inverse Inv $(\gamma) = \gamma^{-1}$ is continuous. Consequently, $GL(H)$ is a topological group. An element $\mu \in GL(H)$ is called unitary or orthogonal if $\|\mu(x)\| = \|x\|$ for all $x \in H$. The unitary elements of $GL(H)$ form a closed subgroup $U(H)$ of $GL(H)$.

If S is a closed linear subspace, then $S^\perp$ denotes the orthogonal complement of S in H, and $\pi_S: H \to H$ the orthogonal projection of H onto S. We notice that $\pi_S \in \mathfrak{L}(H)$, $\|\pi_S\| = 1$, and $\pi_{S^\perp} = \mathbf{1} - \pi_S$.

THEOREM 2.1. *If* $\gamma \in \mathfrak{L}(H, K)$ *maps* H *one-to-one onto* K *then* $\gamma^{-1} \in \mathfrak{L}(K, H)$.

Proof. See, for example, [11, p. 18].

For each $\alpha \in \mathfrak{L}(H, K)$ the adjoint $\alpha^* \in \mathfrak{L}(K, H)$ is defined. A map $\alpha \in \mathfrak{L}(H)$ is called self-adjoint, if $\alpha^* = \alpha$. We recall a few elementary properties of the adjoint operation which will be frequently used.

LEMMA 2.1. $(\alpha(x), y) = (x, \alpha^*(y))$ *for each* $x \in H, y \in K$ *and* $\alpha \in \mathfrak{L}(H, K)$ *(definition). If* $\alpha \in \mathfrak{L}(H, K)$, *then* $\|\alpha^*\| = \|\alpha\|$, *kernel* $(\alpha) = (\alpha^*(K))^\perp$, *kernel* $(\alpha^*) = (\alpha(H))^\perp$, *and* $\alpha(H)$ *is closed if and only if* $\alpha^*(K)$ *is closed. If* $\alpha \in \mathfrak{L}(H, K)$ *and* $\beta \in \mathfrak{L}(K, L)$, *then*

$(\beta \cdot \alpha)^* = \alpha^* \cdot \beta^*$. *The orthogonal projection π_S of the Hilbert space H onto the closed linear subspace S of H is self-adjoint.*

LEMMA 2.2. *If $\alpha \in \mathfrak{L}(H)$ and S is a closed linear subspace with $\alpha(S)$ is closed, then there exists a $\beta \in \mathfrak{L}(H)$ such that*

$$\pi_{\alpha(S)} = \alpha \cdot \pi_S \cdot \beta \cdot \pi_S \cdot \alpha^*.$$

Proof. We consider the map $\tilde{\alpha} = \alpha \cdot \pi_S \colon H \to H$. We have kernel $(\tilde{\alpha}) = (\tilde{\alpha}^*(H))^\perp$ and kernel $(\tilde{\alpha}^*) = (\tilde{\alpha}(H))^\perp$. Since $\tilde{\alpha}(H) = \alpha(S)$ is closed, we conclude that $(\text{kernel } (\tilde{\alpha}^*))^\perp = \alpha(S)$ and $(\text{kernel } (\tilde{\alpha}))^\perp = \tilde{\alpha}^*(H)$. The restrictions $\tilde{\alpha}' \colon \tilde{\alpha}^*(H) \to \alpha(S)$ of $\tilde{\alpha}$ and $\tilde{\alpha}^{*\prime} \colon \alpha(S) \to \tilde{\alpha}^*(H)$ of $\tilde{\alpha}^*$ are therefore well defined and isomorphisms by Theorem 2.1. We observe that $\tilde{\alpha}^* = \tilde{\alpha}^* \cdot \pi_{\alpha(S)}$. Consider $(\tilde{\alpha}')^{-1} \cdot (\tilde{\alpha}^{*\prime})^{-1} \colon \tilde{\alpha}^*(H) \to \tilde{\alpha}^*(H)$, and let $\beta = \iota \cdot ((\tilde{\alpha}')^{-1} \cdot (\tilde{\alpha}^{*\prime})^{-1}) \cdot \pi \in \mathfrak{L}(H)$, where $\iota \colon \tilde{\alpha}^*(H) \to H$ is the natural inclusion and $\pi \colon H \to \tilde{\alpha}^*(H)$ is the orthogonal projection onto the closed linear subspace $\tilde{\alpha}^*(H)$. From the construction follows immediately

$$\pi_{\alpha(S)} = \tilde{\alpha} \cdot \beta \cdot \tilde{\alpha}^* \cdot \pi_{\alpha(S)} = \tilde{\alpha} \cdot \beta \cdot \tilde{\alpha}^* = \alpha \cdot \pi_S \cdot \beta \cdot \pi_S \cdot \alpha^*.$$

The following definition will be very useful.

DEFINITION 2.1. Let $\alpha \in \mathfrak{L}(H, K)$ and let S be a linear subspace of H, then

$$c(S, \alpha) = \inf \{ \|\alpha(x)\| \, ; \, x \in S \text{ and } \|x\| = 1 \}.$$

LEMMA 2.3. *Let $\alpha \in \mathfrak{L}(H, K)$ and let S be a closed linear subspace of H. Then kernel $(\alpha) \cap S = \{o\}$, and $\alpha(S)$ is closed if and only if $c(S, \alpha) > 0$. If $c(S, \alpha) > 0$, then the restriction $\alpha' \colon S \to \alpha(S)$ is an isomorphism and*

$$c(S, \alpha) = (\|\alpha'^{-1}\|)^{-1}.$$

Proof. Suppose $c(S, \alpha) > 0$. Then certainly kernel $(\alpha) \cap S = \{o\}$. Now let $y \in K$ and let $\{\alpha(x_n)\}_{n=1}^\infty$ be a sequence with $\lim \alpha(x_n) = y$. The sequence $\{x_n\}_{n=1}^\infty$ is then a Cauchy sequence since

$$\|x_n - x_m\| \leq (c(S, \alpha))^{-1} \cdot \|\alpha(x_n) - \alpha(x_m)\|.$$

Let $x = \lim x_n$. Then $\alpha(x) = y$. Therefore $\alpha(S)$ is closed. Assume now kernel $(\alpha) \cap S = \{o\}$ and $\alpha(S)$ is closed. $\alpha(S)$ is a Hilbert space and we consider the restriction $\alpha' \colon S \to \alpha(S)$. By Theorem 2.1, $\alpha'^{-1} \in \mathfrak{L}(\alpha(S), S)$. We have

$$\frac{\|\alpha'^{-1}(\alpha'(x))\|}{\|\alpha'(x)\|} = \left(\frac{\|\alpha'(x)\|}{\|x\|} \right)^{-1} \leq (c(S, \alpha))^{-1} \quad \text{for } x \in S \text{ and } x \neq o.$$

With the first expression we can approximate $\|\alpha'^{-1}\|$.

LEMMA 2.4. *Suppose $\alpha \in \mathfrak{L}(H, K)$ with $\alpha(H)$ is closed. Then we have*

$$c(\alpha(H), \alpha^*) = c(\alpha^*(H), \alpha).$$

Proof. Since $\alpha(H)$ is closed, $\alpha^*(K)$ is closed also (Lemma 2.1). The restrictions $\alpha': \alpha^*(K) \to \alpha(H)$ of α and $\alpha^{*\prime}: \alpha(H) \to \alpha^*(K)$ of α^* are well defined and isomorphisms (Theorem 2.1).

$$c(\alpha(H), \alpha^*) = (\|\alpha^{*\prime -1}\|)^{-1} = (\|\alpha'^{*-1}\|)^{-1} = (\|(\alpha'^{-1})^*\|)^{-1} = (\|\alpha'^{-1}\|)^{-1}$$
$$= c(\alpha^*(H), \alpha).$$

DEFINITION 2.2. $\mathfrak{A} \subset \mathfrak{L}(H)$ is called a C^*-algebra, if $\mathfrak{A}$ is a subalgebra of $\mathfrak{L}(H)$, a closed subset of $\mathfrak{L}(H)$, and $(\mathfrak{A})^* = \mathfrak{A}$. If $1 \in \mathfrak{A}$, then we have the closed subgroups

$$\mathrm{GL}(\mathfrak{A}) = \{\gamma; \gamma \in \mathrm{GL}(H) \text{ with } \gamma, \gamma^{-1} \in \mathfrak{A}\} \text{ and}$$

$$\mathrm{U}(\mathfrak{A}) = \mathrm{U}(H) \cap \mathfrak{A}$$

of the topological group $\mathrm{GL}(H)$.

$\mathrm{GL}(\mathfrak{A})$ is an open subset of the Banach space $\mathfrak{A}$. (There is an open neighborhood of 1 in $\mathfrak{A}$ contained in $\mathrm{GL}(\mathfrak{A})$, since for γ sufficiently close to 1 the inverse exists and can be represented by a geometric series which is convergent in $\mathfrak{A}$.)

LEMMA 2.5. Let $\mathfrak{A} \subset \mathfrak{L}(H)$ be a C^*-algebra with $1 \in \mathfrak{A}$. If $\gamma \in \mathrm{GL}(H) \cap \mathfrak{A}$ is self-adjoint, then $\gamma^{-1} \in \mathrm{GL}(H) \cap \mathfrak{A}$.

Proof. Consider $c \cdot \gamma^2$, where c is a real number. We compute

$$((c \cdot \gamma^2 - 1)(x), (c \cdot \gamma^2 - 1)(x)) = c^2 \cdot (\gamma^2(x), \gamma^2(x)) - 2c \cdot (\gamma(x), \gamma(x)) + (x, x)$$

$$\leq c^2 \cdot \|\gamma\|^4 \cdot \|x\|^2 - 2c \cdot (\|\gamma^{-1}\|)^{-2} \cdot \|x\|^2 + \|x\|^2$$

$$= [1 + \|\gamma\|^4 \cdot (c - (\|\gamma\|^2 \cdot \|\gamma^{-1}\|)^{-2})^2 - (\|\gamma\| \cdot \|\gamma^{-1}\|)^{-4}] \cdot \|x\|^2.$$

We conclude that there exists a $c \neq 0$ such that $\|c \cdot \gamma^2 - 1\| < 1$. Since we have $c \cdot \gamma^2 = 1 + (c \cdot \gamma^2 - 1)$ and $\|c \cdot \gamma^2 - 1\| < 1$, the geometric series $\sum_{n=0}^{\infty} (-1)^n \cdot (c \cdot \gamma^2 - 1)^n$ is convergent in $\mathfrak{A}$ and gives $(c \cdot \gamma^2)^{-1} \in \mathfrak{A}$. Then also $(\gamma^2)^{-1} \in \mathfrak{A}$ and $\gamma^{-1} = \gamma \cdot (\gamma^2)^{-1} \in \mathfrak{A}$.

LEMMA 2.6. Recall that a self-adjoint $\alpha \in \mathfrak{L}(H)$ is said to be positive, if $(\alpha(x), x) \geq 0$ for all $x \in H$. Let $\mathfrak{A} \subset \mathfrak{L}(H)$ be a C^*-algebra with $1 \in \mathfrak{A}$. Suppose $\alpha \in \mathfrak{A}$ is self-adjoint and positive. Then there exists a unique self-adjoint and positive $\beta \in \mathfrak{A}$ with $\beta^2 = \alpha$. If further $S \subset H$ is a closed linear subspace with $\alpha(S) \subset S$, then also $\beta(S) \subset S$, and if $\alpha|_S = id$, then also $\beta|_S = id$.

Proof. We may assume that $1 - \alpha$ is positive. We consider the sequence $\{\beta_n\}_{n=0}^{\infty}$ in $\mathfrak{A}$ defined by $\beta_{n+1} = \beta_n + \frac{1}{2}(\alpha - \beta_n^2)$ and $\beta_0 = o$. This sequence is convergent. Let $\beta = \lim \beta_n \in \mathfrak{A}$. Then β has the desired properties. See, for example, [12, p. 15].

LEMMA 2.7. Suppose $\alpha \in \mathfrak{L}(H)$ is self-adjoint and positive. Then $1 + \alpha \in \mathrm{GL}(H)$ and $\|(1 + \alpha)^{-1}\| \leq 1$.

Proof. By Lemma 2.6 there is a $\beta \in \mathfrak{L}(H)$ with $\beta^* = \beta$ and $\beta^2 = \alpha$. We compute for $x \in H$

$$\|(1+\alpha)(x)\|^2 = (x, x) + 2 \cdot (\beta(x), \beta(x)) + (\alpha(x), \alpha(x)) \geq \|x\|^2.$$

This implies $c(H, (1+\alpha)) \geq 1$. Therefore $1+\alpha$ is injective and $(1+\alpha)(H)$ is closed (Lemma 2.3). Since $(1+\alpha)^* = 1+\alpha$, it follows that $1+\alpha$ is also surjective (Lemma 2.1). By Theorem 2.1 then $1+\alpha \in GL(H)$. Finally, by Lemma 2.3

$$\|(1+\alpha)^{-1}\| = (c(H, (1+\alpha)))^{-1} \leq 1.$$

THEOREM 2.2. *Let $\mathfrak{A} \subset \mathfrak{L}(H)$ be a C*-algebra with $1 \in \mathfrak{A}$. For each $\gamma \in GL(H) \cap \mathfrak{A}$ there is a unique decomposition $\gamma = \mu \cdot \kappa$ such that $\mu \in U(H) \cap \mathfrak{A}$ and $\kappa \in GL(H) \cap \mathfrak{A}$ is self-adjoint and positive. The map*

$$u: GL(H) \cap \mathfrak{A} \to U(H) \cap \mathfrak{A},$$
$$u(\gamma) = \mu,$$

is continuous.

Proof. Consider $\gamma^* \cdot \gamma \in GL(H) \cap \mathfrak{A}$. We have $\gamma^* \cdot \gamma$ is self-adjoint and positive. By Lemma 2.6 there exists a positive self-adjoint $\beta \in \mathfrak{A}$ with $\beta^2 = \gamma^* \cdot \gamma$. Obviously β is injective and surjective since $\beta^2 \in GL(H)$. Therefore by Theorem 2.1 $\beta \in GL(H) \cap \mathfrak{A}$, and by Lemma 2.5 also $\beta^{-1} \in GL(H) \cap \mathfrak{A}$. Let $\kappa = \beta$ and $\mu = \gamma \cdot \beta^{-1}$. Then $\gamma = \mu \cdot \kappa$, and κ is self-adjoint and positive. We compute $\mu^* \cdot \mu = \beta^{-1} \cdot \gamma^* \cdot \gamma \cdot \beta^{-1} = \beta^{-1} \cdot \beta^2 \cdot \beta^{-1} = 1$, i.e., $\mu \in U(H) \cap \mathfrak{A}$.

To prove the uniqueness of the decomposition, we assume another representation $\gamma = \tilde{\mu} \cdot \tilde{\kappa}$ with the above properties. Then $\mu \cdot \kappa = \tilde{\mu} \cdot \tilde{\kappa}$, or $\kappa = \mu^* \cdot \tilde{\mu} \cdot \tilde{\kappa}$, and hence $\kappa = \kappa^* = \tilde{\kappa} \cdot \tilde{\mu}^* \cdot \mu$. It follows that $\kappa^2 = \tilde{\kappa}^2 = \gamma^* \cdot \gamma$. From the uniqueness of β in Lemma 2.6, we conclude $\kappa = \tilde{\kappa}$ and $\mu = \tilde{\mu}$.

To prove the continuity of u, it is sufficient to show that the map

$$w: GL(H) \cap \mathfrak{A} \to GL(H) \cap \mathfrak{A},$$

$w(\gamma) = \beta$, where β is the unique self-adjoint and positive element with $\beta^2 = \gamma^* \cdot \gamma$, is continuous. Let $w(\gamma_0) = \beta_0$, $w(\gamma) = \beta$.

We apply to β_0 and β Lemma 2.6, and we obtain $\beta_0 = \delta_0^2$, $\beta = \delta^2$ with $\delta_0^* = \delta_0$, $\delta^* = \delta$, and again $\delta_0, \delta \in GL(H)$. Then we decompose

$$\beta_0 + \beta = \delta_0 \cdot (1 + (\delta_0^{-1} \cdot \delta) \cdot (\delta \cdot \delta_0^{-1})) \cdot \delta_0.$$

Let $\varepsilon = \delta \cdot \delta_0^{-1}$. Now $\varepsilon^* \cdot \varepsilon$ is self-adjoint and positive. By Lemma 2.7 then $1 + \varepsilon^* \cdot \varepsilon \in GL(H)$ and $\|(1 + \varepsilon^* \cdot \varepsilon)^{-1}\| \leq 1$. We compute

$$\|\beta_0 - \beta\| = \|(\beta_0 - \beta) \cdot (\beta_0 + \beta) \cdot (\beta_0 + \beta)^{-1}\| \leq \|\beta_0^2 - \beta^2\| \cdot \|(\beta_0 + \beta)^{-1}\|$$
$$= \|\gamma_0^* \cdot \gamma_0 - \gamma^* \cdot \gamma\| \cdot \|\delta_0^{-1} \cdot (1 + \varepsilon^* \cdot \varepsilon)^{-1} \cdot \delta_0^{-1}\|$$
$$\leq \|\gamma_0^* \cdot \gamma_0 - \gamma^* \cdot \gamma\| \cdot \|\delta_0^{-1}\|^2.$$

Which proves the continuity of w.

COROLLARY 2.1. *Let* $\mathfrak{A} \subset \mathfrak{L}(H)$ *be a C*-algebra with* $1 \in \mathfrak{A}$. *If* $\gamma \in GL(H) \cap \mathfrak{A}$, *then also* $\gamma^{-1} \in GL(H) \cap \mathfrak{A}$, *i.e.*,

$$GL(\mathfrak{A}) = GL(H) \cap \mathfrak{A}.$$

Proof. By Theorem 2.2, $\gamma = \mu \cdot \kappa$ with $\mu \in U(H) \cap \mathfrak{A}$, $\kappa \in GL(H) \cap \mathfrak{A}$, and κ is self-adjoint. Then $\gamma^{-1} = \kappa^{-1} \cdot \mu^*$. By Lemma 2.5, $\kappa^{-1} \in GL(H) \cap \mathfrak{A}$, which proves $\gamma^{-1} \in GL(H) \cap \mathfrak{A}$.

COROLLARY 2.2. *Let* $\mathfrak{A} \subset \mathfrak{L}(H)$ *be a C*-algebra with* $1 \in \mathfrak{A}$. *For each* $\gamma \in GL(\mathfrak{A})$ *there is a unique decomposition* $\gamma = \mu \cdot \kappa$ *such that* $\mu \in U(\mathfrak{A})$ *and* $\kappa \in GL(\mathfrak{A})$ *is self-adjoint and positive. The map*

$$u\colon GL(\mathfrak{A}) \to U(\mathfrak{A}),$$

$$u(\gamma) = \mu,$$

is continuous. We notice further the following property: *Let* $S \subset H$ *be a closed linear subspace. Suppose* $\gamma \in GL(\mathfrak{A})$ *satisfies* $\gamma(S^\perp) = (\gamma(S))^\perp$, *then* $u(\gamma)(S) = \gamma(S)$. *And if* $\gamma_1, \gamma_2 \in GL(\mathfrak{A})$ *satisfy* $\gamma_i(S^\perp) = (\gamma_i(S))^\perp$, $i = 1, 2$, *and* $\gamma_1|_S = \gamma_2|_S$, *then* $u(\gamma_1)|_S = u(\gamma_2)|_S$.

Proof. Theorem 2.2 and Corollary 2.1 prove the first part of the corollary. $\gamma(S^\perp) = (\gamma(S))^\perp$ implies $\gamma^* \cdot \gamma(S) = S$ and $\gamma^* \cdot \gamma(S^\perp) = S^\perp$. The corresponding β satisfies then also $\beta(S) = S$ and $\beta(S^\perp) = S^\perp$ (Lemma 2.6). Hence $u(\gamma)(S) = \mu(S) = (\gamma \cdot \beta^{-1})(S) = \gamma(S)$.

To prove the last property, observe again $\gamma_i^* \cdot \gamma_i(S) = S$ and $\gamma_i^* \cdot \gamma_i(S^\perp) = S^\perp$, $i = 1, 2$. We compute the corresponding β_i by computing $\beta_i|_S$ and $\beta_i|_{S^\perp}$ separately and forming $\beta_i = \beta_i|_S \oplus \beta_i|_{S^\perp}$, $i = 1, 2$. But $\gamma_1^* \cdot \gamma_1|_S = \gamma_2^* \cdot \gamma_2|_S$, therefore $\beta_1|_S = \beta_2|_S$, and hence $\mu_1|_S = \mu_2|_S$.

THEOREM 2.3. *Let* $\mathfrak{A} \subset \mathfrak{L}(H)$ *be a C*-algebra with* $1 \in \mathfrak{A}$, *and let* $S \subset H$ *be a closed linear subspace with* $\pi_S \in \mathfrak{A}$. *We consider the closed subgroups*

$$GL(\mathfrak{A})_S = \{\gamma; \gamma \in GL(\mathfrak{A}) \text{ with } \gamma(S) = S\},$$

$$U(\mathfrak{A})_S = GL(\mathfrak{A})_S \cap U(H)$$

of the topological groups $GL(\mathfrak{A})$ *and* $U(\mathfrak{A})$. *Then* $U(\mathfrak{A})_S \subset GL(\mathfrak{A})_S$ *is a strong deformation retract of the space* $GL(\mathfrak{A})_S$. *In particular* $U(\mathfrak{A})_S$ *and* $GL(\mathfrak{A})_S$ *are of the same homotopy type.*

Proof. We introduce the closed subgroup

$$GL(\mathfrak{A})_{S,S^\perp} = \{\gamma; \gamma \in GL(\mathfrak{A}) \text{ with } \gamma(S) = S \text{ and } \gamma(S^\perp) = S^\perp\}$$

of the group $GL(\mathfrak{A})_S$. We have the strong deformation retract map

$$r\colon GL(\mathfrak{A})_S \to GL(\mathfrak{A})_{S,S^\perp},$$

$$r(\gamma) = \gamma \cdot \pi_S + \pi_{S^\perp} \cdot \gamma \cdot \pi_{S^\perp} = \gamma - \pi_S \cdot \gamma \cdot \pi_{S^\perp}.$$

(Certainly $r(\gamma)(S) \subset S$ and $r(\gamma)(S^\perp) \subset S^\perp$. We compute directly $r(\gamma) \cdot r(\gamma^{-1}) = 1$ and $r(\gamma^{-1}) \cdot r(\gamma) = 1$. Therefore $r(\gamma) \in \mathrm{GL}(\mathfrak{A})_{S,S^\perp}$, and r is well defined.)

A deformation retract homotopy is given by

$$\hat{r} : \mathrm{GL}(\mathfrak{A})_S \times [0, 1] \to \mathrm{GL}(\mathfrak{A})_S,$$

$$\hat{r}(\gamma, t) = r_t(\gamma) = \gamma - t \cdot \pi_S \cdot \gamma \cdot \pi_{S^\perp}.$$

(Again $r_t(\gamma)(S) \subset S$, $r_t(\gamma)(S^\perp) \subset S^\perp$, $r_t(\gamma) \cdot r_t(\gamma^{-1}) = 1$, $r_t(\gamma^{-1}) \cdot r_t(\gamma) = 1$, and thus r_t is well defined.) Then $r_0 = id$, $r_1 = i \cdot r$, where $i : \mathrm{GL}(\mathfrak{A})_{S,S^\perp} \to \mathrm{GL}(\mathfrak{A})_S$ is the inclusion. Further, $r_t(\gamma) = \gamma$ for $\gamma \in \mathrm{GL}(\mathfrak{A})_{S,S^\perp}$ and $0 \leq t \leq 1$. Thus, $\mathrm{GL}(\mathfrak{A})_{S,S^\perp} \subset \mathrm{GL}(\mathfrak{A})_S$ is a strong deformation retract of $\mathrm{GL}(\mathfrak{A})_S$.

Next we show that $\mathrm{U}(\mathfrak{A})_S \subset \mathrm{GL}(\mathfrak{A})_{S,S^\perp}$ is a strong deformation retract of $\mathrm{GL}(\mathfrak{A})_{S,S^\perp}$. The map u of Theorem 2.2 defines a deformation retract map

$$u : \mathrm{GL}(\mathfrak{A})_{S,S^\perp} \to \mathrm{U}(\mathfrak{A})_S,$$

$u(\gamma) = \mu$,   where $\gamma = \mu \cdot \kappa$ is the unique decomposition of Theorem 2.2.

u is well defined. (By Corollary 2.2, $u(\gamma)(S) = S$, and hence $u(\gamma)(S^\perp) = S^\perp$. Therefore, $u(\gamma) \in \mathrm{U}(\mathfrak{A})_S$.) A deformation retract homotopy is given by

$$\hat{u} : \mathrm{GL}(\mathfrak{A})_{S,S^\perp} \times [0, 1] \to \mathrm{GL}(\mathfrak{A})_{S,S^\perp},$$

$$\hat{u}(\gamma, t) = u_t(\gamma) = \mu \cdot (t \cdot 1 + (1 - t) \cdot \kappa).$$

Again the map u_t is well defined. (Namely $t \cdot 1 + (1 - t) \cdot \kappa \in \mathrm{GL}(\mathfrak{A})$ by Lemma 2.7, and $\kappa = \beta \in \mathrm{GL}(\mathfrak{A})_{S,S^\perp}$ gives $t \cdot 1 + (1 - t) \cdot \kappa \in \mathrm{GL}(\mathfrak{A})_{S,S^\perp}$.) We have $u_0 = id$ and $u_1 = i \cdot u$, where $i : \mathrm{U}(\mathfrak{A})_S \to \mathrm{GL}(\mathfrak{A})_{S,S^\perp}$ is the inclusion. Finally, $u_t(\mu) = \mu$ for $\mu \in \mathrm{U}(\mathfrak{A})_S$ and $0 \leq t \leq 1$. Which proves the theorem.

COROLLARY 2.3. *Let $\mathfrak{A}$ be a C*-algebra with $1 \in \mathfrak{A}$. Then $\mathrm{U}(\mathfrak{A}) \subset \mathrm{GL}(\mathfrak{A})$ is a strong deformation retract of the space $\mathrm{GL}(\mathfrak{A})$. In particular, $\mathrm{U}(\mathfrak{A})$ and $\mathrm{GL}(\mathfrak{A})$ are of the same homotopy type. The strong deformation retract map*

$$u_t : \mathrm{GL}(\mathfrak{A}) \to \mathrm{GL}(\mathfrak{A}),$$

$$u_t(\gamma) = \mu \cdot (t \cdot 1 + (1 - t) \cdot \kappa), \qquad 0 \leq t \leq 1,$$

of Theorem 2.3 has further the following property: If $S \subset H$ is a closed linear subspace, and if $\gamma_1, \gamma_2 \in \mathrm{GL}(\mathfrak{A})$ with $\gamma_i(S^\perp) = (\gamma_i(S))^\perp$, $i = 1$, 2, and with $\gamma_1|_S = \gamma_2|_S$, then $u_t(\gamma_1)|_S = u_t(\gamma_2)|_S$ for $0 \leq t \leq 1$.

Proof. We have only to observe that $\kappa_1|_S = \kappa_2|_S$ and $\mu_1|_S = \mu_2|_S$. Compare Corollary 2.2.

3. **The metric space $\mathfrak{X}(H)$ of all closed linear subspaces of a Hilbert space H.**   Let $\mathfrak{X}(H) = \{S; S \subset H$ and S a closed linear subspace of $H\}$. We recall the definition of

various metrics defined on $\mathfrak{X}(H)$ and some of their properties. If $x \in H$ and $S \in \mathfrak{X}(H)$, then $d(x, S) = \inf \{\|x - y\|; y \in S\} = \|(1 - \pi_S)(x)\|$.

DEFINITION 3.1. For $S, T \in \mathfrak{X}(H)$ we define

$$\delta(S, T) = \sup \{d(x, T); x \in S \text{ and } \|x\| = 1\},$$
$$d(S, T) = \delta(S, T) + \delta(T, S), \text{ and}$$
$$g(S, T) = \|\pi_S - \pi_T\|.$$

LEMMA 3.1.

$0 \leq \delta(S, T) \leq 1$, and $0 \leq g(S, T) \leq 1$.

$\delta(S, T) = 0$ if and only if $S \subset T$.

If $S \subset S'$, and $T' \subset T$, then $\delta(S, T) \leq \delta(S', T')$.

$\delta(S, T) = \|(1 - \pi_T) \cdot \pi_S\|$.

$\delta(S, T) \leq \delta(S, R) + \delta(R, T)$.

$\delta(S, T) = \delta(T^\perp, S^\perp)$, $g(S, T) = g(S^\perp, T^\perp)$.

$\delta(S, T) \leq g(S, T)$ and $g(S, T) \leq \delta(S, T) + \delta(T, S)$.

Proof. These properties follow immediately from the definitions. See, for example, [3].

COROLLARY 3.1. *The functions d and g on $\mathfrak{X}(H) \times \mathfrak{X}(H)$ define complete metrics on $\mathfrak{X}(H)$. We have*

$$g(S, T) \leq d(S, T) \leq 2 \cdot g(S, T) \quad \text{for } S, T \in \mathfrak{X}(H),$$

and these two metrics are therefore equivalent.

Proof. The completeness is easily proved by using the g-metric.

Observe that δ itself does not define a metric.

The metric space $\mathfrak{X}(H)$ is the set $\mathfrak{X}(H)$ with the equivalence class of metrics containing d and g.

LEMMA 3.2. *Let $S, T \in \mathfrak{X}(H)$. Then the following relation holds*

$$(\delta(S, T))^2 + (c(S, \pi_T))^2 = 1.$$

In particular $\pi_T(S)$ is closed if and only if $\delta(S, T) < 1$.

Proof. Observe that $\|(1 - \pi_T)(x)\|^2 = \|x\|^2 - \|\pi_T(x)\|^2$ for $x \in H$. Therefore

$$(\delta(S, T))^2 = \sup \{\|(1 - \pi_T)(x)\|^2; x \in S \text{ and } \|x\| = 1\}$$
$$= 1 - \inf \{\|\pi_T(x)\|^2; x \in S \text{ and } \|x\| = 1\}$$
$$= 1 - (c(S, \pi_T))^2.$$

Lemma 2.3 implies therefore $\pi_T(S)$ is closed if and only if $\delta(S, T) < 1$.

Of crucial importance is the following theorem established by T. Kato [8].

THEOREM 3.1. *If* $S, T \in \mathfrak{X}(H)$ *with* $\delta(S, T) < 1$, *then* $T' = \pi_T(S)$ *is in* $\mathfrak{X}(H)$ *and satisfies*

$$\delta(S, T') = \delta(T', S) = \delta(S, T) = g(S, T').$$

If further $\delta(T, S) < 1$, *then* $T' = T$.

Proof. From Lemma 3.2 follows $T' = \pi_T(S)$ is closed and therefore in $\mathfrak{X}(H)$. Since $\pi_T \cdot \pi_S = \pi_{T'} \cdot \pi_S$, we have $(1 - \pi_T) \cdot \pi_S = (1 - \pi_{T'}) \cdot \pi_S$ and therefore

$$\delta(S, T) = \|(1 - \pi_T) \cdot \pi_S\| = \|(1 - \pi_{T'}) \cdot \pi_S\| = \delta(S, T').$$

Consider the map $\alpha = \pi_{T'} \cdot \pi_S$. Then $\alpha(H) = T'$ is closed and we can apply Lemma 2.4.

$$c(\alpha(H), \alpha^*) = c(T', \pi_S \cdot \pi_{T'}) = c(T', \pi_S) = c(\alpha^*(H), \pi_{T'} \cdot \pi_S).$$

But $\alpha^*(H) = (\text{kernel } (\alpha))^\perp = (S^\perp)^\perp = S$. Therefore

$$c(T', \pi_S) = c(S, \pi_{T'} \cdot \pi_S) = c(S, \pi_{T'}).$$

By Lemma 3.2 again $\delta(T', S) = \delta(S, T')$. We have $\delta(T', S) = \|(1 - \pi_S) \cdot \pi_{T'}\|$ $= \|(\pi_{T'} - \pi_S) \cdot \pi_{T'}\| \leqq \|\pi_{T'} - \pi_S\| = g(S, T')$. On the other hand,

$$(\pi_S - \pi_{T'})(x) = (1 - \pi_{T'}) \cdot \pi_S(x) - \pi_{T'} \cdot (1 - \pi_S)(x)$$

for $x \in H$. We obtain

$$\|(\pi_S - \pi_{T'})(x)\|^2 = \|(1 - \pi_{T'}) \cdot \pi_S(x)\|^2 + \|\pi_{T'} (1 - \pi_S)(x)\|^2$$
$$\leqq \delta(S, T')^2 \cdot \|\pi_S(x)\|^2 + \delta(T', S)^2 \cdot \|(1 - \pi_S)(x)\|^2 = \delta(S, T')^2 \cdot \|x\|^2.$$

And, therefore, $g(S, T') = \|\pi_S - \pi_{T'}\| \leqq \delta(S, T')$. I.e., $g(S, T') = \delta(S, T')$. If further $\delta(T, S) < 1$, then again $c(T, \pi_S) > 0$, and therefore kernel $(\pi_S \cdot \pi_T) = T^\perp$. But also

$$\text{kernel } (\pi_T \cdot \pi_S)^* = \text{kernel } (\pi_S \cdot \pi_T) = (\pi_T \cdot \pi_S(H))^\perp = T'^\perp.$$

Hence $T'^\perp = T^\perp$, which implies $T' = T$.

COROLLARY 3.2.

$$g(S, T) = \max \{\delta(S, T), \delta(T, S)\} \quad \textit{for } S, T \in \mathfrak{X}(H).$$

If $g(S, T) < 1$, *then* $\delta(S, T) = \delta(T, S) = g(S, T)$.

Proof. If $\delta(S, T) < 1$ and $\delta(T, S) < 1$, then by Theorem 3.1 $g(S, T) = \delta(S, T)$ $= \delta(T, S)$. If $\delta(S, T) = 1$ or $\delta(T, S) = 1$, then $\delta(S, T), \delta(T, S) \leqq g(S, T) \leqq 1$ implies $g(S, T) = 1$. For another proof see [1].

COROLLARY 3.3. *Suppose* $S, T \in \mathfrak{X}(H)$ *with* $\delta(S, T) < 1$. *Then there exists an* $S' \in \mathfrak{X}(H)$ *with* $S \subset S'$ *and* $g(S', T) = \delta(S, T)$.

Proof. We apply Theorem 3.1 to $\delta(T^\perp, S^\perp) = \delta(S, T) < 1$. Let $\tilde{S} = \pi_{S^\perp}(T^\perp) \subset S^\perp$.

$$g(\tilde{S}^\perp, T) = g(T^\perp, \tilde{S}) = \delta(T^\perp, S^\perp) = \delta(S, T).$$

Let $S' = \tilde{S}^\perp$.

LEMMA 3.3. *Let* $\alpha, \beta \in GL(H)$ *and let* $S, T \in \mathfrak{T}(H)$. *Then*

$$\delta(\alpha(S), \beta(T)) \leqq (\|\alpha - \beta\| + \delta(S, T)) \cdot \|\alpha^{-1}\|, \quad and$$

$$d(\alpha(S), \beta(T)) \leqq (\|\alpha - \beta\| + d(S, T)) \cdot (\|\alpha^{-1}\| + \|\beta^{-1}\|).$$

Proof. By definition, $\delta(\alpha(S), \beta(T)) = \sup \{d(\alpha(x), \beta(T)); x \in S \text{ and } \|\alpha(x)\| = 1\}$, and $d(\alpha(x), \beta(T)) = \inf \{\|\alpha(x) - \beta(y)\|; y \in T\}$. We compute

$$\|\alpha(x) - \beta(y)\| \leqq \|\alpha(x) - \beta(x)\| + \|\beta(x) - \beta(y)\|$$

$$\leqq \|\alpha - \beta\| \cdot \|x\| + \|\beta\| \cdot \|[(\|x\|)^{-1} \cdot x - (\|x\|)^{-1} \cdot y]\| \cdot \|x\|.$$

Therefore, $d(\alpha(x), \beta(T)) \leqq \|\alpha - \beta\| \cdot \|x\| + \|\beta\| \cdot \delta(S, T) \cdot \|x\|$. But $\|x\| = \|\alpha^{-1} \cdot \alpha(x)\| \leqq \|\alpha^{-1}\| \cdot \|\alpha(x)\| = \|\alpha^{-1}\|$ for $\|\alpha(x)\| = 1$. And we obtain

$$\delta(\alpha(S), \beta(T)) \leqq (\|\alpha - \beta\| + \delta(S, T)) \cdot \|\alpha^{-1}\|.$$

LEMMA 3.4. *Let* $\alpha, \beta \in \mathfrak{L}(H)$, $S \in \mathfrak{T}(H)$, $c(S, \alpha) > 0$, *and let* $\beta(S)$ *be closed. Then*

$$\delta(\alpha(S), \beta(S)) \leqq c(S, \alpha)^{-1} \cdot \|\alpha - \beta\|.$$

Proof. Again by definition

$$\delta(\alpha(S), \beta(S)) = \sup \{d(\alpha(x), \beta(S)); x \in S \text{ and } \|\alpha(x)\| = 1\},$$

and $d(\alpha(x), \beta(S)) = \inf \{\|\alpha(x) - \beta(y)\|; y \in S\}$. In particular

$$d(\alpha(x), \beta(S)) \leqq \|\alpha(x) - \beta(x)\| \leqq \|\alpha - \beta\| \cdot \|x\|.$$

Since $1 = \|\alpha(x)\| \geqq c(S, \alpha) \cdot \|x\|$, we obtain $\|x\| \leqq c(S, \alpha)^{-1}$. Therefore $\delta(\alpha(x), \beta(S)) \leqq c(S, \alpha)^{-1} \cdot \|\alpha - \beta\|$, which proves the lemma.

LEMMA 3.5. *Let* $S, T \in \mathfrak{T}(H)$ *with* $g(S, T) < 1$. *Consider* $\alpha_t = 1 - t \cdot \pi_{T^\perp} \in \mathfrak{L}(H)$, $0 \leqq t \leqq 1$. *Then* $\alpha_t(S) \in \mathfrak{T}(H)$, $0 \leqq t \leqq 1$, *and* $t \to \alpha_t(S) \in \mathfrak{T}(H)$, $0 \leqq t \leqq 1$, *is a continuous path in* $\mathfrak{T}(H)$ *which connects* S *with* T *such that*

$$g(S, \alpha_t(S)) = t \cdot g(S, T), \qquad 0 \leqq t \leqq 1.$$

Proof. For $x \in H$ we compute

$$\|\alpha_t(x)\|^2 = ((1 - t \cdot \pi_{T^\perp})(x), (1 - t \cdot \pi_{T^\perp})(x))$$

$$= ((\pi_T + (1 - t) \cdot \pi_{T^\perp})(x), (\pi_T + (1 - t) \cdot \pi_{T^\perp})(x))$$

$$= \|\pi_T(x)\|^2 + (1 - t)^2 \cdot \|\pi_{T^\perp}(x)\|^2 \geqq \|\pi_T(x)\|^2.$$

Therefore, $c(S, \alpha_t) \geq c(S, \pi_T) = (1 - \delta(S, T)^2)^{1/2} = (1 - g(S, T)^2)^{1/2} > 0$. By Lemma 2.3, then $\alpha_t(S) \in \mathfrak{T}(H)$ for $0 \leq t \leq 1$. Certainly $\alpha_0(S) = S$, and $\alpha_1(S) = T$ by Theorem 3.1.

The path $t \to \alpha_t(S) \in \mathfrak{T}(H)$ is continuous. Namely,

$$d(\alpha_{t_1}(S), \alpha_{t_2}(S)) \leq \|\alpha_{t_1} - \alpha_{t_2}\| \cdot (c(S, \alpha_{t_1})^{-1} + c(S, \alpha_{t_2})^{-1})$$

$$\leq 2 \cdot (1 - g(S, T)^2)^{-1/2} \cdot |t_1 - t_2| \qquad \text{(Lemma 3.4)}.$$

Finally, we prove $g(S, \alpha_t(S)) = t \cdot g(S, T)$, $0 \leq t \leq 1$, by a continuity argument. First we conclude from Lemma 3.4 and Corollary 3.2 that $\delta(S, \alpha_t(S)) \leq \|\pi_S - \alpha_t \cdot \pi_S\| = t \cdot \delta(S, T) = t \cdot g(S, T)$. For short, let $T_t = \alpha_t(S)$. We claim $\delta(T_t, S) = \delta(S, T_t)$ for $0 \leq t \leq 1$. [Namely, let $\sigma = \sup \{s; 0 \leq s \leq 1 \text{ with } \delta(T_t, S) = \delta(S, T_t) \text{ for } 0 \leq t \leq s\}$. Suppose $0 \leq \sigma < 1$. Since δ is continuous on $\mathfrak{T}(H) \times \mathfrak{T}(H)$, also $\delta(T_\sigma, S) = \delta(S, T_\sigma) = \sigma \cdot g(S, T) < 1$, and there exists $\varepsilon > 0$ with $\delta(T_t, S) < 1$ and $\delta(S, T_t) < 1$ for $0 \leq t \leq \sigma + \varepsilon$. By Corollary 3.2, $\delta(T_t, S) = \delta(S, T_t)$ for $0 \leq t \leq \sigma + \varepsilon$, which contradicts the definition of σ.] By Corollary 3.2, then $g(S, T_t) = \delta(T_t, S) = \delta(S, T_t) = t \cdot g(S, T)$ for $0 \leq t \leq 1$.

LEMMA 3.6. *Let* $S, T \in \mathfrak{T}(H)$ *with* $g(S, T) < 1$. *Then* $\gamma = \pi_T \cdot \pi_S + (1 - \pi_T) \cdot (1 - \pi_S) \in \mathrm{GL}(H)$ *and* $\gamma(S) = T$.

Proof. $g(S, T) < 1$ implies $\delta(S, T) = \delta(T^\perp, S^\perp) < 1$ and $\delta(T, S) = \delta(S^\perp, T^\perp) < 1$. By Theorem 3.1, $\gamma|_S$ is an isomorphism which maps S onto T, and $\gamma|_{S^\perp}$ is an isomorphism which maps $S^\perp$ onto $T^\perp$. This proves $\gamma \in \mathrm{GL}(H)$. Certainly

$$\gamma(S) = \pi_T(S) = T.$$

LEMMA 3.7. *Let* $R, S, T \in \mathfrak{T}(H)$ *with* $R \subset S$ *and* $R \subset T$, *and let* $S' = R^\perp \cap S$ *and* $T' = R^\perp \cap T$. *Then*

$$\delta(S', T') = \delta(S, T) \quad \text{and} \quad g(S', T') = g(S, T).$$

Proof. We have $\pi_T = \pi_R + \pi_{T'}$ and $\pi_S = \pi_R + \pi_{S'}$. Substitution into $\delta(S', T') = \|(1 - \pi_{T'}) \cdot \pi_{S'}\|$ and $g(S', T') = \|\pi_{S'} - \pi_{T'}\|$ proves immediately the lemma.

4. *R*-sets and maximal *R*-sets. We recall the well known Rellich criterion for compact linear maps of a Hilbert space which states that a linear map of a Hilbert space is compact if and only if for every $\varepsilon > 0$ there is a closed linear subspace of finite codimension such that the norm of the restriction of the linear map to this subspace becomes smaller than ε. Various examples in analysis motivate a generalization of compact linear maps via the collection of closed linear subspaces of finite codimension in this criterion. A proper generalization of this collection is the concept of *R*-sets.

DEFINITION 4.1. A subset $\mathfrak{R} \subset \mathfrak{T}(H)$ is called an *R*-set, if the following condition is satisfied: For any pair $S, T \in \mathfrak{R}$ and every $\varepsilon > 0$ there exists an $R \in \mathfrak{R}$ with

$$\delta(R, S) < \varepsilon \quad \text{and} \quad \delta(R, T) < \varepsilon.$$

Examples 4.1.

(1) Any subset $\mathfrak{R} \subset \mathfrak{T}(H)$ with the property that for any pair $S, T \in \mathfrak{R}$, also $S \cap T \in \mathfrak{R}$, is an R-set.

(2) The subset $\mathfrak{R} = \{S; S \in \mathfrak{T}(H) \text{ with } S_{\min} \subset S\}$ of $\mathfrak{T}(H)$, where $S_{\min} \in \mathfrak{T}(H)$ is a fixed element, is an R-set.

(3) $\mathfrak{T}(H)$ itself is an R-set.

(4) Any set $\{S\}$ consisting of a single element $S \in \mathfrak{T}(H)$ only is an R-set.

(5) A set $\{S_1, \ldots, S_n\}$ consisting of n elements of $\mathfrak{T}(H)$ is an R-set if and only if an $S_{i_0} = S_{\min} \subset S_i$, $i = 1, \ldots, n$.

(6) $\mathfrak{T}(H)^c = \{S; S \in \mathfrak{T}(H) \text{ and codim } (S) < c\}$, where $c \geq \aleph_0$ is a cardinal number, is an R-set. (Namely, for $S, T \in \mathfrak{T}(H)^c$ also $S \cap T \in \mathfrak{T}(H)^c$. This follows from $S \cap T = S \cap (\pi_S(T^\perp))^\perp$ and hence $(S \cap T)^\perp = S^\perp \oplus \mathrm{Cl}(\pi_S(T^\perp))$.) In particular:

$$\mathfrak{T}(H)^f = \{S; S \in \mathfrak{T}(H) \text{ and codim } (S) \text{ is finite}\} \text{ is an } R\text{-set.}$$

(7) If $\sigma = \{S_n\}_{n=1}^\infty$ is a sequence of closed linear subspaces with $S_{n+1} \subset S_n$, $n = 1, 2, \ldots$, then $\mathfrak{R} = \{S_n; n = 1, 2, \ldots\}$ is an R-set.

(8) Further if $\sigma = \{S_n\}_{n=1}^\infty$ is a sequence of closed linear subspaces as in (7), then

$$M(\sigma) = \{S; S \in \mathfrak{T}(H) \text{ with } \lim_{n \to \infty} \delta(S_n, S) = 0\} \text{ is an } R\text{-set.}$$

(9) For explicit examples of R-sets in analysis we refer to [4].

Definition 4.2. For any subset $\mathfrak{S} \subset \mathfrak{T}(H)$ we define

$$\mathfrak{S}^\vee = \{T; T \in \mathfrak{T}(H) \text{ such that there is an } R \in \mathfrak{S} \text{ with } R \subset T\},$$

$$\mathrm{Cl}(\mathfrak{S}) = \text{the closure of } \mathfrak{S} \text{ in the metric space } \mathfrak{T}(H).$$

Lemma 4.1. *If $\mathfrak{R}$ is an R-set, then $\mathfrak{R}^\vee$ and $\mathrm{Cl}(\mathfrak{R})$ are R-sets again.*

Proof. If $S, T \in \mathfrak{R}^\vee$ and $\varepsilon > 0$ are given, then there are $S', T' \in \mathfrak{R}$ with $S' \subset S$, $T' \subset T$ and there is an $R \in \mathfrak{R}$ with $\delta(R, S') < \varepsilon$, $\delta(R, T') < \varepsilon$. But then also $\delta(R, S) \leq \delta(R, S') < \varepsilon$, $\delta(R, T) \leq \delta(R, T') < \varepsilon$. If $S, T \in \mathrm{Cl}(\mathfrak{R})$ and $\varepsilon > 0$ are given, then there are $S', T' \in \mathfrak{R}$ with $d(S', S) < \varepsilon/2$, $d(T', T) < \varepsilon/2$ and there is an $R \in \mathfrak{R}$ with $\delta(R, S') < \varepsilon/2$, $\delta(R, T') < \varepsilon/2$. We conclude $\delta(R, S) \leq \delta(R, S') + \delta(S', S) < \varepsilon$, $\delta(R, T) \leq \delta(R, T') + \delta(T', T) < \varepsilon$.

Lemma 4.2. *For any subset $\mathfrak{S} \subset \mathfrak{T}(H)$ we have $\mathrm{Cl}(\mathfrak{S})^\vee \subset \mathrm{Cl}(\mathfrak{S}^\vee)$.*

Proof. Suppose $T \in \mathrm{Cl}(\mathfrak{S})^\vee$. Then there is a $T' \in \mathrm{Cl}(\mathfrak{S})$ with $T' \subset T$, and there is further a sequence $\{S_n\}_{n=1}^\infty$, $S_n \in \mathfrak{S}$, with $g(S_n, T') < 1/n$. But $\delta(S_n, T) \leq \delta(S_n, T') \leq g(S_n, T') < 1/n$. By Corollary 3.3 there is an S'_n with $S_n \subset S'_n$ and

$$g(S'_n, T) = \delta(S_n, T) < 1/n.$$

Since $S'_n \in \mathfrak{S}^\vee$, this implies $T \in \mathrm{Cl}(\mathfrak{S}^\vee)$.

DEFINITION 4.3. A subset $\mathfrak{M} \subset \mathfrak{T}(H)$ is a maximal R-set, if it is an R-set and if

$$\mathfrak{M}^\vee = \mathfrak{M} \quad \text{and} \quad \text{Cl}(\mathfrak{M}) = \mathfrak{M}.$$

EXAMPLES 4.2. In Examples 4.1, the R-sets in (2), (3), (6), and (8) are maximal.

THEOREM 4.1. *Each R-set $\mathfrak{R}$ is contained in a uniquely determined smallest maximal R-set $M(\mathfrak{R})$.*

Proof. Let $M(\mathfrak{R}) = \text{Cl}(\mathfrak{R}^\vee)$. Then

$$M(\mathfrak{R})^\vee = \text{Cl}(\mathfrak{R}^\vee)^\vee \subset \text{Cl}(\mathfrak{R}^{\vee\vee}) = \text{Cl}(\mathfrak{R}^\vee) = M(\mathfrak{R}) \subset M(\mathfrak{R})^\vee$$

implies $M(\mathfrak{R})^\vee = M(\mathfrak{R})$. Of course $\text{Cl}(M(\mathfrak{R})) = M(\mathfrak{R})$. Obviously $M(\mathfrak{R})$ is the smallest maximal R-set containing the R-set $\mathfrak{R}$.

In associating algebraic structures to an R-set there is a freedom to enlarge the R-set to a maximal R-set (see next chapter). This motivates the introduction of maximal R-sets. Hence it is sufficient to consider maximal R-sets only.

Repeatedly we will apply the following argument:

LEMMA 4.3. *Let $\mathfrak{M}$ be a maximal R-set, let $T \in \mathfrak{T}(H)$ and suppose for each $\varepsilon > 0$ there exists an $S \in \mathfrak{M}$ with $\delta(S, T) < \varepsilon$, then $S \in \mathfrak{M}$.*

Proof. Let $\varepsilon > 0$ be given. Then, by Corollary 3.3, there exists for the S of the hypothesis an $S' \in \mathfrak{T}(H)$ with $S \subset S'$ and $g(S', T) = \delta(S, T) < \varepsilon$. Since $S' \in \mathfrak{M}$, this implies $T \in \overline{\mathfrak{M}} = \mathfrak{M}$.

LEMMA 4.4. *The maximal R-set $\mathfrak{M} = \{S; S \in \mathfrak{T}(H)$ with $S_{\min} \subset S\}$, where $S_{\min} \in \mathfrak{T}(H)$ is a fixed element, has the following representation: We introduce the Hilbert space $H' = (S_{\min})^\perp$. Then the map*

$$c: \mathfrak{M} \to \mathfrak{T}(H'),$$

$$c(S) = (S_{\min})^\perp \cap S,$$

is an isometry onto $\mathfrak{T}(H')$ with respect to the g-metric and δ-structure of $\mathfrak{M}$ and $\mathfrak{T}(H')$.

Proof. Certainly c is bijective and Lemma 3.7.

THEOREM 4.2. *Let $\mathfrak{M}$ be a maximal R-set. If there exists a finite-dimensional $S_0 \in \mathfrak{M}$, then there is an $S_{\min} \in \mathfrak{M}$ such that*

$$\mathfrak{M} = \{S; S \in \mathfrak{T}(H) \text{ with } S_{\min} \subset S\}.$$

Proof. Consider an $S_{\min} \in \mathfrak{M}$ such that $\dim(S_{\min})$ is minimal. Let $S \in \mathfrak{M}$ be

given. Since $\mathfrak{M}$ is an R-set there exists for each $n = 1, 2, \ldots$, an $R_n \in \mathfrak{M}$ such that

$$\delta(R_n, S_{\min}) < \frac{1}{n}, \qquad \delta(R_n, S) < \frac{1}{n}.$$

By Corollary 3.3, there is an $R'_n \in \mathfrak{M}$ with $R_n \subset R'_n$ and

$$g(R'_n, S_{\min}) = \delta(R_n, S_{\min}) < \frac{1}{n}.$$

By Theorem 3.1, $S_{\min}$ and R'_n are isomorphic. Since $\dim(S_{\min}) \leqq \dim(R_n) \leqq \dim(R'_n) = \dim(S_{\min})$ and $\dim(S_{\min})$ is finite, we have $R_n = R'_n$.

We conclude

$$\delta(S_{\min}, S) \leqq \delta(S_{\min}, R_n) + \delta(R_n, S) < \frac{2}{n}, \qquad n = 1, 2, \ldots.$$

Therefore, $\delta(S_{\min}, S) = 0$ and, by Lemma 3.1, $S_{\min} \subset S$.

CoROLLARY 4.1. *If H is a finite-dimensional Hilbert space, then the maximal R-sets $\mathfrak{M}$ of $\mathfrak{T}(H)$ are the spaces $\mathfrak{T}(H')$, $0 \leqq \dim(H') \leqq \dim(H)$. The concept of maximal R-sets becomes in this case trivial, it does not introduce any new structure.*

5. C^*-algebras associated with R-sets.

DEFINITION 5.1. For an R-set $\mathfrak{R}$ we define (compare [4] and [5])

$$I(\mathfrak{R}) = \{\alpha; \alpha \in \mathfrak{L}(H) \text{ such that for each } \varepsilon > 0 \text{ there is an } S \in \mathfrak{R}$$
$$\text{with } \|\alpha \cdot \pi_S\| < \varepsilon \text{ and } \|\alpha^* \cdot \pi_S\| < \varepsilon\}, \text{ and}$$
$$A(\mathfrak{R}) = \{\alpha; \alpha \in \mathfrak{L}(H) \text{ such that for each } S \in \mathfrak{R} \text{ and } \varepsilon > 0 \text{ there is}$$
$$\text{a } T \in \mathfrak{R} \text{ with } \|(1 - \pi_S) \cdot \alpha \cdot \pi_T\| < \varepsilon \text{ and } \|(1 - \pi_S) \cdot \alpha^* \cdot \pi_T\| < \varepsilon\}.$$

THEOREM 5.1. *$I(\mathfrak{R})$ is a C^*-algebra in $\mathfrak{L}(H)$, $A(\mathfrak{R})$ is a C^*-algebra with unit $\mathbf{1}$ in $\mathfrak{L}(H)$, $I(\mathfrak{R}) \subset A(\mathfrak{R})$, and $I(\mathfrak{R})$ is a closed twosided $*$-ideal in $A(\mathfrak{R})$.*

Proof. Immediately from the definitions, compare [4] and [5].

In [4] and [5] it was also shown that $I(\mathfrak{R}) = I(M(\mathfrak{R}))$ and $A(\mathfrak{R}) = A(M(\mathfrak{R}))$, where $M(\mathfrak{R})$ is the uniquely determined maximal R-set associated with $\mathfrak{R}$. This implies that one can restrict oneself to maximal R-sets. The classical situation is obtained, if we choose as a R-set the maximal R-set $\mathfrak{M} = \mathfrak{T}(H)^f$ (Examples 4.1 (6)). Then

$$I(\mathfrak{T}(H)^f) = \mathfrak{C}(H), \text{ the ideal of compact linear maps in } \mathfrak{L}(H).$$
$$A(\mathfrak{T}(H)^f) = \mathfrak{L}(H).$$

LEMMA 5.1. *For $S \in \mathfrak{T}(H)$ let $\mathfrak{L}(H)_{S^\perp} = \{\alpha; \alpha \in \mathfrak{L}(H) \text{ with } \alpha \cdot \pi_S = o \text{ and } \pi_{S^\perp} \cdot \alpha = \alpha\}$. Certainly $\mathfrak{L}(H)_{S^\perp}$ is isometric to $\mathfrak{L}(S^\perp)$. If $\mathfrak{R}$ is an R-set and $S \in \mathfrak{R}$, then $\mathfrak{L}(H)_{S^\perp} \subset I(\mathfrak{R})$. In particular $\pi_{S^\perp} \in I(\mathfrak{R})$, and hence $\pi_S, \pi_{S^\perp} \in A(\mathfrak{R})$.*

Proof. A direct consequence of Definition 5.1.

DEFINITION 5.2. Let $\mathfrak{R}$ be an *R*-set. Then we consider also the following closed invariant subgroups of the groups $\mathrm{GL}(A(\mathfrak{R}))$ and $\mathrm{U}(A(\mathfrak{R}))$ (Definition 2.2):

$$\mathrm{GL}_q(I(\mathfrak{R})) = \{1+\alpha; 1+\alpha \in \mathrm{GL}(H) \text{ and } \alpha \in I(\mathfrak{R})\}, \quad \text{and}$$

$$\mathrm{U}_q(I(\mathfrak{R})) = \mathrm{GL}_q(I(\mathfrak{R})) \cap \mathrm{U}(H).$$

DEFINITION 5.3. If $S \in \mathfrak{T}(H)$ and $G \subset \mathrm{GL}(H)$ a subset, then we denote

$$G_S = \{\gamma; \gamma \in G \text{ with } \gamma(S) = S\},$$

$$G_{(S)} = \{\gamma; \gamma \in G \text{ with } \gamma|_S = id\}, \quad \text{and}$$

$$G|_S = \{\gamma|_S; \gamma \in G \text{ with } \gamma(S) = S\}.$$

COROLLARY 5.1. *Let $\mathfrak{R}$ be an R-set, and $S \in \mathfrak{R}$. Then we have the representations*

$$\mathrm{U}(A(\mathfrak{R}))_S = \mathrm{U}(A(\mathfrak{R}))|_S \times \mathrm{U}(S^\perp),$$

$$\mathrm{U}_q(I(\mathfrak{R}))_S = \mathrm{U}_q(I(\mathfrak{R}))|_S \times \mathrm{U}(S^\perp), \quad \text{and}$$

$$\mathrm{U}(A(\mathfrak{R}))_{(S)} = \mathrm{U}_q(I(\mathfrak{R}))_{(S)} = \mathrm{U}(S^\perp).$$

Further $\mathrm{U}(A(\mathfrak{R}))_S \subset \mathrm{GL}(A(\mathfrak{R}))_S$, and $\mathrm{U}(I(\mathfrak{R}))_S \subset \mathrm{U}(I(\mathfrak{R}))_S$ are strong deformation retracts. In particular $\mathrm{U}(A(\mathfrak{R}))_S$ and $\mathrm{GL}(A(\mathfrak{R}))_S$, and $\mathrm{U}(I(\mathfrak{R}))_S$ and $\mathrm{GL}(I(\mathfrak{R}))_S$ are of the same homotopy type.

Proof. The first part follows directly from Lemma 5.1. The second part from Theorem 2.3 in the case of $A(\mathfrak{R})$, and by the same construction as in the proof of Theorem 2.3 in the case of $I(\mathfrak{R})$. (We have $\pi_{S^\perp} \in I(\mathfrak{R})$ and $\pi_S = 1 - \pi_{S^\perp} \in A(\mathfrak{R})$.)

DEFINITION 5.4. Let $\mathfrak{S} \subset \mathfrak{T}(H)$. If $\gamma \in \mathrm{GL}(H)$, then let $\gamma(\mathfrak{S}) = \{\gamma(S); S \in \mathfrak{S}\}$. We denote

$$\mathrm{GL}(\mathfrak{S}) = \{\gamma; \gamma \in \mathrm{GL}(H) \text{ with } \gamma(\mathfrak{S}) = \mathfrak{S} \text{ and } \gamma^*(\mathfrak{S}) = \mathfrak{S}\}, \quad \text{and}$$

$$\mathrm{U}(\mathfrak{S}) = \{\mu; \mu \in \mathrm{U}(H) \text{ with } \mu(\mathfrak{S}) = \mathfrak{S}\}.$$

If $\mathfrak{S}$ is closed in $\mathfrak{T}(H)$, then $\mathrm{GL}(\mathfrak{S})$ and $\mathrm{U}(\mathfrak{S})$ are closed subgroups of $\mathrm{GL}(H)$.

LEMMA 5.2. *Let $\mathfrak{R}$ be an R-set and let $\gamma \in \mathrm{GL}(H)$. Then $\gamma \in A(\mathfrak{R})$ if and only if for each $S \in \mathfrak{R}$ we have $\gamma^{-1}(S) \in M(\mathfrak{R})$ and $\gamma^{*-1}(S) \in M(\mathfrak{R})$.*

Proof. Observe first that by the definition of the adjoint map $\gamma(S) = (\gamma^{*-1}(S^\perp))^\perp$ for any $S \in \mathfrak{T}(H)$. Suppose now $\gamma \in A(\mathfrak{R})$. Let $S \in \mathfrak{R}$ and $\varepsilon > 0$ be arbitrarily assigned. From Lemma 2.2 we have the representations

$$\pi_{\gamma(S^\perp)} = \gamma \cdot \pi_{S^\perp} \cdot \beta \cdot \pi_{S^\perp} \cdot \gamma^* \quad \text{and} \quad \pi_{\gamma^*(S^\perp)} = \gamma^* \cdot \pi_{S^\perp} \cdot \tilde{\beta} \cdot \pi_{S^\perp} \cdot \gamma.$$

By the definition of $A(\mathfrak{R})$ there is a $T \in \mathfrak{R}$ with

$$\|(1-\pi_S) \cdot \gamma \cdot \pi_T\| < (\|\gamma\| \cdot \|\tilde{\beta}\|)^{-1} \cdot \varepsilon, \quad \text{and}$$

$$\|(1-\pi_S) \cdot \gamma^* \cdot \pi_T\| < (\|\gamma\| \cdot \|\beta\|)^{-1} \cdot \varepsilon.$$

We conclude

$$\delta(T, \gamma^{-1}(S)) = \delta(T, (\gamma^*(S^\perp))^\perp) = \|\pi_{\gamma^*(S^\perp)} \cdot \pi_T\| = \|\gamma^* \cdot \pi_{S^\perp} \cdot \tilde{\beta} \cdot (\pi_{S^\perp} \cdot \gamma \cdot \pi_T)\| < \varepsilon,$$

and similarly $\delta(T, \gamma^{*-1}(S)) = \delta(T, (\gamma(S^\perp))^\perp) < \varepsilon$. By Lemma 4.3,

$$\gamma^{-1}(S) \in M(\mathfrak{R}) \text{ and } \gamma^{*-1}(S) \in M(\mathfrak{R}).$$

Suppose now $\gamma \in GL(H)$ satisfies $\gamma^{-1}(S)$, $\gamma^{*-1}(S) \in M(\mathfrak{R})$ for all $S \in \mathfrak{R}$. We turn to the definition of $A(\mathfrak{R})$ and assume $S \in \mathfrak{R}$ and $\varepsilon > 0$ be given. Let $T' = \gamma^{-1}(S)$ and $T'' = \gamma^{*-1}(S)$. Since $\mathfrak{R}$ is an R-set, there is an $R \in \mathfrak{R}$ with $\delta(R, T') < (\|\gamma\|)^{-1} \cdot \varepsilon$ and $\delta(R, T'') < (\|\gamma\|)^{-1} \cdot \varepsilon$. Observe that with Lemma 2.2 we can conclude that $\pi_{S^\perp} \cdot \gamma \cdot \pi_{T'} = o$ and $\pi_{S^\perp} \cdot \gamma^* \cdot \pi_{T''} = o$. We compute

$$\|(1-\pi_S) \cdot \gamma \cdot \pi_R\| \leq \|(1-\pi_S) \cdot \gamma \cdot (1-\pi_{T'}) \cdot \pi_R\| + \|(1-\pi_S) \cdot \gamma \cdot \pi_{T'} \cdot \pi_R\|$$

$$\leq \|\gamma\| \cdot \delta(R, T') < \varepsilon,$$

and similarly $\|(1-\pi_S) \cdot \gamma^* \cdot \pi_R\| < \varepsilon$. Which proves $\gamma \in A(\mathfrak{R})$.

THEOREM 5.2 (COMPARE [5]). *Let $\mathfrak{M}$ be a maximal R-set. Then*

$$GL(\mathfrak{M}) = GL(H) \cap A(\mathfrak{M}) = GL(A(\mathfrak{M})), \quad \text{and}$$

$$U(\mathfrak{M}) = U(H) \cap A(\mathfrak{M}) = U(A(\mathfrak{M})).$$

Proof. If $\gamma \in GL(\mathfrak{M})$, then $\gamma^{-1}(\mathfrak{M}) = \mathfrak{M}$ and $\gamma^{*-1}(\mathfrak{M}) = \mathfrak{M}$ and, by Lemma 5.2, $\gamma \in GL(H) \cap A(\mathfrak{M})$. If $\gamma \in GL(H) \cap A(\mathfrak{M})$, then by Corollary 2.1, $\gamma^{-1} \in GL(H) \cap A(\mathfrak{M})$, and we conclude $\gamma(\mathfrak{M}) = \mathfrak{M}$ and $\gamma^*(\mathfrak{M}) = \mathfrak{M}$, and therefore $\gamma \in GL(\mathfrak{M})$. This proves also $U(\mathfrak{M}) = U(H) \cap A(\mathfrak{M})$.

LEMMA 5.3. *Let $\mathfrak{M}$ be a maximal R-set. For any pair $S, T \in \mathfrak{M}$ and every $\varepsilon > 0$ there exists an $R \in \mathfrak{M}$ with $R \subset S$ and $\delta(R, T) < \varepsilon$ (compare Definition 4.1).*

Proof. We may assume $0 < \varepsilon \leq 1$. First there is an $\tilde{R} \in \mathfrak{M}$ with $\delta(\tilde{R}, S) < \varepsilon/8$ and $\delta(\tilde{R}, T) < \varepsilon/8$. By Corollary 3.3 there are $S', T' \in \mathfrak{M}$ with $\tilde{R} \subset S'$, $\tilde{R} \subset T'$, and $g(S', S) < \varepsilon/8$, $g(T', T) < \varepsilon/8$. Consider now $\gamma = \pi_S \cdot \pi_{S'} + (1-\pi_S) \cdot (1-\pi_{S'}) \in GL(\mathfrak{M})$ (Lemma 3.6). We have $\gamma(S') = S$, and we compute

$$\|\gamma - 1\| = \|\pi_S \cdot \pi_{S'} - \pi_S - \pi_{S'} + \pi_S \cdot \pi_{S'}\| \leq \delta(S, S') + \delta(S', S) \leq 2 \cdot g(S', S) < \varepsilon/4,$$

and $\|\gamma^{-1}\| = \|(1 + (\gamma - 1))^{-1}\| \leq 1/(1 - \|\gamma - 1\|) < 2$. Let $R = \gamma(\tilde{R}) \in \mathfrak{M}$. Certainly $R \subset \gamma(S') = S$. Now $\delta(R, T) \leq \delta(\gamma(\tilde{R}), T') + \delta(T', T)$. But by Lemma 3.3, $\delta(\gamma(\tilde{R}), T') \leq \|\gamma - 1\| \cdot \|\gamma^{-1}\| < \varepsilon/2$. And $\delta(T', T) \leq g(T', T) < \varepsilon/8$. Hence $\delta(R, T) < \varepsilon$.

DEFINITION 5.5. *Let* $\mathfrak{S} \subset \mathfrak{T}(H)$ *and let* $S_0 \in \mathfrak{T}(H)$. *Then we denote*

$$\mathfrak{S}|_{S_0} = \mathfrak{S} \cap \mathfrak{T}(S_0) = \{S; S \in \mathfrak{S} \quad \text{with} \quad S \subset S_0\}.$$

THEOREM 5.3. *Let* $\mathfrak{M}$ *be a maximal R-set and let* $S_0 \in \mathfrak{M}$. *Then* $\mathfrak{M}|_{S_0}$ *is a maximal R-set again with respect to the Hilbert space* S_0.

Proof. Lemma 5.3 gives $\mathfrak{M}|_{S_0}$ is an R-set. Evidently $(\mathfrak{M}|_{S_0})^{\vee} = \mathfrak{M}|_{S_0}$ (see Definition 4.3). And $\mathfrak{M}|_{S_0} = \mathfrak{M} \cap \mathfrak{T}(S_0)$ is closed, since $\mathfrak{T}(S_0) \subset \mathfrak{T}(H)$ is closed.

DEFINITION 5.6. *Let* $\mathfrak{A} \subset \mathfrak{L}(H)$ *and* $S_0 \in \mathfrak{T}(H)$. *Then we denote by*

$$\mathfrak{A}|_{S_0} = \{\alpha|_{S_0}; \alpha|_{S_0} \in \mathscr{L}(S_0) \text{ where } \alpha \in \mathfrak{A} \text{ with } \alpha(S_0) \subset S_0\}.$$

THEOREM 5.4. *Let* $\mathfrak{M}$ *be a maximal R-set and* $S_0 \in \mathfrak{M}$. *Then*

$$I(\mathfrak{M}|_{S_0}) = I(\mathfrak{M})|_{S_0} \quad \text{and} \quad A(\mathfrak{M}|_{S_0}) = A(\mathfrak{M})|_{S_0}.$$

Proof. Directly from Definition 5.1 with the use of Lemma 5.3.

THEOREM 5.5. *Let* $\mathfrak{M}$ *be a maximal R-set and let* $S \in \mathfrak{M}$. *Then we have*

$$\mathrm{GL}(A(\mathfrak{M}|_S)) = \mathrm{GL}(A(\mathfrak{M}))_S / \mathrm{GL}(A(\mathfrak{M}))_{\langle S \rangle},$$

$$\mathrm{U}(A(\mathfrak{M}|_S)) = \mathrm{U}(A(\mathfrak{M}))_S / \mathrm{U}(A(\mathfrak{M}))_{\langle S \rangle},$$

$$\mathrm{GL}_q(I(\mathfrak{M}|_S)) = \mathrm{GL}_q(I(\mathfrak{M}))_S / \mathrm{GL}_q(I(\mathfrak{M}))_{\langle S \rangle},$$

$$\mathrm{U}_q(I(\mathfrak{M}|_S)) = \mathrm{U}_q(I(\mathfrak{M}))_S / \mathrm{U}_q(I(\mathfrak{M}))_{\langle S \rangle},$$

$$\mathrm{U}(A(\mathfrak{M}))_S = \mathrm{U}(A(\mathfrak{M}|_S)) \times \mathrm{U}(S^{\perp}), \quad \text{and}$$

$$\mathrm{U}_q(I(\mathfrak{M}))_S = \mathrm{U}_q(I(\mathfrak{M}|_S)) \times \mathrm{U}(S^{\perp}).$$

Proof. Direct consequence of Theorem 5.4.

6. The path components of maximal R-sets.

LEMMA 6.1 (COMPARE [4]). *Let* $\mathfrak{M}$ *be a maximal R-set. Each open ball* $\mathfrak{B}_{\langle S, r \rangle}$ $= \{T; T \in \mathfrak{M} \text{ and } g(S, T) < r\}$ *in* $\mathfrak{M}$ *with center* S *and radius* r, $0 < r \leqq 1$, *is path connected.*

Proof. Let $T \in \mathfrak{B}_{\langle S, r \rangle}$ be given. We consider first $S^{\perp}, T^{\perp} \in \mathfrak{T}(H)$. Since $g(S^{\perp}, T^{\perp})$ $= g(S, T) < r$, we can apply Lemma 3.5 to the pair $S^{\perp}, T^{\perp}$. Consider $\alpha_t = 1 - t \cdot \pi_T$ $\in \mathfrak{L}(H)$, $0 \leqq t \leqq 1$. Then $t \to \alpha_t(S^{\perp}) \in \mathfrak{T}(H)$ is a continuous path in $\mathfrak{T}(H)$, which connects $S^{\perp}$ with $T^{\perp}$ and with $g(S^{\perp}, \alpha_t(S^{\perp})) = t \cdot g(S^{\perp}, T^{\perp})$ for $0 \leqq t \leqq 1$.

Now we turn to the path $t \to T_t = (\alpha_t(S^{\perp}))^{\perp} \in \mathfrak{T}(H)$, $0 \leqq t \leqq 1$. The complement map $\perp : \mathfrak{T}(H) \to \mathfrak{T}(H)$ is continuous by Lemma 3.1. The new path is therefore continuous again, and connects S with T in $\mathfrak{T}(H)$. We have $g(S, T_t) = g(S^{\perp}, \alpha_t(S^{\perp}))$ $= t \cdot g(S, T) < r$, $0 \leqq t \leqq 1$.

Finally, $T_t \in \mathfrak{M}, 0 \leqq t \leqq 1$. We apply Lemma 4.3. Let $\varepsilon > 0$ be given. By Lemma 2.2, $\pi_{\alpha_t(S^\perp)} = \alpha_t \cdot \pi_{S^\perp} \cdot \beta \cdot \pi_{S^\perp} \cdot \alpha_t^*$. There exists an $R \in \mathfrak{M}$ with $\delta(R, S) < (\|\alpha_t\| \cdot \|\beta\|)^{-1} \cdot \varepsilon$ and $\delta(R, T) < (\|\alpha_t\| \cdot \|\beta\|)^{-1} \cdot \varepsilon$. Then

$$
\begin{aligned}
\delta(R, T_t) = \delta(R, (\alpha_t(S^\perp))^\perp) &= \|(1 - \pi_{(\alpha_t(S^\perp))^\perp}) \cdot \pi_R\| = \|\pi_{\alpha_t(S^\perp)} \cdot \pi_R\| \\
&\leqq \|\alpha_t\| \cdot \|\beta\| \cdot \|\pi_{S^\perp} \cdot ((1-t) \cdot 1 + t \cdot \pi_{T^\perp}) \cdot \pi_R\| \\
&\leqq \|\alpha_t\| \cdot \|\beta\| \cdot ((1-t) \cdot \delta(R, S) + t \cdot \delta(R, T)) < \varepsilon.
\end{aligned}
$$

COROLLARY 6.1. *A maximal R-set $\mathfrak{M}$ is locally path connected. Path components and connected components coincide therefore, and they are open and closed in $\mathfrak{M}$.*

DEFINITION 6.1. First we introduce the following notation: If G is a topological group, then G^1 denotes the path component of its unit $1 \in G$. Let $\mathfrak{B}_{(S,1)} = \{T; T \in \mathfrak{M} \text{ with } g(S, T) < 1\}$ be the open ball in $\mathfrak{M}$ in the g-metric with radius 1 and center $S \in \mathfrak{M}$. Then we have the following cross-section map

$$
s: \mathfrak{B}_{(S,1)} \to \mathrm{GL}_q^1(I(\mathfrak{M})) \subset \mathrm{GL}^1(\mathfrak{M})
$$
$$
s(T) = \pi_T \cdot \pi_S + (1 - \pi_T) \cdot (1 - \pi_S).
$$

LEMMA 6.2. *The map s is well defined, continuous, and satisfies $s(T)(S) = T$.*

Proof. By Lemma 3.6, $s(T) \in \mathrm{GL}(H)$ and $s(T)(S) = \pi_T(S) = T$. Further, $s(T) = (1 - \pi_{T^\perp}) \cdot (1 - \pi_{S^\perp}) + \pi_{T^\perp} \cdot \pi_{S^\perp} \in \mathrm{GL}_q(I(\mathfrak{M}))$. To show the continuity of s, we compute

$$
\begin{aligned}
\|s(T_1) - s(T_2)\| &= \|\pi_{T_1} \cdot \pi_S - \pi_{T_2} \cdot \pi_S + \pi_{T_1^\perp} \cdot \pi_{S^\perp} - \pi_{T_2} \cdot \pi_{S^\perp}\| \\
&\leqq \|\pi_{T_1} - \pi_{T_2}\| + \|\pi_{T_1^\perp} - \pi_{T_2^\perp}\| = 2 \cdot g(T_1, T_2).
\end{aligned}
$$

By Lemma 6.1, $\mathfrak{B}_{(S,1)}$ is path connected. Since $1 = s(S) \in s(\mathfrak{B}_{(S,1)})$, it follows $s(\mathfrak{B}_{(S,1)}) \subset \mathrm{GL}_q^1(I(\mathfrak{M}))$.

DEFINITION 6.2. We have the action of the group $\mathrm{GL}(\mathfrak{M})$ on $\mathfrak{M}$

$$
a: \mathrm{GL}(\mathfrak{M}) \times \mathfrak{M} \to \mathfrak{M},
$$
$$
a(\gamma, S) = \gamma(S).
$$

LEMMA 6.3. *a is continuous. If $\mathfrak{C} \in \pi_0(\mathfrak{M})$ is a path component of $\mathfrak{M}$, we have in particular the action of $\mathrm{GL}^1(\mathfrak{M})$ on $\mathfrak{C}$*

$$
a: \mathrm{GL}^1(\mathfrak{M}) \times \mathfrak{C} \to \mathfrak{C}.
$$

And from this action the maps

$$
a_\gamma: \mathfrak{C} \to \mathfrak{C},
$$
$$
a_\gamma(S) = \gamma(S) \text{ for a fixed } \gamma \in \mathrm{GL}^1(\mathfrak{M}), \quad \text{and}
$$
$$
a_S: \mathrm{GL}^1(\mathfrak{M}) \to \mathfrak{C},
$$
$$
a_S(\gamma) = \gamma(S) \text{ for a fixed } S \in \mathfrak{C}.
$$

a_γ is a homeomorphism of $\mathfrak{C}$ onto itself for each $\gamma \in \mathrm{GL}^1(\mathfrak{M})$. a_S is a continuous map onto $\mathfrak{C}$. The same holds for the group $\mathrm{GL}_q^1(I(\mathfrak{M}))$.

Proof. Let (γ_1, S_1), $(\gamma_2, S_2) \in \mathrm{GL}(\mathfrak{M}) \times \mathfrak{M}$. Then from Lemma 3.3

$$d(a(\gamma_1, S_1), a(\gamma_2, S_2)) = d(\gamma_1(S_1), \gamma_2(S_2))$$

$$\leqq (\|\gamma_1 - \gamma_2\| + d(S_1, S_2)) \cdot (\|\gamma_1^{-1}\| + \|\gamma_2^{-1}\|),$$

which proves the continuity of a, and therefore of a_γ and a_S. a_γ is a homeomorphism, since $a_\gamma \cdot a_\gamma^{-1} = id$ and $a_\gamma^{-1} \cdot a_\gamma = id$.

To show that a_S is a map onto $\mathfrak{C}$, let $T \in \mathfrak{C}$ be given. Then there exists a continuous path S_t, $0 \leqq t \leqq 1$, in $\mathfrak{C}$, which connects $S_0 = S$ with $S_1 = T$. We can find an integer N with $g(S_{k/N}, S_{(k+1)/N}) < 1$ for $k = 0, 1, \ldots, N-1$. It is therefore enough to show that for $S, T \in \mathfrak{C}$ with $g(S, T) < 1$ there exists a $\gamma \in \mathrm{GL}^1(\mathfrak{M})$ with $\gamma(S) = T$. We can take $s(T) \in \mathrm{GL}^1(\mathfrak{M})$ of Lemma 6.1 as such an element γ.

LEMMA 6.4. *The map $a_S \colon \mathrm{GL}^1(\mathfrak{M}) \to \mathfrak{C}$, where $S \in \mathfrak{C}$, is open. The same holds for $\mathrm{GL}_q^1(I(\mathfrak{M}))$.*

Proof. If $\gamma \in \mathrm{GL}^1(\mathfrak{M})$, then we denote by $l_\gamma \colon \mathrm{GL}^1(\mathfrak{M}) \to \mathrm{GL}^1(\mathfrak{M})$ the left translation $l_\gamma(\alpha) = \gamma \cdot \alpha$, $\alpha \in \mathrm{GL}^1(\mathfrak{M})$. Observe that $a_S = a_\gamma \cdot a_S \cdot l_{\gamma^{-1}}$ for $\gamma \in \mathrm{GL}^1(\mathfrak{M})$. Also that $\mathfrak{B}_{(S,1)} \subset \mathfrak{C}$ and $(a_S \cdot s)(T) = T$ for all $T \in \mathfrak{B}_{(S,1)}$ by Lemma 6.2.

Suppose now $\mathfrak{D} \subset \mathrm{GL}^1(\mathfrak{M})$ is an open subset, and consider $\gamma \in \mathfrak{D}$. Then there is an open neighborhood $\mathfrak{D}_1 \subset \mathrm{GL}^1(\mathfrak{M})$ of 1 with $\mathfrak{D}_1 \subset l_{\gamma^{-1}}(\mathfrak{D}) \cap (a_S)^{-1}(\mathfrak{B}_{(S,1)})$. Now $a_S(\mathfrak{D}) = (a_\gamma \cdot a_S \cdot l_{\gamma^{-1}})(\mathfrak{D})$. But $s^{-1}(\mathfrak{D}_1) \subset a_S(\mathfrak{D}_1)$ (if $T \in s^{-1}(\mathfrak{D}_1)$, then $s(T) \in \mathfrak{D}_1$ and $T = (a_S \cdot s)(T)$; hence $T \in a_S(\mathfrak{D}_1)$). Further $S \in s^{-1}(\mathfrak{D}_1)$ (namely $1 \in \mathfrak{D}_1$). Therefore, $a_S(\gamma) = \gamma(S) \in a_\gamma(s^{-1}(\mathfrak{D}_1)) \subset a_\gamma(a_S(l_{\gamma^{-1}}(\mathfrak{D}))) = a_S(\mathfrak{D})$. Since a_γ is a homeomorphism, $a_\gamma(s^{-1}(\mathfrak{D}_1))$ is open, which proves the lemma.

DEFINITION 6.3. A Banach manifold is a topological Hausdorff space such that each point has an open neighborhood homeomorphic to an open subset of a Banach space. (See for example [10] and [13].)

THEOREM 6.1. *A maximal R-set $\mathfrak{M}$ is a Banach manifold. In particular the path components of $\mathfrak{M}$ are connected Banach manifolds.*

Proof. Let $S \in \mathfrak{M}$ be given, and let $\mathfrak{L} \in \pi_0(\mathfrak{M})$ be the path component of $\mathfrak{M}$ with $S \in \mathfrak{L}$. We construct a coordinate system $(\mathfrak{U}, f)$ at S: Consider the open map of the preceding Lemma 6.4

$$a_S \colon \mathrm{GL}^1(\mathfrak{M}) \to \mathfrak{C},$$

$$a_S(\gamma) = \gamma(S).$$

Let $\tilde{\mathfrak{U}} = \{\gamma; \gamma \in \mathrm{GL}^1(\mathfrak{M}) \text{ with } \pi_S' \cdot \gamma \cdot \iota_S \in \mathrm{GL}(S)\}$, where $\iota_S \colon S \to H$ is the inclusion and $\pi_S' \colon H \to S$ is the projection. $\tilde{\mathfrak{U}}$ is open in $\mathrm{GL}^1(\mathfrak{M})$. (Proof: The map $p_S \colon \mathrm{GL}^1(\mathfrak{M}) \to$

$\mathfrak{L}(S)$ defined by $p_S(\alpha) = \pi'_S \cdot \alpha \cdot \iota_S$ is continuous. GL(S) is open in $\mathfrak{L}(S)$, and $\tilde{\mathfrak{u}}$ $= (p_S)^{-1}(\mathrm{GL}(S))$.) We define $\mathfrak{u} = a_S(\tilde{\mathfrak{u}})$. $\mathfrak{u}$ is open in $\mathfrak{C}$ and therefore in $\mathfrak{M}$.

Next let $A(\mathfrak{M})_{S,S^\perp} = \{\alpha; \alpha \in A(\mathfrak{M}) \text{ with } \alpha \cdot \pi_S = \alpha \text{ and } \pi_{S^\perp} \cdot \alpha = \alpha\}$. $A(\mathfrak{M})_{S,S^\perp}$ is a closed linear subspace of the Banach space $A(\mathfrak{M})$ and hence a Banach space.

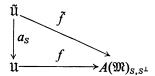Before we construct the coordinate homeomorphism f, we notice the following: Let

$$\mathfrak{L}(H)_{S,S^\perp} = \{\alpha; \alpha \in \mathfrak{L}(H) \text{ with } \alpha \cdot \pi_S = \alpha \text{ and } \pi_{S^\perp} \cdot \alpha = \alpha\}.$$

If $\alpha \in \mathfrak{L}(H)_{S,S^\perp}$, then $1 + \alpha \in \mathrm{GL}(H)$ (namely $(1+\alpha) \cdot (1-\alpha) = 1$ and $(1-\alpha) \cdot (1+\alpha) = 1$). And if $\alpha_1, \alpha_2 \in \mathfrak{L}(H)_{S,S^\perp}$, then $(1+\alpha_1)(S) = (1+\alpha_2)(S)$ if and only if $\alpha_1 = \alpha_2$.

Consider now the continuous map

$$\tilde{f}: \tilde{\mathfrak{u}} \to A(\mathfrak{M})_{S,S^\perp},$$
$$\tilde{f}(\gamma) = \pi_{S^\perp} \cdot \gamma \cdot \pi_S \cdot (\pi_S \cdot \gamma \cdot \pi_S + \pi_{S^\perp})^{-1} \cdot \pi_S$$
$$= \gamma \cdot \pi_S \cdot (\pi_S \cdot \gamma \cdot \pi_S + \pi_{S^\perp})^{-1} \cdot \pi_S - \pi_S.$$

Observe that $(1 + \tilde{f}(\gamma))(S) = \gamma(S)$. We define f via the commutative diagram

$$
\begin{array}{ccc}
\tilde{\mathfrak{u}} & \xrightarrow{\ \tilde{f}\ } & \\
\downarrow{\scriptstyle a_S} & \searrow & \\
\mathfrak{u} & \xrightarrow[\ f\ ]{} & A(\mathfrak{M})_{S,S^\perp}
\end{array}
$$

by $f = \tilde{f} \cdot (a_S)^{-1}$. The map f is well defined and therefore continuous. (Namely if $a_S(\gamma_1) = a_S(\gamma_2)$ for $\gamma_1, \gamma_2 \in \tilde{\mathfrak{u}}$, then $\gamma_1(S) = \gamma_2(S)$ and $(1+\tilde{f}(\gamma_1))(S) = (1+\tilde{f}(\gamma_2))(S)$, and hence $\tilde{f}(\gamma_1) = \tilde{f}(\gamma_2)$ by the remark above.)

Next we consider the continuous map

$$\tilde{h}: A(\mathfrak{M})_{S,S^\perp} \to \mathrm{GL}^1(\mathfrak{M}),$$
$$\tilde{h}(\alpha) = 1 + \alpha.$$

$\tilde{h}$ is well defined. $(\tilde{h}(\alpha) \in \mathrm{GL}(H) \cap A(\mathfrak{M}) = \mathrm{GL}(\mathfrak{M})$, and $1 + t \cdot \alpha$, $0 \leq t \leq 1$, is a path which connects 1 with $\tilde{h}(\alpha)$. Therefore, $\tilde{h}(\alpha) \in \mathrm{GL}^1(\mathfrak{M})$.) Since $\pi'_S \cdot (\tilde{h}(\alpha)) \cdot \iota_S = 1_S$, we have further $\tilde{h}(\alpha) \in \tilde{\mathfrak{u}}$. We compute

$$(\tilde{f} \cdot \tilde{h})(\alpha) = \alpha \quad \text{for} \quad \alpha \in A(\mathfrak{M})_{S,S^\perp}, \quad \text{and}$$
$$(\tilde{h} \cdot \tilde{f})(\gamma) = 1 + \tilde{f}(\gamma) \quad \text{for} \quad \gamma \in \tilde{\mathfrak{u}}.$$

And finally we define

$$h: A(\mathfrak{M})_{S,S^\perp} \to \mathfrak{u},$$
$$h = a_S \cdot \tilde{h}.$$

Then we have $(f \cdot h)(\alpha) = \alpha$ for $\alpha \in A(\mathfrak{M})_{S,S^\perp}$, and $(h \cdot f)(T) = (1 + \tilde{f}(\gamma))(S) = \gamma(S) = T$ for $T \in \mathfrak{u}$, where $\gamma \in \tilde{\mathfrak{u}}$ with $a_S(\gamma) = \gamma(S) = T$. This proves $f: \mathfrak{u} \to A(\mathfrak{M})_{S,S^\perp}$ is a homeomorphism onto the Banach space $A(\mathfrak{M})_{S,S^\perp}$ and $h = f^{-1}$.

DEFINITION 6.4. Let $\mathfrak{M}$ be a maximal R-set, let $\mathfrak{C} \in \pi_0(\mathfrak{M})$ be a path component of $\mathfrak{M}$, and let $S \in \mathfrak{C}$ be fixed. We consider the closed subgroup

$$\mathrm{GL}^1(\mathfrak{M})_S = \{\gamma; \gamma \in \mathrm{GL}^1(\mathfrak{M}) \text{ with } \gamma(S) = S\},$$

and we form the quotient space

$$\mathrm{GL}^1(\mathfrak{M})/\mathrm{GL}^1(\mathfrak{M})_S = \{[\gamma]; \gamma \in \mathrm{GL}^1(\mathfrak{M})\}$$

of left cosets $[\gamma] = \gamma \cdot \mathrm{GL}^1(\mathfrak{M})_S$. This quotient space is given the topology induced by the projection map

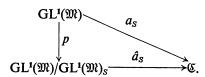$$p: \mathrm{GL}^1(\mathfrak{M}) \to \mathrm{GL}^1(\mathfrak{M})/\mathrm{GL}^1(\mathfrak{M})_S.$$

We can lift the map $a_S: \mathrm{GL}^1(\mathfrak{M}) \to \mathfrak{C}$ onto the quotient space.

$$\hat{a}_S: \mathrm{GL}^1(\mathfrak{M})/\mathrm{GL}^1(\mathfrak{M})_S \to \mathfrak{C},$$
$$\hat{a}_S([\gamma]) = a_S(\gamma) = \gamma(S).$$

Similarly for the group $\mathrm{GL}_q^1(I(\mathfrak{M}))$.

THEOREM 6.2. $\hat{a}_S: \mathrm{GL}^1(\mathfrak{M})/\mathrm{GL}^1(\mathfrak{M})_S \to \mathfrak{C}$ *is a homeomorphism. The spaces $\mathfrak{C}$ and $\mathrm{GL}^1(\mathfrak{M})/\mathrm{GL}^1(\mathfrak{M})_S$ can therefore be identified.*

Proof. Consider the commutative diagram

$$
\begin{array}{ccc}
\mathrm{GL}^1(\mathfrak{M}) & \xrightarrow{\quad a_S \quad} & \\
\downarrow p & & \\
\mathrm{GL}^1(\mathfrak{M})/\mathrm{GL}^1(\mathfrak{M})_S & \xrightarrow[\hat{a}_S]{} & \mathfrak{C}.
\end{array}
$$

Certainly $\hat{a}_S$ is bijective. Since a_S is continuous and open, $\hat{a}_S$ is a homeomorphism.

THEOREM 6.2$_q$. $\hat{a}_S: \mathrm{GL}_q^1(I(\mathfrak{M}))/\mathrm{GL}_q^1(I(\mathfrak{M}))_S \to \mathfrak{C}$ *is a homeomorphism. The spaces $\mathfrak{C}$ and $\mathrm{GL}_q^1(I(\mathfrak{M}))/\mathrm{GL}_q^1(I(\mathfrak{M}))_S$ can be identified also.*

DEFINITION 6.5. We have a canonical local cross-section of $\mathrm{GL}^1(\mathfrak{M})_S$ in $\mathrm{GL}^1(\mathfrak{M})$. $\mathfrak{B}' = (\hat{a}_S)^{-1}(\mathfrak{B}_{(S,1)})$ is an open neighborhood of the point $[1] \in \mathrm{GL}^1(\mathfrak{M})/\mathrm{GL}^1(\mathfrak{M})_S$. We define

$$s': \mathfrak{B}' \to \mathrm{GL}^1(\mathfrak{M})$$
$$s' = s \cdot \hat{a}_S, \quad \text{i.e.,} \quad s'([\gamma]) = s(\gamma(S)).$$

We have $p \cdot s'([\gamma]) = [s(\gamma(S))]$. But $(s(\gamma(S)))(S) = \gamma(S)$ and therefore $[s(\gamma(S))] = [\gamma]$. Or $p \cdot s'([\gamma]) = [\gamma]$ for $[\gamma] \in \mathfrak{B}'$. Similarly for $\mathrm{GL}_q^1(I(\mathfrak{M}))_S$ and $\mathrm{GL}^1(I(\mathfrak{M}))$.

The Steenrod construction [14, p. 30] can now be applied, and we obtain:

THEOREM 6.3. *If K is a closed subgroup of $\mathrm{GL}^1(\mathfrak{M})_S$,*

$$p: \mathrm{GL}^1(\mathfrak{M})/K \to \mathfrak{C} = \mathrm{GL}^1(\mathfrak{M})/\mathrm{GL}^1(\mathfrak{M})_S$$

the projection map induced by the inclusion of cosets, then a bundle structure can be assigned to $\mathrm{GL}^1(\mathfrak{M})/K$ *relative to* p *such that the fibre of the bundle is* $\mathrm{GL}^1(\mathfrak{M})_S/K$ *and the group of the bundle is* $\mathrm{GL}^1(\mathfrak{M})_S/K_0$ *acting on* $\mathrm{GL}^1(\mathfrak{M})_S/K$ *as left translations, where* K_0 *is the largest subgroup of* K *invariant in* $\mathrm{GL}^1(\mathfrak{M})_S$. *The same holds for the group* $\mathrm{GL}^1(I(\mathfrak{M}))$.

COROLLARY 6.2. $\mathrm{GL}^1(\mathfrak{M})$ *is a principal fibre bundle over*

$$\mathfrak{C} = \mathrm{GL}^1(\mathfrak{M})/\mathrm{GL}^1(\mathfrak{M})_S$$

with fibre and structure group $\mathrm{GL}^1(\mathfrak{M})_S$, *which acts on the fibres by left translations. In particular we have an exact sequence for the homotopy groups*

$$\cdots \to \pi_{n+1}(\mathfrak{C}) \to \pi_n(\mathrm{GL}^1(\mathfrak{M})_S) \to \pi_n(\mathrm{GL}^1(\mathfrak{M})) \to \pi_n(\mathfrak{C}) \to \cdots.$$

COROLLARY 6.2_q. $\mathrm{GL}^1_q(I(\mathfrak{M}))$ *is a principal fibre bundle over*

$$\mathfrak{C} = \mathrm{GL}^1_q(I(\mathfrak{M}))/\mathrm{GL}^1_q(I(\mathfrak{M}))_S$$

with fibre and structure group $\mathrm{GL}^1_q(I(\mathfrak{M}))_S$, *which acts on the fibres by left translations. In particular we have an exact sequence for the homotopy groups*

$$\cdots \to \pi_{n+1}(\mathfrak{C}) \to \pi_n(\mathrm{GL}^1_q(I(\mathfrak{M}))_S) \to \pi_n(\mathrm{GL}^1_q(I(\mathfrak{M}))) \to \pi_n(\mathfrak{C}) \to \cdots.$$

The preceding constructions and arguments can also be applied to the unitary groups $\mathrm{U}^1(\mathfrak{M})$ and $\mathrm{U}^1_q(I(\mathfrak{M}))$.

DEFINITION 6.1′. Let $\mathfrak{B}_{(S,1)} = \{T; T \in \mathfrak{M}$ with $g(S, T) < 1\}$ be again the open ball in $\mathfrak{M}$ in the g-metric with radius 1 and center $S \in \mathfrak{M}$. Then we have the cross-section map

$$v \colon \mathfrak{B}_{(S,1)} \to \mathrm{U}^1_q(I(\mathfrak{M})) \subset \mathrm{U}^1(\mathfrak{M}),$$
$$v = u \cdot s, \quad \text{i.e.,} \quad v(T) = u(\pi_T \cdot \pi_S + (1 - \pi_T) \cdot (1 - \pi_S))$$

where $u \colon \mathrm{GL}(\mathfrak{M}) \to \mathrm{U}(\mathfrak{M})$ is the map of Theorem 2.2.

LEMMA 6.2′. *The map* v *is well defined, continuous, and satisfies* $v(T)(S) = T$.

Proof. $v(T)(S) = T$ follows from Corollary 2.2. We show

$$u(\mathrm{GL}^1_q(I(\mathfrak{M}))) \subset \mathrm{GL}^1_q(I(\mathfrak{M})).$$

Recall that if $\gamma \in \mathrm{GL}^1_q(I(\mathfrak{M}))$, then $u(\gamma) = \mu = \gamma \cdot \beta^{-1}$, where $\beta \in \mathrm{GL}(\mathfrak{M})$ is the uniquely determined positive and self-adjoint element with $\beta^2 = \gamma^* \cdot \gamma$. We write $\beta = 1 + \alpha$, $\alpha \in A(\mathfrak{M})$, and show $\alpha \in I(\mathfrak{M})$. First $\beta^2 = 1 + 2\alpha + \alpha^2 \in \mathrm{GL}^1_q(I(\mathfrak{M}))$ implies $2\alpha + \alpha^2 = 2\alpha \cdot (1 + \tfrac{1}{2}\alpha) \in I(\mathfrak{M})$. We have now $1 + \tfrac{1}{2}\alpha \in \mathrm{GL}(\mathfrak{M})$. (Namely $1 + \beta \in \mathrm{GL}(\mathfrak{M})$ by Lemma 2.7, and therefore $1 + \tfrac{1}{2}\alpha = \tfrac{1}{2} \cdot (1 + \beta) \in \mathrm{GL}(\mathfrak{M})$.) Since $I(\mathfrak{M})$ is an ideal in $A(\mathfrak{M})$, we conclude that $\alpha \in I(\mathfrak{M})$. This shows that β and hence $\mu = \gamma \cdot \beta^{-1}$ are in $\mathrm{GL}^1_q(I(\mathfrak{M}))$.

From here on, Definition 6.2, Lemma 6.3, Lemma 6.4, Definition 6.4, and Definition 6.5 carry over word by word to the unitary groups $U^1(\mathfrak{M})$ and $U_q^1(I(\mathfrak{M}))$, and we obtain:

THEOREM 6.2'. $\hat{a}_S: U^1(\mathfrak{M})/U^1(\mathfrak{M})_S \to \mathfrak{C}$ *is a homeomorphism. The spaces* $\mathfrak{C}$ *and* $U^1(\mathfrak{M})/U^1(\mathfrak{M})_S$ *can therefore be identified.*

THEOREM 6.2'$_q$. $\hat{a}_S: U_q^1(I(\mathfrak{M}))/U_q^1(I(\mathfrak{M}))_S \to \mathfrak{C}$ *is a homeomorphism. The spaces and* $U_q^1(I(\mathfrak{M}))/U_q^1(I(\mathfrak{M}))_S$ *can be identified also.*

THEOREM 6.3'. *If K is a closed subgroup of* $U^1(\mathfrak{M})_S$,

$$p: U^1(\mathfrak{M})/K \to \mathfrak{C} = U^1(\mathfrak{M})/U^1(\mathfrak{M})_S$$

the projection map induced by the inclusion of cosets, then a bundle structure can be assigned to $U^1(\mathfrak{M})/K$ *relative to p such that the fibre of the bundle is* $U^1(\mathfrak{M})_S/K$ *and the group of the bundle is* $U^1(\mathfrak{M})_S/K_0$ *acting on* $U^1(\mathfrak{M})_S/K$ *as left translations, where K_0 is the largest subgroup of K invariant in* $U^1(\mathfrak{M})_S$. *The same holds for the group* $U_q^1(I(\mathfrak{M}))$.

COROLLARY 6.2'. $U^1(\mathfrak{M})$ *is a principal fibre bundle over*

$$\mathfrak{C} = U^1(\mathfrak{M})/U^1(\mathfrak{M})_S$$

with fibre and structure group $U^1(\mathfrak{M})_S$, *which acts on the fibres by left translations. In particular we have the exact sequence for the homotopy groups*

$$\cdots \to \pi_{n+1}(\mathfrak{C}) \to \pi_n(U^1(\mathfrak{M})_S) \to \pi_n(U^1(\mathfrak{M})) \to \pi_n(\mathfrak{C}) \to \cdots$$

COROLLARY 6.2'$_q$. $U_q^1(I(\mathfrak{M}))$ *is a principal fibre bundle over*

$$\mathfrak{C} = U_q^1(I(\mathfrak{M}))/U_q^1(I(\mathfrak{M}))_S$$

with fibre and structure group $U_q^1(I(\mathfrak{M}))_S$, *which acts on the fibres by left translations. In particular, we have the exact sequence for the homotopy groups*

$$\cdots \to \pi_{n+1}(\mathfrak{C}) \to \pi_n(U_q^1(I(\mathfrak{M})_S) \to \pi_n(U_q^1(I(\mathfrak{M})) \to \pi_n(\mathfrak{C}) \to \cdots$$

7. Stiefel spaces associated with maximal R-sets. In the following we introduce homogeneous spaces, which are direct analogues of the Stiefel manifolds of finite-dimensional vectorspaces. We consider the group $GL(A(\mathfrak{M}))$ only. Everything holds also for the group $GL_q(I(\mathfrak{M}))$.

DEFINITION 7.1. Let $\mathfrak{M}$ be a maximal R-set, $\mathfrak{C} \in \pi_0(\mathfrak{M})$ a path component of $\mathfrak{M}$, and let $S \in \mathfrak{C}$ be fixed. The Stiefel space $St(\mathfrak{C})_S$ associated with the path component $\mathfrak{C}$ (relative to S) is defined as

$$St(\mathfrak{C})_S = \{\lambda; \lambda \in \mathfrak{L}(S, H) \text{ such that there is a } \gamma \in GL^1(\mathfrak{M}) \text{ with } \lambda = \gamma \cdot \iota_S\},$$

where $\iota_S \colon S \to H$ is the inclusion. $\mathrm{St}(\mathfrak{C})_S$ is a closed subset of $\mathfrak{L}(S, H)$ and is given the relative topology.

LEMMA 7.1. *If $S_1, S_2 \in \mathfrak{C}$, then the spaces $\mathrm{St}(\mathfrak{C})_{S_1}$ and $\mathrm{St}(\mathfrak{C})_{S_2}$ are homeomorphic. The reference element $S \in \mathfrak{C}$ is therefore irrelevant. The Stiefel spaces $\mathrm{St}(\mathfrak{C})_S$ are path connected.*

Proof. There exists a $\beta \in \mathrm{GL}^1(\mathfrak{M})$ with $\beta(S_1) = S_2$ (Lemma 6.3). Consider the map $i_\beta \colon \mathrm{St}(\mathfrak{C})_{S_1} \to \mathrm{St}(\mathfrak{C})_{S_2}$, $i_\beta(\lambda) = \gamma \cdot \beta^{-1} \cdot \iota_{S_2}$, where $\lambda = \gamma \cdot \iota_{S_1}$. Then i_β is a homeomorphism. $\mathrm{GL}^1(\mathfrak{M})$ is path connected implies directly $\mathrm{St}(\mathfrak{C})_S$ is path connected.

DEFINITION 7.2. We have the natural projection map

$$q \colon \mathrm{St}(\mathfrak{C})_S \to \mathfrak{C},$$

$$q(\lambda) = \lambda(S).$$

q is continuous (Lemma 3.4) and maps $\mathrm{St}(\mathfrak{C})_S$ onto $\mathfrak{C}$.

DEFINITION 7.3. Let $\mathfrak{D}(\iota_S) = q^{-1}(\mathfrak{B}_{(S,1)}) = \{\lambda; \lambda \in \mathrm{St}(\mathfrak{C})_S \text{ with } g(S, \lambda(S)) < 1\}$. We have the cross-section map

$$r \colon \mathfrak{D}(\iota_S) \to \mathrm{GL}^1(\mathfrak{M})$$

$$r(\lambda) = \gamma \cdot \pi_S + (1 - \pi_{\gamma(S)}) \cdot (1 - \pi_S), \qquad \text{where } \lambda = \gamma \cdot \iota_S.$$

LEMMA 7.2. *r is well defined, continuous, and satisfies $r(\lambda) \cdot \iota_S = \lambda$.*

Proof. The continuity follows from Lemma 3.4.

DEFINITION 7.4. We have the action of the group $\mathrm{GL}^1(\mathfrak{M})$ on $\mathrm{St}(\mathfrak{C})_S$

$$b \colon \mathrm{GL}^1(\mathfrak{M}) \times \mathrm{St}(\mathfrak{C})_S \to \mathrm{St}(\mathfrak{C})_S,$$

$$b(\gamma, \lambda) = \gamma \cdot \lambda.$$

b is continuous. It determines the maps

$$b_\gamma \colon \mathrm{St}(\mathfrak{C})_S \to \mathrm{St}(\mathfrak{C})_S,$$

$$b_\gamma(\lambda) = \gamma \cdot \lambda \quad \text{for a fixed } \gamma \in \mathrm{GL}^1(\mathfrak{M}), \quad \text{and}$$

$$b_\lambda \colon \mathrm{GL}^1(\mathfrak{M}) \to \mathrm{St}(\mathfrak{C})_S,$$

$$b_\lambda(\gamma) = \gamma \cdot \lambda \quad \text{for a fixed } \lambda \in \mathrm{St}(\mathfrak{C})_S.$$

b_γ is a homeomorphism of $\mathrm{St}(\mathfrak{C})_S$ onto itself for each $\gamma \in \mathrm{GL}^1(\mathfrak{M})$. b_λ is a continuous map onto $\mathrm{St}(\mathfrak{C})_S$.

LEMMA 7.3. *The map $b_\lambda \colon \mathrm{GL}^1(\mathfrak{M}) \to \mathrm{St}(\mathfrak{C})_S$ is open.*

Proof. First b_{ι_S} is open by the same arguments as in Lemma 6.4 using the

cross-section r of Definition 7.3. And $b_\lambda = b_{\iota_S} \cdot k_\beta$, where $\lambda = \beta \cdot \iota_S$ and $k: \mathrm{GL}^1(\mathfrak{M}) \to$ $\mathrm{GL}^1(\mathfrak{M})$, the right translation $k_\beta(\gamma) = \gamma \cdot \beta$.

THEOREM 7.1. *The Stiefel spaces* $\mathrm{St}(\mathfrak{C})_S$ *are Banach manifolds.*

Proof. There is a natural connection with Theorem 6.1. In the following we refer to the notation introduced in the proof of this theorem. We construct a coordinate system $(\mathfrak{B}, g)$ at the point ι_S. Consider the open map of the preceding Lemma 7.3

$$b_{\iota_S}: \mathrm{GL}^1(\mathfrak{M}) \to \mathrm{St}(\mathfrak{C})_S,$$

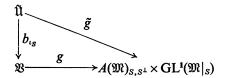$$b_{\iota_S}(\gamma) = \gamma \cdot \iota_S.$$

Let $\tilde{\mathfrak{u}} = \{\gamma; \gamma \in \mathrm{GL}^1(\mathfrak{M}) \text{ with } (\pi_S \cdot \gamma)|_S = \pi'_S \cdot \gamma \cdot \iota_S \in \mathrm{GL}^1(\mathfrak{M}|_S)\}$. $\tilde{\mathfrak{u}}$ is an open subset of $\mathrm{GL}^1(\mathfrak{M})$. Let $\mathfrak{B} = b_{\iota_S}(\tilde{\mathfrak{u}})$. Then $\mathfrak{B}$ is open in $\mathrm{St}(\mathfrak{C})_S$.

Consider next the continuous map

$$\tilde{g}: \tilde{\mathfrak{u}} \to A(\mathfrak{M})_{S,S^\perp} \times \mathrm{GL}^1(\mathfrak{M}|_S)$$

$$\tilde{g}(\gamma) = (\tilde{f}(\gamma), \pi'_S \cdot \gamma \cdot \iota_S).$$

We define g via the commutative diagram

by $g = \tilde{g} \cdot (b_{\iota_S})^{-1}$. The map g is well defined and therefore continuous. (Namely if $b_{\iota_S}(\gamma_1) = b_{\iota_S}(\gamma_2)$ for $\gamma_1, \gamma_2 \in \tilde{\mathfrak{u}}$, then $\gamma_1 \cdot \iota_S = \gamma_2 \cdot \iota_S$ and $\tilde{f}(\gamma_1) = \tilde{f}(\gamma_2)$, and therefore $\tilde{g}(\gamma_1) = \tilde{g}(\gamma_2)$.)

We consider the continuous map

$$\tilde{k}: A(\mathfrak{M})_{S,S^\perp} \times \mathrm{GL}^1(\mathfrak{M}|_S) \to \mathrm{GL}^1(\mathfrak{M}),$$

$$\tilde{k}(\alpha, \gamma') = (1+\alpha) \cdot (\iota_S \cdot \gamma' \cdot \pi'_S + \pi_{S^\perp}).$$

$\tilde{k}$ is well defined and continuous. Further $\pi'_S \cdot \tilde{k}(\alpha, \gamma') \cdot \iota_S = \gamma'$ implies $\tilde{k}(\alpha, \gamma') \in \tilde{\mathfrak{u}}$. We compute

$$(\tilde{g} \cdot \tilde{k})(\alpha, \gamma') = (\alpha, \gamma') \quad \text{for} \quad (\alpha, \gamma') \in A(\mathfrak{M})_{S,S^\perp} \times \mathrm{GL}^1(\mathfrak{M}|_S), \quad \text{and}$$

$$(\tilde{k} \cdot \tilde{g})(\gamma) = \gamma \cdot \pi_S + \pi_{S^\perp} \quad \text{for} \quad \gamma \in \tilde{\mathfrak{u}}.$$

Finally we define

$$k: A(\mathfrak{M})_{S,S^\perp} \times \mathrm{GL}^1(\mathfrak{M}|_S) \to \mathfrak{B},$$

$$k = b_{\iota_S} \cdot \tilde{k}.$$

Then we have $(g \cdot k)(\alpha, \gamma') = (\alpha, \gamma')$ for $(\alpha, \gamma') \in A(\mathfrak{M})_{S,S^\perp} \times \mathrm{GL}^1(\mathfrak{M}|_S)$, and $(k \cdot g)(\lambda)$ $= (\gamma \cdot \pi_S + \pi_{S^\perp}) \cdot \iota_S = \lambda$ for $\lambda \in \mathfrak{B}$, where $\gamma \in \tilde{\mathfrak{U}}$ with $\gamma \cdot \iota_S = \lambda$. This proves $g \colon \mathfrak{B} \to$ $A(\mathfrak{M})_{S,S^\perp} \times \mathrm{GL}^1(\mathfrak{M}|_S)$ is a homeomorphism onto the open subset $A(\mathfrak{M})_{S,S^\perp}$ $\times \mathrm{GL}^1(\mathfrak{M}|_S)$ of the Banach space $A(\mathfrak{M})_{S,S^\perp} \times A(\mathfrak{M}|_S)$.

DEFINITION 7.5. Let $\mathfrak{M}$ be a maximal R-set, let $\mathfrak{C} \in \pi_0(\mathfrak{M})$ be a path component of $\mathfrak{M}$, and let $S \in \mathfrak{C}$ be fixed. We consider the closed subgroup

$$\mathrm{GL}^1(\mathfrak{M})_{\langle S \rangle} = \{ \gamma; \gamma \in \mathrm{GL}^1(\mathfrak{M}) \text{ and } \gamma|_S = id \},$$

and we form the quotient space

$$\mathrm{GL}^1(\mathfrak{M})/\mathrm{GL}^1(\mathfrak{M})_{\langle S \rangle} = \{ [\gamma]; \gamma \in \mathrm{GL}^1(\mathfrak{M}) \}$$

of left cosets $[\gamma] = \gamma \cdot \mathrm{GL}^1(\mathfrak{M})_{\langle S \rangle}$. This quotient space is given the topology induced by the projection map

$$p \colon \mathrm{GL}^1(\mathfrak{M}) \to \mathrm{GL}^1(\mathfrak{M})/\mathrm{GL}^1(\mathfrak{M})_{\langle S \rangle}.$$

We can lift the map $b_{\iota_S} \colon \mathrm{GL}^1(\mathfrak{M}) \to \mathrm{St}(\mathfrak{C})_S$ onto the quotient space.

$$\hat{b}_{\iota_S} \colon \mathrm{GL}^1(\mathfrak{M})/\mathrm{GL}^1(\mathfrak{M})_{\langle S \rangle} \to \mathrm{St}(\mathfrak{C})_S,$$

$$\hat{b}_{\iota_S}([\gamma]) = b_{\iota_S}(\gamma) = \gamma \cdot \iota_S.$$

THEOREM 7.2. $\hat{b}_{\iota_S} \colon \mathrm{GL}^1(\mathfrak{M})/\mathrm{GL}^1(\mathfrak{M})_{\langle S \rangle} \to \mathrm{St}(\mathfrak{C})_S$ is a homeomorphism. The spaces $\mathrm{St}(\mathfrak{C})_S$ and $\mathrm{GL}^1(\mathfrak{M})/\mathrm{GL}^1(\mathfrak{M})_{\langle S \rangle}$ can be identified.

Proof. The same as that of Theorem 6.2.

DEFINITION 7.6. We have a canonical local cross-section of $\mathrm{GL}^1(\mathfrak{M})_{\langle S \rangle}$ in $\mathrm{GL}^1(\mathfrak{M})$. $\mathfrak{O}' = (\hat{b}_{\iota_S})^{-1}(\mathfrak{O}(\iota_S))$ is an open neighborhood of the point $[1] \in \mathrm{GL}^1(\mathfrak{M})/\mathrm{GL}^1(\mathfrak{M})_{\langle S \rangle}$. We define

$$r' \colon \mathfrak{O}' \to \mathrm{GL}^1(\mathfrak{M}),$$

$$r' = r \cdot \hat{b}_{\iota_S}, \quad \text{i.e.,} \quad r'([\gamma]) = r(\gamma \cdot \iota_S).$$

We have $p \cdot r'([\gamma]) = [r(\gamma \cdot \iota_S)]$. But $r(\gamma \cdot \iota_S) \cdot \iota_S = \gamma \cdot \iota_S$ and therefore $[r(\gamma \cdot \iota_S)] = [\gamma]$. Hence $p \cdot r'([\gamma]) = [\gamma]$ for $[\gamma] \in \mathfrak{O}'$.

The Steenrod construction [14, p. 30] can be applied, and we obtain:

THEOREM 7.3. *If K is a closed subgroup of $\mathrm{GL}^1(\mathfrak{M})_{\langle S \rangle}$,*

$$p \colon \mathrm{GL}^1(\mathfrak{M})/K \to \mathrm{St}(\mathfrak{C})_S = \mathrm{GL}^1(\mathfrak{M})/\mathrm{GL}^1(\mathfrak{M})_{\langle S \rangle}$$

the projection map induced by the inclusion of cosets, then a bundle structure can be assigned to $\mathrm{GL}^1(\mathfrak{M})_{\langle S \rangle}/K$ and the group of the bundle is $\mathrm{GL}^1(\mathfrak{M})_{\langle S \rangle}/K_0$ acting on $\mathrm{GL}^1(\mathfrak{M})_{\langle S \rangle}/K$ as left translations, where K_0 is the largest subgroup of K invariant in $\mathrm{GL}^1(\mathfrak{M})_{\langle S \rangle}$.

COROLLARY 7.1. $GL^1(\mathfrak{M})$ *is a principal fibre bundle over*

$$St(\mathfrak{C})_S = GL^1(\mathfrak{M})/GL^1(\mathfrak{M})_{\{S\}}$$

with fibre and structure group $GL^1(\mathfrak{M})_{\{S\}}$ *which acts on the fibres by left translations.*

COROLLARY 7.2. *The projection*

$$q\colon St(\mathfrak{C})_S = GL^1(\mathfrak{M})/GL^1(\mathfrak{M})_{\{S\}} \to \mathfrak{C} = GL^1(\mathfrak{M})/GL^1(\mathfrak{M})_S$$

(*compare Definition 7.2*) *defines a principal bundle over* $\mathfrak{C}$ *with fibre and structure group* $GL^1(\mathfrak{M}|_S) = GL^1(\mathfrak{M})_S/GL^1(\mathfrak{M})_{\{S\}}$, *which acts on the fibres by left translations.*

The same constructions and arguments apply also to the orthogonal case.

DEFINITION 7.1′. Let $\mathfrak{M}$ be a maximal R-set, let $\mathfrak{C} \in \pi_0(\mathfrak{M})$ be a path component of $\mathfrak{M}$, and let $S \in \mathfrak{C}$ be fixed. The orthogonal Stiefel space $U(\mathfrak{C})_S$ associated with the path component $\mathfrak{C}$ (relative to S) is defined as

$$U(\mathfrak{C})_S = \{\nu;\ \nu \in \mathfrak{L}(S, H) \text{ such that there is a } \mu \in U^1(\mathfrak{M}) \text{ with } \nu = \mu \cdot \iota_S\},$$

where $\iota_S\colon S \to H$ is the inclusion. $U(\mathfrak{C})_S$ is a closed subset of $\mathfrak{L}(S, H)$ and is given the relative topology.

LEMMA 7.1′. *If* $S_1, S_2 \in \mathfrak{C}$, *then the spaces* $U(\mathfrak{C})_{S_1}$ *and* $U(\mathfrak{C})_{S_2}$ *are homeomorphic. The reference element* $S \in \mathfrak{C}$ *is therefore irrelevant. The orthogonal Stiefel spaces* $U(\mathfrak{C})_S$ *are path connected.*

Proof. The same as of Lemma 7.1.

DEFINITION 7.2′. We have the natural projection map

$$q\colon U(\mathfrak{C})_S \to \mathfrak{C},$$
$$q(\nu) = \nu(S).$$

q is continuous and maps $U(\mathfrak{C})_S$ onto $\mathfrak{C}$.

DEFINITION 7.3′. Let $\mathfrak{B}(\iota_S) = q^{-1}(\mathfrak{B}_{(S,1)}) = \{\nu;\ \nu \in U(\mathfrak{C})_S \text{ with } g(S, \nu(S)) < 1\}$. We have the cross-section map

$$w\colon \mathfrak{B}(\iota_S) \to U^1(\mathfrak{M}),$$

$$w = u \cdot r, \quad \text{i.e.,} \quad w(\nu) = u(\mu \cdot \pi_S + (1 - \pi_{\mu(S)}) \cdot (1 - \pi_S)) \qquad \text{where } \nu = \mu \cdot \iota_S,$$

where $u\colon GL(\mathfrak{M}) \to U(\mathfrak{M})$ is from Theorem 2.2, and r is the cross-section map of Definition 7.3.

LEMMA 7.2′. w *is well defined, continuous, and satisfies* $w(\nu) \cdot \iota_S = \nu$.

Proof. The last property follows from Corollary 2.2.

Definition 7.4 and Lemma 7.3 carry over word by word to the orthogonal case.

THEOREM 7.1'. *The orthogonal Stiefel spaces* $U(\mathbb{C})_S$ *are Banach manifolds.*

Proof. The same construction as in Theorem 7.1. The group $GL^1(\mathfrak{M})$ is replaced by the group $U^1(\mathfrak{M})$, which is a Banach manifold. (The well-known exponential map $\exp: H(A(\mathfrak{M})) \to U^1(\mathfrak{M})$, where $H(A(\mathfrak{M}))$ is the closed real linear subspace of the hermitian elements of $A(\mathfrak{M})$, determines a homeomorphism from an open neighborhood of o in $H(A(\mathfrak{M}))$ to an open neighborhood of $\mathbf{1}$ in $U^1(\mathfrak{M})$. The inverse to the exponential map is provided by a log-map.)

Definitions 7.5 and 7.6 carry over again without any change to the orthogonal case. We conclude:

THEOREM 7.2'. $\hat{b}_{t_S}: U^1(\mathfrak{M})/U^1(\mathfrak{M})_{\{S\}} \to U(\mathbb{C})_S$ *is a homeomorphism. The spaces* $U(\mathbb{C})_S$ *and* $U^1(\mathfrak{M})/U^1(\mathfrak{M})_{\{S\}}$ *can be identified.*

THEOREM 7.3'. *If K is a closed subgroup of* $U^1(\mathfrak{M})_{\{S\}}$,

$$p: U^1(\mathfrak{M})/K \to U(\mathbb{C})_S = U^1(\mathfrak{M})/U^1(\mathfrak{M})_{\{S\}}$$

the projection map induced by the inclusion of cosets, then a bundle structure can be assigned to $U^1(\mathfrak{M})/K$ *relative to p such that the fibre of the bundle is* $U^1(\mathfrak{M})_{\{S\}}/K$ *and the group of the bundle is* $U^1(\mathfrak{M})_{\{S\}}/K_0$ *acting on* $U^1(\mathfrak{M})_{\{S\}}/K$ *as left translations, where K_0 is the largest subgroup of K invariant in* $U^1(\mathfrak{M})_{\{S\}}$.

COROLLARY 7.1'. $U^1(\mathfrak{M})$ *is a principal fibre bundle over* $U(\mathbb{C})_S = U^1(\mathfrak{M})/U^1(\mathfrak{M})_{\{S\}}$ *with fibre and structure group* $U^1(\mathfrak{M})_{\{S\}} \subset U(S^\perp)$, *which acts on the fibres by left translations.*

COROLLARY 7.2'. *The projection map*

$$q: U(\mathbb{C})_S = U^1(\mathfrak{M})/U^1(\mathfrak{M})_{\{S\}} \to \mathbb{C} = U^1(\mathfrak{M})/U^1(\mathfrak{M})_S$$

defines a principal bundle over $\mathbb{C}$ with fibre and structure group

$$U^1(\mathfrak{M}|_S) = U^1(\mathfrak{M})_S/U^1(\mathfrak{M})_{\{S\}},$$

which acts on the fibres by left translations.

THEOREM 7.4. *The orthogonal Stiefel space* $U(\mathbb{C})_S \subset St(\mathbb{C})_S$ *is a strong deformation retract of the Stiefel space* $St(\mathbb{C})_S$. *In particular* $U(\mathbb{C})_S$ *and* $St(\mathbb{C})_S$ *are of the same homotopy type.*

Proof. We introduce first the map

$$e: GL(\mathfrak{M}) \to GL(\mathfrak{M}),$$

$$e(\gamma) = \gamma \cdot \pi_S + \pi_{(\gamma(S))^\perp} \cdot \gamma \cdot \pi_{S^\perp} = \gamma - \pi_{\gamma(S)} \cdot \gamma \cdot \pi_{S^\perp}.$$

e is well defined. Namely we compute

$$(\gamma - \pi_{\gamma(S)} \cdot \gamma \cdot \pi_{S^\perp}) \cdot (\gamma^{-1} + \pi_S \cdot \gamma^{-1} \cdot \pi_{(\gamma(S))^\perp}) = 1, \quad \text{and}$$

$$(\gamma^{-1} + \pi_S \cdot \gamma^{-1} \cdot \pi_{(\gamma(S))^\perp}) \cdot (\gamma - \pi_{\gamma(S)} \cdot \gamma \cdot \pi_{S^\perp}) = 1.$$

This proves $e(\gamma) \in \mathrm{GL}(\mathfrak{M})$. The continuity of e follows from Lemma 2.3.
  Finally we consider the map

$$j: \mathrm{GL}^1(\mathfrak{M}) \times [0, 1] \to \mathrm{GL}^1(\mathfrak{M}),$$

$$j(\gamma, t) = j_t(\gamma) = u_t(e(\gamma)), \qquad 0 \leqq t \leqq 1$$

where u_t is the strong deformation retract map of Corollary 2.3. We turn to the commutative diagram

$$
\begin{array}{ccc}
\mathrm{GL}^1(\mathfrak{M}) \times [0, 1] & \xrightarrow{\ j\ } & \mathrm{GL}^1(\mathfrak{M}) \\
\downarrow{\scriptstyle b_{\iota_S} \times id} & & \downarrow{\scriptstyle b_{\iota_S}} \\
\mathrm{St}(\mathfrak{C})_S \times [0, 1] & \xrightarrow[\ \hat{j}\ ]{} & \mathrm{St}(\mathfrak{C})_S.
\end{array}
$$

Let $\hat{j} = b_{\iota_S} \cdot j \cdot (b_{\iota_S} \times id)^{-1}$. The map $\hat{j}$ is well defined by the particular property of u_t established in Corollary 2.3. It is continuous, since b_{ι_S} is open.
  We have $\hat{j}_0 = id$, $\hat{j}_1$ maps $\mathrm{St}(\mathfrak{C})_S$ onto $\mathrm{U}(\mathfrak{C})_S$, and $\hat{j}_t|_{\mathrm{U}(\mathfrak{C})_S} = id$ for $0 \leqq t \leqq 1$. Hence $\hat{j}_t$, $0 \leqq t \leqq 1$, is a strong deformation retract homotopy for $\mathrm{U}(\mathfrak{C})_S \subset \mathrm{St}(\mathfrak{C})_S$.

  8. Grassmann spaces and Stiefel spaces of a Hilbert space. We consider now the maximal R-set $\mathfrak{M} = \mathfrak{X}(H)$. In this case $A(\mathfrak{M}) = \mathfrak{L}(H)$ and $I(\mathfrak{M}) = \mathfrak{L}(H)$, and the theory simplifies essentially. It does not depend on Chapters 4 and 5 at all. Since $\mathrm{GL}(\mathfrak{M}) = \mathrm{GL}(H)$ and $\mathrm{U}(\mathfrak{M}) = \mathrm{U}(H)$, one has not to take particular care of these groups any more.
  This case establishes also a direct connection with the "ordinary theory" of Grassmann and Stiefel spaces of a Hilbert space. In the following we present this ordinary theory independent of the dimension of the Hilbert space. We investigate then the Grassmann and Stiefel spaces of infinite-dimensional Hilbert spaces, and we will obtain a certain complete characterization.
  Crucial is the following theorem:

  THEOREM 8.1 (SEE [9]). *Let H be an infinite-dimensional Hilbert space. The groups $\mathrm{GL}(H)$ and $\mathrm{U}(H)$ are contractible. In particular $\mathrm{GL}(H)$ and $\mathrm{U}(H)$ are connected, and all homotopy groups are trivial.*

$$\pi_i(\mathrm{GL}(H)) = 0 \quad and \quad \pi_i(\mathrm{U}(H)) = 0 \quad for\ i = 0, 1, 2, \ldots.$$

  DEFINITION 8.1. Let $\mathfrak{k}$ and $\mathfrak{l}$ be two cardinal numbers with $\mathfrak{k} + \mathfrak{l} = \dim(H)$. The

Grassmann space $\mathfrak{G}_{\langle \mathfrak{k}, \mathfrak{l} \rangle}$ of type $\mathfrak{k}$, $\mathfrak{l}$ of the Hilbert space H is the set

$$\mathfrak{G}_{\langle \mathfrak{k}, \mathfrak{l} \rangle} = \{S; S \subset H \text{ a closed linear subspace with dim } (S) = \mathfrak{k} \text{ and codim } (S) = \mathfrak{l}\}$$

with the relative topology induced by the inclusion $\mathfrak{G}_{\langle \mathfrak{k}, \mathfrak{l} \rangle} \subset \mathfrak{T}(H)$.

If H is finite dimensional, we obtain the ordinary Grassmann manifolds. In this case we can of course dispense with the double index.

If H is a separable infinite dimensional Hilbert space, we denote dim $(H) = \aleph_0 = \infty$, and we have the following types of Grassmann spaces

$$\mathfrak{G}_{\langle n, \infty \rangle}, \quad \mathfrak{G}_{\langle \infty, \infty \rangle}, \quad \mathfrak{G}_{\langle \infty, n \rangle} \quad \text{where } n = 0, 1, 2, \ldots.$$

THEOREM 8.2. *The Grassmann spaces $\mathfrak{G}_{\langle \mathfrak{k}, \mathfrak{l} \rangle}$ are the path components $\mathfrak{C}$ of the space $\mathfrak{T}(H)$. The Grassmann spaces $\mathfrak{G}_{\langle \mathfrak{k}, \mathfrak{l} \rangle}$ are therefore connected Banach manifolds (Theorem 6.1), and homogeneous spaces*

$$\mathfrak{G}_{\langle \mathfrak{k}, \mathfrak{l} \rangle} = \mathrm{GL}(H)/\mathrm{GL}(H)_S = \mathrm{U}(H)/\mathrm{U}(H)_S,$$

where $S \in \mathfrak{G}_{\langle \mathfrak{k}, \mathfrak{l} \rangle}$.

Proof. Let $S \in \mathfrak{G}_{\langle \mathfrak{k}, \mathfrak{l} \rangle}$ and let $\mathfrak{C}$ be the path component of $\mathfrak{T}(H)$ with $S \in \mathfrak{C}$. First we have $\mathfrak{C} \subset \mathfrak{G}_{\langle \mathfrak{k}, \mathfrak{l} \rangle}$. Namely if $T \in \mathfrak{C}$, then there exists a $\gamma \in \mathrm{GL}(H)$ with $\gamma(S) = T$ and therefore dim $(T) = \mathfrak{k}$ and codim $(T) = \mathfrak{l}$. Hence $T \in \mathfrak{G}_{\langle \mathfrak{k}, \mathfrak{l} \rangle}$. Now let $T \in \mathfrak{G}_{\langle \mathfrak{k}, \mathfrak{l} \rangle}$ be given. We show that there is a continuous path in $\mathfrak{T}(H)$ which connects S with T. Since dim $(S) = \dim (T) = \mathfrak{k}$ and codim $(S) = \mathrm{codim} (T) = \mathfrak{l}$, we can construct a $\gamma \in \mathrm{GL}(H)$ with $\gamma(S) = T$. If H is a finite-dimensional real Hilbert space, we may assume $\gamma \in \mathrm{GL}^{\mathfrak{l}}(H)$. In all other cases we have $\mathrm{GL}^{\mathfrak{l}}(H) = \mathrm{GL}(H)$. Therefore there exists a continuous path $\gamma_t \in \mathrm{GL}(H)$, $0 \leq t \leq 1$, with $\gamma_0 = 1$ and $\gamma_1 = \gamma$. Then $\gamma_t(S)$, $0 \leq t \leq 1$, is a continuous path in $\mathfrak{T}(H)$ which connects S with T (Lemma 3.4). This shows $T \in \mathfrak{C}$. Hence $\mathfrak{C} = \mathfrak{G}_{\langle \mathfrak{k}, \mathfrak{l} \rangle}$.

COROLLARY 8.1. *Let $S \in \mathfrak{G}_{\langle \mathfrak{k}, \mathfrak{l} \rangle}$ be fixed. $\mathrm{GL}(H)$ is a principal bundle over $\mathfrak{G}_{\langle \mathfrak{k}, \mathfrak{l} \rangle}$ with fibre and structure group $\mathrm{GL}(H)_S$, and also $\mathrm{U}(H)$ is a principal bundle over $\mathfrak{G}_{\langle \mathfrak{k}, \mathfrak{l} \rangle}$ with fibre and structure group $\mathrm{U}(H)_S = \mathrm{U}(S) \times \mathrm{U}(S^\perp)$ (Corollaries 5.1 and 5.1').*

THEOREM 8.3. *The map $\perp: \mathfrak{G}_{\langle \mathfrak{k}, \mathfrak{l} \rangle} \to \mathfrak{G}_{\langle \mathfrak{l}, \mathfrak{k} \rangle}$, $\perp(S) = S^\perp$, is an isometry with respect to the d-metric and to the g-metric.*

Proof. Lemma 3.1.

DEFINITION 8.2. Let $\mathfrak{k}$ and $\mathfrak{l}$ be two cardinal numbers with $k + l = \dim (H)$. An orthonormal $\langle \mathfrak{k}, \mathfrak{l} \rangle$-frame ϕ in H is an ordered set $\{e_i\}$ of power k of pairwise orthogonal vectors of unit length, and if $[\phi] = \overline{[\{e_i\}]}$ is the closed linear subspace determined by the set ϕ, then dim $([\phi]) = \mathfrak{k}$ and codim $([\phi]) = \mathfrak{l}$.

A $\langle \mathfrak{k}, \mathfrak{l} \rangle$-frame ψ in H is an ordered set $\{v_i\}$ of power $\mathfrak{k}$ of linearly independent vectors, such that if $[\psi] = \overline{[\{v_i\}]}$ is the closed linear subspace determined by the set

ψ, we have dim $([\psi]) = \mathfrak{k}$ and codim $([\psi]) = \mathfrak{l}$, and further there exists an orthonormal $\langle \mathfrak{k}, \mathfrak{l} \rangle$-frame $\phi_0 = \{e_i\}$ such that the linear map

$$\alpha_{(\phi_0, \psi)} : [\phi_0] \to [\psi] \quad \text{determined by}$$

$$\alpha_{(\phi_0, \psi)}(e_i) = v_i$$

is an isomorphism onto $[\psi]$. This last condition is of course not necessary if $\mathfrak{k}$ is finite.

The Stiefel space $\mathfrak{S}_{\langle \mathfrak{k}, \mathfrak{l} \rangle}$ of type $\mathfrak{k}, \mathfrak{l}$ of the Hilbert space H is the set

$$\mathfrak{S}_{\langle \mathfrak{k}, \mathfrak{l} \rangle} = \{\psi; \psi \text{ a } \langle \mathfrak{k}, \mathfrak{l} \rangle\text{-frame in } H\}.$$

And the orthogonal Stiefel space $\mathfrak{U}_{\langle \mathfrak{k}, \mathfrak{l} \rangle}$ of type $\mathfrak{k}, \mathfrak{l}$ of the Hilbert space H is the set

$$\mathfrak{U}_{\langle \mathfrak{k}, \mathfrak{l} \rangle} = \{\phi; \phi \text{ an orthogonal } \langle \mathfrak{k}, \mathfrak{l} \rangle\text{-frame in } H\}.$$

A natural topology on the sets $\mathfrak{S}_{\langle \mathfrak{k}, \mathfrak{l} \rangle}$ and $\mathfrak{U}_{\langle \mathfrak{k}, \mathfrak{l} \rangle}$ will be subsequently introduced. Obviously $\mathfrak{U}_{\langle \mathfrak{k}, \mathfrak{l} \rangle} \subseteq \mathfrak{S}_{\langle \mathfrak{k}, \mathfrak{l} \rangle}$.

We obtain the ordinary Stiefel manifolds if H is a finite-dimensional vectorspace. In this case we need of course again one index only.

If H is a separable infinite-dimensional Hilbert space, we denote again dim $(H) = \aleph_0 = \infty$, and we have the following types of Stiefel spaces

$$\mathfrak{S}_{\langle n, \infty \rangle}, \quad \mathfrak{S}_{\langle \infty, \infty \rangle}, \quad \mathfrak{S}_{\langle \infty, n \rangle},$$

and the orthogonal Stiefel spaces

$$\mathfrak{U}_{\langle n, \infty \rangle}, \quad \mathfrak{U}_{\langle \infty, \infty \rangle}, \quad \mathfrak{U}_{\langle \infty, n \rangle} \quad \text{where } n = 0, 1, \ldots.$$

LEMMA 8.1. *Let* $\phi_0 = \{e_i\}$ *be a fixed orthonormal* $\langle \mathfrak{k}, \mathfrak{l} \rangle$-*frame of* H, *and let* $S = [\phi_0]$. *If* H *is a finite-dimensional real vectorspace, we assume* $\mathfrak{k} < $ dim (H). *We consider the following natural map (compare Chapter 7)*

$$i_{\phi_0} : \mathfrak{S}_{\langle \mathfrak{k}, \mathfrak{l} \rangle} \to \text{St}(\mathfrak{G}_{\langle \mathfrak{k}, \mathfrak{l} \rangle})_S,$$

$i_{\phi_0}(\psi) : S \to H$ *is the linear map uniquely determined*

by $i_{\phi_0}(\psi)(e_i) = v_i$ *where* $\psi = \{v_i\} \in \mathfrak{S}_{\langle \mathfrak{k}, \mathfrak{l} \rangle}$.

This map i_{ϕ_0} *defines also*

$$i_{\phi_0} : \mathfrak{U}_{\langle \mathfrak{k}, \mathfrak{l} \rangle} \to \text{U}(\mathfrak{G}_{\langle \mathfrak{k}, \mathfrak{l} \rangle})_S.$$

The map i_{ϕ_0} *is bijective in both cases.*

DEFINITION 8.3. Let $S \in \mathfrak{G}_{\langle \mathfrak{k}, \mathfrak{l} \rangle}$. Then we identify $\mathfrak{S}_{\langle \mathfrak{k}, \mathfrak{l} \rangle}$ with $\text{St}(\mathfrak{G}_{\langle \mathfrak{k}, \mathfrak{l} \rangle})_S$, and $\mathfrak{U}_{\langle \mathfrak{k}, \mathfrak{l} \rangle}$ with $\text{U}(\mathfrak{G}_{\langle \mathfrak{k}, \mathfrak{l} \rangle})_S$. If H is a finite-dimensional real vectorspace and $\mathfrak{k} = $ dim H, then we identify $\mathfrak{S}_{\langle \dim (H), 0 \rangle}$ with $\text{GL}(H)$ and $\mathfrak{U}_{\langle \dim (H), 0 \rangle}$ with $\text{U}(H)$.

THEOREM 8.4. *The Stiefel spaces* $\mathfrak{S}_{\langle \mathfrak{k}, \mathfrak{l} \rangle}$ *and* $\mathfrak{U}_{\langle \mathfrak{k}, \mathfrak{l} \rangle}$ *are Banach manifolds. They are connected except for* H *a finite-dimensional real vectorspace and* $\mathfrak{k} = $ dim H.

(*In this case* $\mathfrak{S}_{\langle \dim(H), 0 \rangle} = \mathrm{GL}(H)$ *and* $\mathfrak{U}_{\langle \dim(H), 0 \rangle} = \mathrm{U}(H)$ *have exactly two path components.*) *The Stiefel spaces are also homogeneous spaces. We have*

$$\mathfrak{S}_{\langle \mathfrak{t}, \mathfrak{l} \rangle} = \mathrm{GL}(H)/\mathrm{GL}(H)_{\langle S \rangle} \quad \text{and} \quad \mathfrak{U}_{\langle \mathfrak{t}, \mathfrak{l} \rangle} = \mathrm{U}(H)/\mathrm{U}(H)_{\langle S \rangle},$$

where $S \in \mathfrak{G}_{\langle \mathfrak{t}, \mathfrak{l} \rangle}$. *Finally* $\mathfrak{U}_{\langle \mathfrak{t}, \mathfrak{l} \rangle} \subset \mathfrak{S}_{\langle \mathfrak{t}, \mathfrak{l} \rangle}$ *is a strong deformation retract of the space* $\mathfrak{S}_{\langle \mathfrak{t}, \mathfrak{l} \rangle}$. *In particular* $\mathfrak{U}_{\langle \mathfrak{t}, \mathfrak{l} \rangle}$ *and* $\mathfrak{S}_{\langle \mathfrak{t}, \mathfrak{l} \rangle}$ *are of the same homotopy type.*

COROLLARY 8.2. *Let* $S \in \mathfrak{G}_{\langle \mathfrak{t}, \mathfrak{l} \rangle}$. *Then* $\mathrm{GL}(H)$ *is a principal fibre bundle over* $\mathfrak{S}_{\langle \mathfrak{t}, \mathfrak{l} \rangle} = \mathrm{GL}(H)/\mathrm{GL}(H)_{\langle S \rangle}$ *with fibre and structure group* $\mathrm{GL}(H)_{\langle S \rangle}$, *which acts on the fibres by left translations. And* $\mathrm{U}(H)$ *is a principal fibre bundle over* $\mathfrak{U}_{\langle \mathfrak{t}, \mathfrak{l} \rangle} = \mathrm{U}(H)/\mathrm{U}(H)_{\langle S \rangle}$ *with fibre and structure group* $\mathrm{U}(H)_{\langle S \rangle} = \mathrm{U}(S^{\perp})$, *which acts on the fibres by left translations.*

THEOREM 8.5. *Let* H *be an infinite-dimensional Hilbert space. The projection map* $p\colon \mathrm{U}(H) \to \mathfrak{U}_{\langle \mathfrak{t}, \mathfrak{l} \rangle} = \mathrm{U}(H)/\mathrm{U}(H)_{\langle S \rangle}$, *where* $S \in \mathfrak{G}_{\langle \mathfrak{t}, \mathfrak{l} \rangle}$, *determines a universal classifying principal bundle for the group* $\mathrm{U}(S^{\perp})$. (*For the definition of universal classifying principal bundles see, for example,* [14, *p.* 100].) $\mathfrak{U}_{\langle \mathfrak{t}, \mathfrak{l} \rangle}$ *is therefore a universal classifying space* $B_{\mathrm{U}(H')}$ *for the group* $\mathrm{U}(H')$, *where* H' *is a Hilbert space with* $\dim(H') = \mathfrak{l}$. *In particular* $\mathfrak{U}_{\langle \mathfrak{t}, n \rangle}$ *is a universal classifying space* $B_{\mathrm{U}(n)}$ *for the orthogonal group* $\mathrm{U}(n)$ *of the* n-*dimensional vectorspace,* $n = 0, 1, \ldots$.

Proof. By Theorem 8.1, $\pi_i(\mathrm{U}(H)) = 0$, $i = 0, 1, \ldots$, and the theorem on p. 102 in [14].

COROLLARY 8.3. *Let* H *be an infinite-dimensional Hilbert space. The homotopy groups of the Stiefel spaces* $\mathfrak{U}_{\langle \mathfrak{t}, \mathfrak{l} \rangle}$ *and* $\mathfrak{S}_{\langle \mathfrak{t}, \mathfrak{l} \rangle}$ *are*

$$\pi_i(\mathfrak{U}_{\langle \mathfrak{t}, \mathfrak{l} \rangle}) = \pi_i(\mathfrak{S}_{\langle \mathfrak{t}, \mathfrak{l} \rangle}) = 0, \quad i = 0, 1, 2, \ldots, \quad \text{if } \mathfrak{l} \text{ is not an integer,}$$

$$\pi_0(\mathfrak{U}_{\langle \mathfrak{t}, n \rangle}) = \pi_0(\mathfrak{S}_{\langle \mathfrak{t}, n \rangle}) = 0, \quad \text{for } n = 0, 1, 2, \ldots,$$

$$\pi_i(\mathfrak{U}_{\langle \mathfrak{t}, n \rangle}) = \pi_i(\mathfrak{S}_{\langle \mathfrak{t}, n \rangle}) = \pi_{i-1}(\mathrm{U}(n)), \quad i = 1, 2, \ldots, \quad \text{for } n = 0, 1, \ldots.$$

And the singular homology and cohomology with coefficients in the ring of integers Z *is*

$$H(\mathfrak{U}_{\langle \mathfrak{t}, \mathfrak{l} \rangle}; Z) = H(\mathfrak{S}_{\langle \mathfrak{t}, \mathfrak{l} \rangle}; Z) = 0, \quad \text{if } \mathfrak{l} \text{ is not an integer,}$$

$$H(\mathfrak{U}_{\langle \mathfrak{t}, n \rangle}; Z) = H(\mathfrak{S}_{\langle \mathfrak{t}, n \rangle}; Z) = H(B_{\mathrm{U}(n)}; Z) \quad \text{for } n = 0, 1, 2, \ldots.$$

For a description of the cohomology ring $H^*(B_{\mathrm{U}(n)}; Z)$ *see, for example,* [2].

Proof. By Theorem 8.4, $\mathfrak{U}_{\langle \mathfrak{t}, \mathfrak{l} \rangle}$ and $\mathfrak{S}_{\langle \mathfrak{t}, \mathfrak{l} \rangle}$ are of the same homotopy type. The exact sequence for the homotopy groups of the principal bundle $p\colon \mathrm{U}(H) \to \mathrm{U}(H)/\mathrm{U}(H)_{\langle S \rangle}$ gives immediately the homotopy groups of the space $\mathfrak{U}_{\langle \mathfrak{t}, \mathfrak{l} \rangle}$. The vanishing of the singular homology groups and therefore of the singular cohomol-

ogy groups in the first case follows from the Hurewicz isomorphism theorem which relates homotopy groups and singular homology groups (see, for example, [14, p. 79]).

COROLLARY 8.4. *In particular let H be a separable infinite-dimensional Hilbert space. Then the homotopy groups of the Stiefel spaces are*

$$\pi_i(\mathfrak{S}_{\langle n,\infty \rangle}) = \pi_i(\mathfrak{U}_{\langle n,\infty \rangle}) = 0, \quad i = 0, 1, 2, \ldots, \quad \text{for } n = 0, 1, 2, \ldots,$$

$$\pi_i(\mathfrak{S}_{\langle \infty,\infty \rangle}) = \pi_i(\mathfrak{U}_{\langle \infty,\infty \rangle}) = 0, \quad i = 0, 1, 2, \ldots,$$

$$\pi_0(\mathfrak{S}_{\langle \infty,n \rangle}) = \pi_0(\mathfrak{U}_{\langle \infty,n \rangle}) = 0, \quad \text{for } n = 0, 1, 2, \ldots,$$

$$\pi_i(\mathfrak{S}_{\langle \infty,n \rangle}) = \pi_i(\mathfrak{U}_{\langle \infty,n \rangle}) = \pi_{i-1}(U(n)), \quad i = 1, 2, \ldots, \quad \text{for } n = 0, 1, 2, \ldots.$$

THEOREM 8.6. *Let $S \in \mathfrak{G}_{\langle \mathfrak{t},\mathfrak{l} \rangle}$. The projection map*

$$q: \mathfrak{S}_{\langle \mathfrak{t},\mathfrak{l} \rangle} = \mathrm{GL}(H)/\mathrm{GL}(H)_{\langle S \rangle} \to \mathfrak{G}_{\langle \mathfrak{t},\mathfrak{l} \rangle} = \mathrm{GL}(H)/\mathrm{GL}(H)_S$$

defines a principal bundle over $\mathfrak{G}_{\langle \mathfrak{t},\mathfrak{l} \rangle}$ with fibre and structure group $\mathrm{GL}(S)$, which acts on the fibres by left translations. And the projection map

$$q: \mathfrak{U}_{\langle \mathfrak{t},\mathfrak{l} \rangle} = \mathrm{U}(H)/\mathrm{U}(H)_{\langle S \rangle} \to \mathfrak{G}_{\langle \mathfrak{t},\mathfrak{l} \rangle} = \mathrm{U}(H)/\mathrm{U}(H)_S$$

defines a principal bundle over $\mathfrak{G}_{\langle \mathfrak{t},\mathfrak{l} \rangle}$ with fibre and structure group $\mathrm{U}(S)$, which acts on the fibres by left translations.

COROLLARY 8.5. *Let H be an infinite-dimensional Hilbert space. Suppose $\mathfrak{l}$ is not an integer. Then the projection $q: \mathfrak{S}_{\langle \mathfrak{t},\mathfrak{l} \rangle} \to \mathfrak{G}_{\langle \mathfrak{t},\mathfrak{l} \rangle}$ determines a universal classifying principal bundle for the group $\mathrm{GL}(H')$, and the projection $q: \mathfrak{U}_{\langle \mathfrak{t},\mathfrak{l} \rangle} \to \mathfrak{G}_{\langle \mathfrak{t},\mathfrak{l} \rangle}$ a universal classifying principal bundle for the group $\mathrm{U}(H')$, where H' is a Hilbert space with $\dim(H') = \mathfrak{t}$. $\mathfrak{G}_{\langle \mathfrak{t},\mathfrak{l} \rangle}$ is therefore a universal classifying space $B_{\mathrm{GL}(H')}$ and $B_{\mathrm{U}(H')}$ for the groups $\mathrm{GL}(H')$ and $\mathrm{U}(H')$. In particular $\mathfrak{G}_{\langle n,\mathfrak{l} \rangle}$ is a universal classifying space $B_{\mathrm{GL}(n)}$ and $B_{\mathrm{U}(n)}$ for the groups $\mathrm{GL}(n)$ and $\mathrm{U}(n)$ of the n-dimensional vectorspace.*

COROLLARY 8.6. *Let H be an infinite-dimensional Hilbert space. The homotopy groups of the Grassmann spaces $\mathfrak{G}_{\langle \mathfrak{t},\mathfrak{l} \rangle}$ are*

$$\pi_i(\mathfrak{G}_{\langle \mathfrak{t},\mathfrak{l} \rangle}) = 0, \quad \text{if neither } \mathfrak{t} \text{ nor } \mathfrak{l} \text{ is an integer,}$$

$$\pi_0(\mathfrak{G}_{\langle n,\mathfrak{l} \rangle}) = \pi_0(\mathfrak{G}_{\langle \mathfrak{t},n \rangle}) = 0, \quad \text{for } n = 0, 1, 2, \ldots,$$

$$\pi_i(\mathfrak{G}_{\langle n,\mathfrak{l} \rangle}) = \pi_i(\mathfrak{G}_{\langle \mathfrak{t},n \rangle}) = \pi_{i-1}(\mathrm{U}(n)), \quad i = 1, 2, \ldots, \quad \text{for } n = 0, 1, 2, \ldots.$$

And the singular homology and cohomology with coefficients in the ring of integers Z is

$$H(\mathfrak{G}_{\langle \mathfrak{t},\mathfrak{l} \rangle}; Z) = 0, \quad \text{if neither } \mathfrak{t} \text{ nor } \mathfrak{l} \text{ is an integer,}$$

$$H(\mathfrak{G}_{\langle n,\mathfrak{l} \rangle}; Z) = H(\mathfrak{G}_{\langle \mathfrak{t},n \rangle}; Z) = H(B_{\mathrm{U}(n)}; Z), \quad \text{for } n = 0, 1, 2, \ldots.$$

COROLLARY 8.7. *Let H be a separable infinite-dimensional Hilbert space. Then the homotopy groups of the Grassmann spaces are*

$$\pi_i(\mathfrak{G}_{\langle \infty, \infty \rangle}) = 0, \quad i = 0, 1, 2, \ldots,$$

$$\pi_0(\mathfrak{G}_{\langle n, \infty \rangle}) = \pi_0(\mathfrak{G}_{\langle \infty, n \rangle}) = 0, \quad for \ n = 0, 1, 2, \ldots,$$

$$\pi_i(\mathfrak{G}_{\langle n, \infty \rangle}) = \pi_i(\mathfrak{G}_{\langle \infty, n \rangle}) = \pi_{i-1}(U(n)), \quad i = 1, 2, \ldots, \quad for \ n = 0, 1, 2, \ldots.$$

REMARKS. It is easy to associate with the various classifying spaces of type $B_{\mathrm{GL}(n)}$ and $B_{\mathrm{U}(n)}$ in this chapter in a natural way universal classifying vector bundles of dimension n. Since all spaces which occur in this chapter are Banach manifolds, it follows from Theorem 15 of [13] that all maps in this chapter which induce isomorphisms for the homotopy groups are homotopy equivalences. In particular all spaces with vanishing homotopy groups are contractible.

9. **The general case.** There appear various examples of maximal R-sets in analysis (see [4]). In a subsequent publication we intend to compute the homotopy type of the path components and of the associated groups of some of these examples. Theorem 5.5, Corollaries 6.2 and 6.2_q, and Theorem 9.1 imply the following theorem, which can be used for computations:

THEOREM 9.1. *Let $\mathfrak{M}$ be a maximal R-set, and let $\mathfrak{C} \in \pi_0(\mathfrak{M})$ be a path component of $\mathfrak{M}$ such that there is a $S \in \mathfrak{C}$ with* codim (S) *infinite. Suppose also that $U(\mathfrak{M}|_S)$ and $U_q(I(\mathfrak{M}|_S))$ are connected. Then we have the exact sequences of homotopy groups*

$$\cdots \to \pi_{n+1}(\mathfrak{C}) \to \pi_n(U(\mathfrak{M}|_S)) \to \pi_n(U^1(\mathfrak{M})) \to \pi_n(\mathfrak{C}) \to \cdots$$

$$\cdots \to \pi_{n+1}(\mathfrak{C}) \to \pi_n(U_q(I(\mathfrak{M}|_S))) \to \pi_n(U_q^1(I(\mathfrak{M}))) \to \pi_n(\mathfrak{C}) \to \cdots.$$

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UNIVERSITY OF BRITISH COLUMBIA,
    VANCOUVER, BRITISH COLUMBIA, CANADA