COMPARISON THEOREMS FOR ELLIPTIC EQUATIONS ON UNBOUNDED DOMAINS

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Comparison theorems of Sturm's type will be obtained for the linear elliptic partial differential equations

$$lu = \sum_{i,j=1}^{n} D_i(a_{ij}D_ju)+2\sum_{i=1}^{n} b_iD_iu+cu = 0, \tag{1}$$

$$Lv = \sum_{i,j=1}^{n} D_i(A_{ij}D_jv)+2\sum_{i=1}^{n} B_iD_iv+Cv = 0 \tag{2}$$

on unbounded domains R in n-dimensional Euclidean space E^n. The boundary P of R is supposed to have a piecewise continuous unit normal vector at each point. Points in E^n are denoted by $x=(x^1, x^2, \ldots, x^n)$ and differentiation with respect to x^i is denoted by D_i, $i=1, 2, \ldots, n$. The coefficients a_{ij}, b_i, c, A_{ij}, B_i, and C in (1) and (2) are assumed to be real and continuous on $R \cup P$, and the matrices (a_{ij}) and (A_{ij}) symmetric and positive definite in R. A "solution" of (1) (or (2)) is supposed to be continuous in $R \cup P$ and have uniformly continuous first partial derivatives in R, and all derivatives involved in (1) (or (2)) are supposed to exist, be continuous, and satisfy the differential equation at every point in R.

Some recent results of Clark and the author [2], [7] apply to bounded domains R in E^n. In the self-adjoint case $b_i=B_i=0$, $i=1, 2, \ldots, n$, the *variation* of lu is defined as

$$V[u] = \int_R \left[\sum_{i,j=1}^{n} (a_{ij}-A_{ij})D_iuD_ju+(C-c)u^2\right] dx. \tag{3}$$

The following result [2] is typical of those to be extended to unbounded domains.

THEOREM A. *Let R be a bounded domain in E^n whose boundary P has a piecewise continuous unit normal. Suppose $b_i=B_i=0$ in* (1), (2), *$i=1, 2, \ldots, n$. If there exists a nontrivial solution u of $lu=0$ in R such that $u=0$ on P and $V[u]\geqq 0$, then every solution of $Lv=0$ vanishes at some point in $\bar{R}$.*

In [7] the author extended this to the general second-order linear elliptic equations (1) and (2). Theorem A generalizes a theorem of Hartman and Wintner [4] in which the condition $V[u]\geqq 0$ is replaced by the pointwise conditions $C\geqq c$ and $(a_{ij}-A_{ij})$ is positive semidefinite in $\bar{R}$. In the case $n=1$, Theorem A reduces to

Presented to the Society, August 30, 1966; received by the editors July 13, 1966.

(1) This research was supported by the United States Air Force Office of Scientific Research under grant AF-AFOSR-379-66.

Leighton's generalization [5] of the classical Sturm-Picone theorem. In this case, because the solutions of second order ordinary linear differential equations have only simple zeros, it is easy to obtain the following modification of Theorem A: If there exists a nontrivial solution of $lu=0$ in (α, β) such that $u(\alpha)=u(\beta)=0$ and $V[u]>0$, then every solution of $Lv=0$ has a zero in (α, β). In the case $n=2$, Protter [6] obtained pointwise conditions on the coefficients in (1) and (2) to ensure the conclusion of Theorem A.

Our purpose here is to extend Theorem A to unbounded domains in E^n. Apparently no general results are known even in the case $n=1$. Our results will constitute an extension of the Sturm-Picone theorem in 4 directions: (i) to n-dimensions; (ii) to nonselfadjoint differential equations; (iii) to coefficients satisfying a general condition of the type $V[u]\geqq 0$; and (iv) to unbounded domains.

However, differential equations of order higher than 2 and general boundary conditions will not be considered here.

Let D_a denote the n-disk $\{x \in E^n: |x-x_0|<a\}$ and let S_a denote the bounding $(n-1)$-sphere, where x_0 is a fixed point in E^n. Define

$$R_a = R \cap D_a, \qquad P_a = P \cap D_a, \qquad C_a = R \cap S_a.$$

Clearly there exists a positive number a_0 such that R_a is a bounded domain with boundary $P_a \cup C_a$ for all $a \geqq a_0$.

Let $Q[z]$ be the quadratic form in $n+1$ variables $z_1, z_2, \ldots, z_{n+1}$ defined by

$$Q[z] = \sum_{i,j=1}^{n} A_{ij} z_i z_j - 2z_{n+1} \sum_{i=1}^{n} B_i z_i + G z_{n+1}^2, \tag{4}$$

where the continuous function G is to be determined so that this form is positive semidefinite. The matrix Q associated with $Q[z]$ has the block form

$$Q = \begin{pmatrix} A & -B \\ -B^T & G \end{pmatrix}, \qquad A = (A_{ij}),$$

where B^T is the n-vector $(B_1, B_2, \ldots, B_n)$. Let B_i^* denote the cofactor of $-B_i$ in Q. Since A is positive definite, a necessary and sufficient condition for Q to be positive semidefinite is $\det Q \geqq 0$, or

$$G \det (A_{ij}) \geqq - \sum_{i=1}^{n} B_i B_i^*. \tag{5}$$

The proof is a slight modification of the well-known proof for positive definite matrices [3].

Let M_a be the quadratic functional defined by

$$M_a[u] = \int_{R_a} F[u]\, dx, \tag{6}$$

where

$$F[u] = \sum_{i,j} A_{ij} D_i u D_j u - 2u \sum_i B_i D_i u + (G-C)u^2. \tag{7}$$

Define $M[u]=\lim_{a\to\infty} M_a[u]$ (whenever the limit exists). The domain $\mathfrak{D}_M$ of M is defined to be the set of all real-valued continuous functions u in $R \cup P$ such that u has uniformly continuous first partial derivatives in R_a for all $a \geqq a_0$, $M[u]$ exists, and u vanishes on P. Define

$$[u, v]_a = \int_{C_a} u \sum_{i,j} A_{ij} n_i D_j v \, ds \tag{8}$$

where (n_i) is the unit normal to C_a;

$$[u, v] = \lim_{a\to\infty} [u, v]_a \tag{9}$$

whenever the limit on the right side exists.

LEMMA 1. *Suppose G satisfies* (5) *in R. If there exists $u \in \mathfrak{D}_M$ not identically zero such that $M[u]<0$, then every solution v of $Lv=0$ for which $[u^2/v, v] \geqq 0$ vanishes at some point of $R \cup P$.*

Proof. Suppose to the contrary that there exists a solution $v \neq 0$ in $R \cup P$. For $u \in \mathfrak{D}_M$ define

$$X^i = vD_i(u/v),$$
$$Y^i = v^{-1} \sum_j A_{ij} D_j v, \qquad i = 1, 2, \ldots, n.$$

The following identity in R will now be established:

$$\sum_{i,j} A_{ij} X^i X^j - 2u \sum_i B_i X^i + Gu^2 + \sum_i D_i(u^2 Y^i) = F[u] + u^2 v^{-1} Lv. \tag{10}$$

The left member of (10) is equal to

$$\frac{1}{v^2} \sum_{i,j} A_{ij}(vD_i u - uD_i v)(vD_j u - uD_j v) - \frac{2u}{v} \sum_i B_i(vD_i u - uD_i v)$$
$$+ Gu^2 + \frac{2u}{v} \sum_{i,j} A_{ij} D_i u D_j v + \frac{u^2}{v^2} \sum_{i,j} (vD_i(A_{ij} D_j v) - A_{ij} D_i v D_j v).$$

Since (A_{ij}) is symmetric, this reduces easily to the right member of (10). Since $Lv=0$ in R,

$$\int_{R_a} F[u]\, dx = \int_{R_a} \left[\sum_{i,j} A_{ij} X^i X^j - 2u \sum_i B_i X^i + Gu^2\right] dx + \int_{R_a} \sum_i D_i(u^2 Y^i)\, dx. \tag{11}$$

The first integrand on the right side is a positive semidefinite form by the hypothesis (5). Since $u=0$ on P_a, it follows from Green's formula that

$$\int_{R_a} \sum_i D_i(u^2 Y^i)\, dx = \int_{P_a \cup C_a} \sum_i u^2 n_i Y^i \, ds$$
$$= \int_{C_a} \frac{u^2}{v} \sum_{i,j} A_{ij} n_i D_j v \, ds.$$

Hence (8) and (11) yield

$$\int_{R_a} F[u]\,dx \geqq [u^2/v, v]_a.$$

Since $[u^2/v, v] \geqq 0$ by hypothesis,

$$M[u] = \lim_{a\to\infty} \int_{R_a} F[u]\,dx \geqq 0.$$

This contradiction establishes Lemma 1.

LEMMA 2 (SELF-ADJOINT CASE). *Suppose $B_i=0$ in (2) and (7), $i=1, 2, \ldots, n$. If there exists $u \in \mathfrak{D}_M$ not identically zero such that $M[u] \leqq 0$, then every solution v of $Lv=0$ for which $[u^2/v, v] \geqq 0$ vanishes at some point of $R \cup P$.*

Proof. In this case we can take $G=0$, and the first integrand on the right side of (11) is a positive definite form. Hence

$$\int_{R_a} \sum_{i,j} A_{ij} X^i X^j \geqq 0,$$

equality holding iff X^i is identically zero for each $i=1, 2, \ldots, n$; i.e., u is a constant multiple of v. The latter cannot occur since $u=0$ on P and $v \neq 0$ on P, and therefore

$$\int_{R_a} F[u]\,dx > [u^2/v, v]_a.$$

It follows that $M[u]>0$, contrary to the hypothesis $M[u] \leqq 0$.

In addition to (6) consider the quadratic functional defined by

$$m_a[u] = \int_{R_a} \left[\sum_{i,j} a_{ij} D_i u D_j u - 2u \sum_i b_i D_i u - cu^2 \right] dx,$$

whose Euler-Jacobi operator is l and let $m[u] = \lim_{a\to\infty} m_a[u]$ (whenever the limit exists). The domain $\mathfrak{D}_m$ of m consists of all real-valued continuous functions u in $R \cup P$ such that u has uniformly continuous first partial derivatives in R_a for all $a \geqq a_0$, $m[u]$ exists, and u vanishes on P. The *variation* of $m[u]$ is defined as $V[u] = m[u] - M[u]$, that is

$$(12) \quad V[u] = \int_R \left[\sum_{i,j} (a_{ij} - A_{ij}) D_i u D_j u - 2u \sum_i (b_i - B_i) D_i u + (C - c - G)u^2 \right] dx,$$

with domain $\mathfrak{D} = \mathfrak{D}_m \cap \mathfrak{D}_M$. The analogues of (8), (9) for the operator l are

$$\{u, v\}_a = \int_{C_a} u \sum_{i,j} a_{ij} n_i D_j v\, ds,$$

$$\{u, v\} = \lim_{a\to\infty} \{u, v\}_a.$$

THEOREM 1. *Suppose G satisfies (5). If there exists a nontrivial solution $u \in \mathfrak{D}$ of $lu=0$ such that $\{u, u\} \leqq 0$ and $V[u]>0$, then every solution v of $Lv=0$ for which*

$[u^2/v, v] \geqq 0$ *vanishes at some point of* $R \cup P$. *The same conclusion holds if the conditions* $V[u]>0$, $[u^2/v, v] \geqq 0$ *are replaced by* $V[u] \geqq 0$, $[u^2/v, v]>0$ *respectively.*

Proof. Since $u=0$ on P_a, it follows from Green's formula that

$$m_a[u] = -\int_{R_a} ulu\, dx + \{u, u\}_a.$$

Since $lu=0$ and $\{u, u\} \leqq 0$, we obtain in the limit $a \to \infty$ that $m[u] \leqq 0$. The hypothesis $V[u]>0$ is equivalent to $M[u]<m[u]$. Hence the condition $M[u]<0$ of Lemma 1 is fulfilled and v vanishes at some point of $R \cup P$. The second statement of Theorem 1 follows upon obvious modification of the inequalities.

THEOREM 2 (SELF-ADJOINT CASE). *Suppose* $b_i=B_i=0$ *in* (1) *and* (2), $i=1, 2, \ldots, n$. *If there exists a nontrivial solution* $u \in \mathfrak{D}$ *of* $lu=0$ *such that* $\{u, u\} \leqq 0$ *and* $V[u] \geqq 0$, *then every solution* v *of* $Lv=0$ *for which* $[u^2/v, v] \geqq 0$ *vanishes at some point of* $R \cup P$.

This follows from Lemma 2 by a proof analogous to that of Theorem 1.

In the case that equality holds in (5), that is

$$G = -\sum_{i=1}^{n} B_i B_i^* / \det(A_{ij}),$$

we define

$$\delta = \sum_{i=1}^{n} D_i(b_i - B_i) + C - c - G.$$

It follows from (12) by partial integration that

$$V[u] = \int_R \left[\sum_{i,j} (a_{ij} - A_{ij}) D_i u D_j u + \delta u^2\right] dx + \Omega, \tag{13}$$

where

$$\Omega = \lim_{a \to \infty} \int_{C_a} \sum_i (B_i - b_i) u^2 n_i\, ds.$$

L is called a "strict Sturmian majorant" of l when the following conditions hold: (i) $(a_{ij}-A_{ij})$ is positive semidefinite and $\delta \geqq 0$ in R; (ii) $\Omega \geqq 0$; and (iii) either $\delta>0$ at some point in R or $(a_{ij}-A_{ij})$ is positive definite and $c \neq 0$ at some point. A function defined in R is said to be of class $C^{2,1}(R)$ when all of its second partial derivatives exist and are Lipschitzian in R.

THEOREM 3. *Suppose L is a strict Sturmian majorant of l and all the coefficients* a_{ij} *involved in l are of class* $C^{2,1}(R)$. *If there exists a nontrivial solution* $u \in \mathfrak{D}$ *of* $lu=0$ *such that* $\{u, u\} \leqq 0$, *then every solution* v *of* $Lv=0$ *for which* $[u^2/v, v] \geqq 0$ *vanishes at some point of* $R \cup P$.

Proof. $V[u]$ exists since $u \in \mathfrak{D}$, and hence each term on the right side of (13) exists by the strict Sturmian hypothesis. Since $a_{ij} \in C^{2,1}(R)$, $i, j = 1, 2, \ldots, n$, Aronszajn's unique continuation theorem [1] guarantees that the nontrivial solution u cannot vanish identically in any open subset of R. In the case that $\delta > 0$ at some point in R it then follows from (13) that $V[u] > 0$. In the case that $\delta \equiv 0$ in R it follows from (13) and the positive definite hypothesis on $(a_{ij} - A_{ij})$ that $V[u] = 0$ only if $D_i u = 0$ for each $i = 1, 2, \ldots, n$ in some open set S of R, that is, u is constant in S. Since $c \neq 0$ at some point $x_0 \in S$, the differential equation (1) would not be satisfied at x_0. Hence $V[u] > 0$ also in the case that $\delta \equiv 0$. The conclusion of Theorem 3 then follows from Theorem 1.

THEOREM 4 (SELF-ADJOINT CASE). *Suppose $b_i = B_i = 0$ in* (1) *and* (2), $i = 1, 2, \ldots, n$, $C \geqq c$, *and $(a_{ij} - A_{ij})$ is positive semidefinite in $R \cup P$. If there exists a nontrivial solution $u \in \mathfrak{D}$ of* (1) *such that $\{u, u\} \leqq 0$, then every solution v of* (2) *for which $[u^2/v, v] \geqq 0$ vanishes at some point of $R \cup P$.*

This is an immediate consequence of Theorem 2. We assert that the same conclusion holds even if (A_{ij}) is only positive *semidefinite*, provided L is a strict Sturmian majorant of l and all the coefficients a_{ij} are of class $C^{2,1}(R)$. In fact, under these assumptions $V[u] > 0$ as in Theorem 3, i.e., $M[u] < 0$ by the proof of Theorem 1, and Lemma 2 remains valid for positive semidefinite (A_{ij}) provided the hypothesis $M[u] \leqq 0$ is replaced by $M[u] < 0$.

With trivial modifications the above theorems and proofs remain valid in the case that R is a *bounded* domain, i.e., C_a is void for $a \geqq a_0$. In particular Theorem 2 implies Theorem A and Theorem 1 implies the author's recent result [7] for the general elliptic equations (1), (2) on bounded domains.

In the case $n = 2$ considered by Protter [6], the condition $\delta \geqq 0$ of Theorem 3 reduces to

$$(A_{11}A_{22} - A_{12}^2)\left(\sum_{i=1}^{2} D_i(b_i - B_i) + C - c\right) \geqq A_{11}B_2^2 - 2A_{12}B_1B_2 + A_{22}B_1^2.$$

If R is a bounded domain, Theorem 3 then reduces (with trivial modifications) to the author's result in [7].

It is interesting to note the following one-dimensional instances of Theorem 2, in which R is an open interval (α, β). When $n = 1$ and $b_1 = B_1 = 0$, the differential equations (1), (2) have the form

$$(au')' + cu = 0, \qquad a > 0, \tag{14}$$

$$(Av')' + Cv = 0, \qquad A > 0. \tag{15}$$

THEOREM 5. *If there exists a nontrivial solution u of* (14) *in* (α, ∞) *such that $u(\alpha) = 0$, $a(x)u(x)u'(x) \to 0$ as $x \to \infty$, and*

$$\int_\alpha^\infty [(a - A)u'^2 + (C - c)u^2]\, dx \geqq 0, \tag{16}$$

then every solution v of (15) *for which $A(x)u^2(x)v'(x)/v(x)$ has a nonnegative limit as $x \to \infty$ has a zero on $[\alpha, \infty)$. Unless v is a constant multiple of u, v has a zero in (α, ∞).*

Proof. The first statement follows immediately from Theorem 2. To prove the second statement, recall from the proof of Lemma 1 that for all $a \geqq a_0$,

$$\int_\alpha^a F[u]\,dx = \left[\frac{A(x)u^2(x)v'(x)}{v(x)}\right]_\alpha^a + \int_\alpha^a Av^2\left(\frac{u}{v}\right)'^2 dx. \tag{17}$$

Since the solutions of second order ordinary linear differential equations have only simple zeros, an application of L'Hospital's rule yields

$$\lim_{x\to\alpha} \frac{A(x)u^2(x)v'(x)}{v(x)} = 0.$$

Thus the limit of the first term on the right side of (17) as $a \to \infty$ is nonnegative. The second term is nonnegative for all a and zero iff u is a constant multiple of v. Hence $M[u] > 0$ unless v is a constant multiple of u. This contradicts the hypothesis (16).

The next result applies to the case that α, β may be singular points of the differential equations (14), (15); the possibility that they are $\pm\infty$ is not excluded. The proof is similar to that of Theorem 5 and will be omitted.

THEOREM 6. *If there exists a nontrivial solution u of* (14) *in (α, β) such that $a(x)u(x)u'(x) \to 0$ as $x \to \alpha$ and as $x \to \beta$, and*

$$\int_\alpha^\beta [(a-A)u'^2 + (C-c)u^2]\,dx > 0, \tag{18}$$

then every solution v of (15) *for which $A(x)u^2(x)v'(x)/v(x)$ has a nonnegative limit as $x \to \beta$ and a nonpositive limit as $x \to \alpha$ has a zero in (α, β). If the left side of* (18) *is only nonnegative, the same conclusion holds unless v is a constant multiple of u.*

In the special case that α, β are ordinary points of (14) and (15), this reduces to the following generalization of the classical Sturm-Picone theorem; our result is a slight extension of Leighton's theorem [5].

THEOREM 7. *If there exists a nontrivial solution u of* (14) *in $[\alpha, \beta]$ such that $u(\alpha) = u(\beta) = 0$ and the left side of* (18) *is nonnegative, then every solution of* (15) *except a constant multiple of u has a zero in (α, β).*

As an example of Theorem 5, consider the differential equations

$$u'' + (2n+1-x^2)u = 0, \tag{19}$$

$$v'' + [2n+1-x^2+p(x)]v = 0, \tag{20}$$

on a half-open interval $[\alpha, \infty)$, where $p(x)$ is a polynomial. Equation (19) has the well-known solution $u(x) = \exp(-x^2/2)H_n(x)$, where $H_n(x)$ denotes the Hermite polynomial of degree n. Clearly $u \in \mathfrak{D}$. Since every solution v of (20) satisfies $v'(a)/v(a) \sim q(a)$ as $a \to \infty$, where $q(a)$ is a polynomial, it follows that the hypothesis $u^2(a)v'(a)/v(a) \to 0$ as $a \to \infty$ is fulfilled. Hence if α is a zero of $H_n(x)$, then every solution of (20) has a zero in (α, ∞) provided

$$\int_\alpha^\infty p(x)u^2(x)\,dx > 0.$$

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