

COMPARISON THEOREMS FOR ELLIPTIC EQUATIONS ON UNBOUNDED DOMAINS

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Comparison theorems of Sturm's type will be obtained for the linear elliptic partial differential equations

$$(1) \quad lu = \sum_{i,j=1}^n D_i(a_{ij}D_ju) + 2 \sum_{i=1}^n b_i D_i u + cu = 0,$$

$$(2) \quad Lv = \sum_{i,j=1}^n D_i(A_{ij}D_jv) + 2 \sum_{i=1}^n B_i D_i v + Cv = 0$$

on unbounded domains R in n -dimensional Euclidean space E^n . The boundary P of R is supposed to have a piecewise continuous unit normal vector at each point. Points in E^n are denoted by $x = (x^1, x^2, \dots, x^n)$ and differentiation with respect to x^i is denoted by D_i , $i = 1, 2, \dots, n$. The coefficients a_{ij} , b_i , c , A_{ij} , B_i , and C in (1) and (2) are assumed to be real and continuous on $R \cup P$, and the matrices (a_{ij}) and (A_{ij}) symmetric and positive definite in R . A "solution" of (1) (or (2)) is supposed to be continuous in $R \cup P$ and have uniformly continuous first partial derivatives in R , and all derivatives involved in (1) (or (2)) are supposed to exist, be continuous, and satisfy the differential equation at every point in R .

Some recent results of Clark and the author [2], [7] apply to bounded domains R in E^n . In the self-adjoint case $b_i = B_i = 0$, $i = 1, 2, \dots, n$, the *variation* of lu is defined as

$$(3) \quad V[u] = \int_R \left[\sum_{i,j=1}^n (a_{ij} - A_{ij}) D_i u D_j u + (C - c) u^2 \right] dx.$$

The following result [2] is typical of those to be extended to unbounded domains.

THEOREM A. *Let R be a bounded domain in E^n whose boundary P has a piecewise continuous unit normal. Suppose $b_i = B_i = 0$ in (1), (2), $i = 1, 2, \dots, n$. If there exists a nontrivial solution u of $lu = 0$ in R such that $u = 0$ on P and $V[u] \geq 0$, then every solution of $Lv = 0$ vanishes at some point in \bar{R} .*

In [7] the author extended this to the general second-order linear elliptic equations (1) and (2). Theorem A generalizes a theorem of Hartman and Wintner [4] in which the condition $V[u] \geq 0$ is replaced by the pointwise conditions $C \geq c$ and $(a_{ij} - A_{ij})$ is positive semidefinite in \bar{R} . In the case $n = 1$, Theorem A reduces to

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Leighton's generalization [5] of the classical Sturm-Picone theorem. In this case, because the solutions of second order ordinary linear differential equations have only simple zeros, it is easy to obtain the following modification of Theorem A: If there exists a nontrivial solution of $lu=0$ in (α, β) such that $u(\alpha)=u(\beta)=0$ and $V[u]>0$, then every solution of $Lv=0$ has a zero in (α, β) . In the case $n=2$, Protter [6] obtained pointwise conditions on the coefficients in (1) and (2) to ensure the conclusion of Theorem A.

Our purpose here is to extend Theorem A to unbounded domains in E^n . Apparently no general results are known even in the case $n=1$. Our results will constitute an extension of the Sturm-Picone theorem in 4 directions: (i) to n -dimensions; (ii) to nonselfadjoint differential equations; (iii) to coefficients satisfying a general condition of the type $V[u] \geq 0$; and (iv) to unbounded domains.

However, differential equations of order higher than 2 and general boundary conditions will not be considered here.

Let D_a denote the n -disk $\{x \in E^n : |x - x_0| < a\}$ and let S_a denote the bounding $(n-1)$ -sphere, where x_0 is a fixed point in E^n . Define

$$R_a = R \cap D_a, \quad P_a = P \cap D_a, \quad C_a = R \cap S_a.$$

Clearly there exists a positive number a_0 such that R_a is a bounded domain with boundary $P_a \cup C_a$ for all $a \geq a_0$.

Let $Q[z]$ be the quadratic form in $n+1$ variables z_1, z_2, \dots, z_{n+1} defined by

$$(4) \quad Q[z] = \sum_{i,j=1}^n A_{ij}z_i z_j - 2z_{n+1} \sum_{i=1}^n B_i z_i + Gz_{n+1}^2,$$

where the continuous function G is to be determined so that this form is positive semidefinite. The matrix Q associated with $Q[z]$ has the block form

$$Q = \begin{pmatrix} A & -B \\ -B^T & G \end{pmatrix}, \quad A = (A_{ij}),$$

where B^T is the n -vector (B_1, B_2, \dots, B_n) . Let B_i^* denote the cofactor of $-B_i$ in Q . Since A is positive definite, a necessary and sufficient condition for Q to be positive semidefinite is $\det Q \geq 0$, or

$$(5) \quad G \det (A_{ij}) \geq - \sum_{i=1}^n B_i B_i^*.$$

The proof is a slight modification of the well-known proof for positive definite matrices [3].

Let M_a be the quadratic functional defined by

$$(6) \quad M_a[u] = \int_{R_a} F[u] dx,$$

where

$$(7) \quad F[u] = \sum_{i,j} A_{ij} D_i u D_j u - 2u \sum_i B_i D_i u + (G - C)u^2.$$

Define $M[u] = \lim_{a \rightarrow \infty} M_a[u]$ (whenever the limit exists). The domain \mathfrak{D}_M of M is defined to be the set of all real-valued continuous functions u in $R \cup P$ such that u has uniformly continuous first partial derivatives in R_a for all $a \geq a_0$, $M[u]$ exists, and u vanishes on P . Define

$$(8) \quad [u, v]_a = \int_{C_a} u \sum_{i,j} A_{ij} n_i D_j v \, ds$$

where (n_i) is the unit normal to C_a ;

$$(9) \quad [u, v] = \lim_{a \rightarrow \infty} [u, v]_a$$

whenever the limit on the right side exists.

LEMMA 1. *Suppose G satisfies (5) in R . If there exists $u \in \mathfrak{D}_M$ not identically zero such that $M[u] < 0$, then every solution v of $Lv = 0$ for which $[u^2/v, v] \geq 0$ vanishes at some point of $R \cup P$.*

Proof. Suppose to the contrary that there exists a solution $v \neq 0$ in $R \cup P$. For $u \in \mathfrak{D}_M$ define

$$\begin{aligned} X^i &= v D_i(u/v), \\ Y^i &= v^{-1} \sum_j A_{ij} D_j v, \quad i = 1, 2, \dots, n. \end{aligned}$$

The following identity in R will now be established:

$$(10) \quad \sum_{i,j} A_{ij} X^i X^j - 2u \sum_i B_i X^i + Gu^2 + \sum_i D_i(u^2 Y^i) = F[u] + u^2 v^{-1} Lv.$$

The left member of (10) is equal to

$$\begin{aligned} &\frac{1}{v^2} \sum_{i,j} A_{ij} (v D_i u - u D_i v) (v D_j u - u D_j v) - \frac{2u}{v} \sum_i B_i (v D_i u - u D_i v) \\ &\quad + Gu^2 + \frac{2u}{v} \sum_{i,j} A_{ij} D_i u D_j v + \frac{u^2}{v^2} \sum_{i,j} (v D_i (A_{ij} D_j v) - A_{ij} D_i v D_j v). \end{aligned}$$

Since (A_{ij}) is symmetric, this reduces easily to the right member of (10). Since $Lv = 0$ in R ,

$$(11) \quad \int_{R_a} F[u] \, dx = \int_{R_a} \left[\sum_{i,j} A_{ij} X^i X^j - 2u \sum_i B_i X^i + Gu^2 \right] \, dx + \int_{R_a} \sum_i D_i(u^2 Y^i) \, dx.$$

The first integrand on the right side is a positive semidefinite form by the hypothesis (5). Since $u = 0$ on P_a , it follows from Green's formula that

$$\begin{aligned} \int_{R_a} \sum_i D_i(u^2 Y^i) \, dx &= \int_{P_a \cup C_a} \sum_i u^2 n_i Y^i \, ds \\ &= \int_{C_a} \frac{u^2}{v} \sum_{i,j} A_{ij} n_i D_j v \, ds. \end{aligned}$$

Hence (8) and (11) yield

$$\int_{R_a} F[u] \, dx \geq [u^2/v, v]_a.$$

Since $[u^2/v, v] \geq 0$ by hypothesis,

$$M[u] = \lim_{a \rightarrow \infty} \int_{R_a} F[u] \, dx \geq 0.$$

This contradiction establishes Lemma 1.

LEMMA 2 (SELF-ADJOINT CASE). *Suppose $B_i=0$ in (2) and (7), $i=1, 2, \dots, n$. If there exists $u \in \mathfrak{D}_M$ not identically zero such that $M[u] \leq 0$, then every solution v of $Lv=0$ for which $[u^2/v, v] \geq 0$ vanishes at some point of $R \cup P$.*

Proof. In this case we can take $G=0$, and the first integrand on the right side of (11) is a positive definite form. Hence

$$\int_{R_a} \sum_{i,j} A_{ij} X^i X^j \geq 0,$$

equality holding iff X^i is identically zero for each $i=1, 2, \dots, n$; i.e., u is a constant multiple of v . The latter cannot occur since $u=0$ on P and $v \neq 0$ on P , and therefore

$$\int_{R_a} F[u] \, dx > [u^2/v, v]_a.$$

It follows that $M[u] > 0$, contrary to the hypothesis $M[u] \leq 0$.

In addition to (6) consider the quadratic functional defined by

$$m_a[u] = \int_{R_a} \left[\sum_{i,j} a_{ij} D_i u D_j u - 2u \sum_i b_i D_i u - cu^2 \right] dx,$$

whose Euler-Jacobi operator is l and let $m[u] = \lim_{a \rightarrow \infty} m_a[u]$ (whenever the limit exists). The domain \mathfrak{D}_m of m consists of all real-valued continuous functions u in $R \cup P$ such that u has uniformly continuous first partial derivatives in R_a for all $a \geq a_0$, $m[u]$ exists, and u vanishes on P . The variation of $m[u]$ is defined as $V[u] = m[u] - M[u]$, that is

$$(12) \quad V[u] = \int_R \left[\sum_{i,j} (a_{ij} - A_{ij}) D_i u D_j u - 2u \sum_i (b_i - B_i) D_i u + (C - c - G)u^2 \right] dx,$$

with domain $\mathfrak{D} = \mathfrak{D}_m \cap \mathfrak{D}_M$. The analogues of (8), (9) for the operator l are

$$\begin{aligned} \{u, v\}_a &= \int_{C_a} u \sum_{i,j} a_{ij} n_i D_j v \, ds, \\ \{u, v\} &= \lim_{a \rightarrow \infty} \{u, v\}_a. \end{aligned}$$

THEOREM 1. *Suppose G satisfies (5). If there exists a nontrivial solution $u \in \mathfrak{D}$ of $lu=0$ such that $\{u, u\} \leq 0$ and $V[u] > 0$, then every solution v of $Lv=0$ for which*

$[u^2/v, v] \geq 0$ vanishes at some point of $R \cup P$. The same conclusion holds if the conditions $V[u] > 0$, $[u^2/v, v] \geq 0$ are replaced by $V[u] \geq 0$, $[u^2/v, v] > 0$ respectively.

Proof. Since $u=0$ on P_a , it follows from Green's formula that

$$m_a[u] = - \int_{R_a} ulu \, dx + \{u, u\}_a.$$

Since $lu=0$ and $\{u, u\} \leq 0$, we obtain in the limit $a \rightarrow \infty$ that $m[u] \leq 0$. The hypothesis $V[u] > 0$ is equivalent to $M[u] < m[u]$. Hence the condition $M[u] < 0$ of Lemma 1 is fulfilled and v vanishes at some point of $R \cup P$. The second statement of Theorem 1 follows upon obvious modification of the inequalities.

THEOREM 2 (SELF-ADJOINT CASE). *Suppose $b_i = B_i = 0$ in (1) and (2), $i = 1, 2, \dots, n$. If there exists a nontrivial solution $u \in \mathfrak{D}$ of $lu=0$ such that $\{u, u\} \leq 0$ and $V[u] \geq 0$, then every solution v of $Lv=0$ for which $[u^2/v, v] \geq 0$ vanishes at some point of $R \cup P$.*

This follows from Lemma 2 by a proof analogous to that of Theorem 1.

In the case that equality holds in (5), that is

$$G = - \sum_{i=1}^n B_i B_i^* / \det (A_{ij}),$$

we define

$$\delta = \sum_{i=1}^n D_i(b_i - B_i) + C - c - G.$$

It follows from (12) by partial integration that

$$(13) \quad V[u] = \int_R \left[\sum_{i,j} (a_{ij} - A_{ij}) D_i u D_j u + \delta u^2 \right] dx + \Omega,$$

where

$$\Omega = \lim_{a \rightarrow \infty} \int_{C_a} \sum_i (B_i - b_i) u^2 n_i \, ds.$$

L is called a "strict Sturmian majorant" of l when the following conditions hold: (i) $(a_{ij} - A_{ij})$ is positive semidefinite and $\delta \geq 0$ in R ; (ii) $\Omega \geq 0$; and (iii) either $\delta > 0$ at some point in R or $(a_{ij} - A_{ij})$ is positive definite and $c \neq 0$ at some point. A function defined in R is said to be of class $C^{2,1}(R)$ when all of its second partial derivatives exist and are Lipschitzian in R .

THEOREM 3. *Suppose L is a strict Sturmian majorant of l and all the coefficients a_{ij} involved in l are of class $C^{2,1}(R)$. If there exists a nontrivial solution $u \in \mathfrak{D}$ of $lu=0$ such that $\{u, u\} \leq 0$, then every solution v of $Lv=0$ for which $[u^2/v, v] \geq 0$ vanishes at some point of $R \cup P$.*

Proof. $V[u]$ exists since $u \in \mathfrak{D}$, and hence each term on the right side of (13)

exists by the strict Sturmian hypothesis. Since $a_{ij} \in C^{2,1}(R)$, $i, j=1, 2, \dots, n$, Aronszajn's unique continuation theorem [1] guarantees that the nontrivial solution u cannot vanish identically in any open subset of R . In the case that $\delta > 0$ at some point in R it then follows from (13) that $V[u] > 0$. In the case that $\delta \equiv 0$ in R it follows from (13) and the positive definite hypothesis on $(a_{ij} - A_{ij})$ that $V[u] = 0$ only if $D_i u = 0$ for each $i = 1, 2, \dots, n$ in some open set S of R , that is, u is constant in S . Since $c \neq 0$ at some point $x_0 \in S$, the differential equation (1) would not be satisfied at x_0 . Hence $V[u] > 0$ also in the case that $\delta \equiv 0$. The conclusion of Theorem 3 then follows from Theorem 1.

THEOREM 4 (SELF-ADJOINT CASE). *Suppose $b_i = B_i = 0$ in (1) and (2), $i = 1, 2, \dots, n$, $C \geq c$, and $(a_{ij} - A_{ij})$ is positive semidefinite in $R \cup P$. If there exists a nontrivial solution $u \in \mathfrak{D}$ of (1) such that $\{u, u\} \leq 0$, then every solution v of (2) for which $[u^2/v, v] \geq 0$ vanishes at some point of $R \cup P$.*

This is an immediate consequence of Theorem 2. We assert that the same conclusion holds even if (A_{ij}) is only positive semidefinite, provided L is a strict Sturmian majorant of l and all the coefficients a_{ij} are of class $C^{2,1}(R)$. In fact, under these assumptions $V[u] > 0$ as in Theorem 3, i.e., $M[u] < 0$ by the proof of Theorem 1, and Lemma 2 remains valid for positive semidefinite (A_{ij}) provided the hypothesis $M[u] \leq 0$ is replaced by $M[u] < 0$.

With trivial modifications the above theorems and proofs remain valid in the case that R is a bounded domain, i.e., C_a is void for $a \geq a_0$. In particular Theorem 2 implies Theorem A and Theorem 1 implies the author's recent result [7] for the general elliptic equations (1), (2) on bounded domains.

In the case $n=2$ considered by Protter [6], the condition $\delta \geq 0$ of Theorem 3 reduces to

$$(A_{11}A_{22} - A_{12}^2) \left(\sum_{i=1}^2 D_i(b_i - B_i) + C - c \right) \geq A_{11}B_2^2 - 2A_{12}B_1B_2 + A_{22}B_1^2.$$

If R is a bounded domain, Theorem 3 then reduces (with trivial modifications) to the author's result in [7].

It is interesting to note the following one-dimensional instances of Theorem 2, in which R is an open interval (α, β) . When $n=1$ and $b_1 = B_1 = 0$, the differential equations (1), (2) have the form

$$(14) \quad (au)' + cu = 0, \quad a > 0,$$

$$(15) \quad (Av)' + Cv = 0, \quad A > 0.$$

THEOREM 5. *If there exists a nontrivial solution u of (14) in (α, ∞) such that $u(\alpha) = 0$, $a(x)u(x)u'(x) \rightarrow 0$ as $x \rightarrow \infty$, and*

$$(16) \quad \int_{\alpha}^{\infty} [(a - A)u'^2 + (C - c)u^2] dx \geq 0,$$

then every solution v of (15) for which $A(x)u^2(x)v'(x)/v(x)$ has a nonnegative limit as $x \rightarrow \infty$ has a zero on $[\alpha, \infty)$. Unless v is a constant multiple of u , v has a zero in (α, ∞) .

Proof. The first statement follows immediately from Theorem 2. To prove the second statement, recall from the proof of Lemma 1 that for all $a \geq a_0$,

$$(17) \quad \int_{\alpha}^a F[u] dx = \left[\frac{A(x)u^2(x)v'(x)}{v(x)} \right]_{\alpha}^a + \int_{\alpha}^a Av^2 \left(\frac{u}{v} \right)'^2 dx.$$

Since the solutions of second order ordinary linear differential equations have only simple zeros, an application of L'Hospital's rule yields

$$\lim_{x \rightarrow \alpha} \frac{A(x)u^2(x)v'(x)}{v(x)} = 0.$$

Thus the limit of the first term on the right side of (17) as $a \rightarrow \infty$ is nonnegative. The second term is nonnegative for all a and zero iff u is a constant multiple of v . Hence $M[u] > 0$ unless v is a constant multiple of u . This contradicts the hypothesis (16).

The next result applies to the case that α, β may be singular points of the differential equations (14), (15); the possibility that they are $\pm \infty$ is not excluded. The proof is similar to that of Theorem 5 and will be omitted.

THEOREM 6. *If there exists a nontrivial solution u of (14) in (α, β) such that $a(x)u(x)u'(x) \rightarrow 0$ as $x \rightarrow \alpha$ and as $x \rightarrow \beta$, and*

$$(18) \quad \int_{\alpha}^{\beta} [(a-A)u'^2 + (C-c)u^2] dx > 0,$$

then every solution v of (15) for which $A(x)u^2(x)v'(x)/v(x)$ has a nonnegative limit as $x \rightarrow \beta$ and a nonpositive limit as $x \rightarrow \alpha$ has a zero in (α, β) . If the left side of (18) is only nonnegative, the same conclusion holds unless v is a constant multiple of u .

In the special case that α, β are ordinary points of (14) and (15), this reduces to the following generalization of the classical Sturm-Picone theorem; our result is a slight extension of Leighton's theorem [5].

THEOREM 7. *If there exists a nontrivial solution u of (14) in $[\alpha, \beta]$ such that $u(\alpha) = u(\beta) = 0$ and the left side of (18) is nonnegative, then every solution of (15) except a constant multiple of u has a zero in (α, β) .*

As an example of Theorem 5, consider the differential equations

$$(19) \quad u'' + (2n + 1 - x^2)u = 0,$$

$$(20) \quad v'' + [2n + 1 - x^2 + p(x)]v = 0,$$

on a half-open interval $[\alpha, \infty)$, where $p(x)$ is a polynomial. Equation (19) has the well-known solution $u(x) = \exp(-x^2/2)H_n(x)$, where $H_n(x)$ denotes the Hermite

polynomial of degree n . Clearly $u \in \mathfrak{D}$. Since every solution v of (20) satisfies $v'(a)/v(a) \sim q(a)$ as $a \rightarrow \infty$, where $q(a)$ is a polynomial, it follows that the hypothesis $u^2(a)v'(a)/v(a) \rightarrow 0$ as $a \rightarrow \infty$ is fulfilled. Hence if α is a zero of $H_n(x)$, then every solution of (20) has a zero in (α, ∞) provided

$$\int_{\alpha}^{\infty} p(x)u^2(x) dx > 0.$$

REFERENCES

1. N. Aronszajn, *A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order*, J. Math. Pures Appl. **36** (1957), 235–249.
2. Colin Clark and C. A. Swanson, *Comparison theorems for elliptic differential equations*, Proc. Amer. Math. Soc. **16** (1965), 886–890.
3. F. R. Gantmacher, *The theory of matrices*, Vol. I, Chelsea, New York, 1959.
4. Philip Hartman and Aurel Wintner, *On a comparison theorem for self-adjoint partial differential equations of elliptic type*, Proc. Amer. Math. Soc. **6** (1955), 862–865.
5. Walter Leighton, *Comparison theorems for linear differential equations of second order*, Proc. Amer. Math. Soc. **13** (1962), 603–610.
6. M. H. Protter, *A comparison theorem for elliptic equations*, Proc. Amer. Math. Soc. **10** (1959), 296–299.
7. C. A. Swanson, *A comparison theorem for elliptic differential equations*, Proc. Amer. Math. Soc. **17** (1966), 611–616.

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