SPACES DETERMINED BY THEIR HOMEOMORPHISM GROUPS(1)

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1. **Introduction.** Let F be a topological space and let G_F denote the group of homeomorphisms of F onto itself. We give G_F the point-open topology (the topology of pointwise convergence on F) and consider it as a topological space. In general, with this topology, G_F is not a topological group; for example, if F is the unit square, inversion is not continuous.

We are concerned with finding conditions such that if F' and F satisfy these conditions then an isomorphism of $G_{F'}$ onto G_F which is also a homeomorphism induces a homeomorphism of F' onto F. Some conditions are needed since there is such an isomorphism between the group of the open unit interval and that of the closed unit interval.

In [2], Wechsler showed that a sufficient condition for Hausdorff, nondiscrete spaces is ω-homogeneity, i.e., for each n, and each pair of n-tuples of distinct points $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$ there is a homeomorphism h such that $h(x_i)=y_i$ for $i=1, \ldots, n$.

Our main result is to replace ω-homogeneity by the following two conditions.

Condition A. *Let P and Q be finite disjoint sets, let y be a point of $F-P$, and let V be an open set in F. If some member of G_F maps y into V then some member of G_F maps y into $V-Q$ leaving P pointwise fixed.*

Condition B. *If $\{x_\lambda\}$ is a net in F not converging to $x \in F$ then there is a subnet $\{y_\mu\}$ of $\{x_\lambda\}$ and a map $g \in G_F$ such that $g(y_\mu)=y_\mu$ for all μ and $g(x) \neq x$.*

Frequently our arguments parallel those in [2] although Condition A is a good deal weaker than ω-homogeneity and applies to a much wider class of spaces.

§2 consists of preliminary results. In §3 we prove the main theorems of this paper, Theorems 3.1 and 3.2. In the second of these we show that Condition B may be replaced by 1-homogeneity. In §4 we give examples of spaces which are not ω-homogeneous but which satisfy A. These include certain manifolds and manifold-like spaces.

Finally, in §5 we introduce the notion of a determining group and using this notion and our previous results we prove that no conditions whatever are needed if F' and F are manifolds of dimension at least 3.

Notation and conventions. Most of our notation can be found in [2]; for completeness a summary is given here.

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All spaces are assumed to be T_1, so that complements of finite sets are open. The point-open topology for G_F has a basis of sets of the form

$$\{f \in G_F \mid f(x_i) \in U_i,\ i = 1, \ldots, n\}$$

where the x_i are in F and the U_i are open in F. If the number of points in the basis element is irrelevant the above set will be denoted $W[x_i, U_i]$. For convenience we adopt the convention that the x_i appearing in the basis element are distinct.

The notation $\bar{x}$ is used for a point of the k-fold Cartesian product F^k, no two coordinates of which are equal (what k is will be clear from the context); the ith coordinate of $\bar{x}$ is denoted x_i. Similarly $\bar{U}$ stands for the open set $U_1 \times \cdots \times U_k$ where each U_i is open in F. Thus $W[\bar{x}, \bar{U}]$ represents a typical basis element in G_F. G_F acts on F^k coordinatewise; i.e., if $H \subseteq G_F$ then $H(x) = \{(h(x_1), \ldots, h(x_k)) \mid h \in H\}$ and if $h \in H$ then $h(\bar{U}) = h(U_1) \times \cdots \times h(U_k)$.

The composition of functions is denoted by juxtaposition; $gf(x) = g(f(x))$.

By "manifold" we mean a topological n-manifold; we make no a priori connectedness or boundary assumptions. Many of the topological concepts we use are to be found in [1].

REMARK. For the applications of the next section the following version of Condition A is more suitable than the simple form given above. *Let $x_1, \ldots, x_n, y_1, \ldots, y_m$ be distinct points and let $V_1, \ldots, V_m$ be open sets such that, for $i = 1, \ldots, m$, $G_F(y_i) \cap V_i \neq \varnothing$. Given any finite set Q, there is h in G_F such that $h(x_i) = x_i$ for $i = 1, \ldots, n$ and $h(y_i) \in V_i - Q$ for $i = 1, \ldots, m$.* This statement follows from repeated applications of the original Condition A.

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2. **Preliminary results.**

Throughout this section, x denotes a point of a space F which satisfies Condition A.

We begin by establishing two lemmas of a technical nature. The first is used throughout the paper, the second in the proof of Theorem 2.3.

LEMMA 2.1. *Let $W = W[\bar{x}, \bar{U}]$ be nonvoid in G_F and let g be a member of G_F such that $g(x), x_1, \ldots, x_k$ are distinct ($\bar{x} = (x_1, \ldots, x_k)$). There is f in W such that $f(g(x)) = x$.*

Proof. Choose $h \in W$ and let $h^{-1}(\bar{U}) = \bar{V}$. Now $W[g^{-1}(\bar{x}), \bar{U}]$ contains hg, hence is nonvoid, and the points $x, g^{-1}(x_1), \ldots, g^{-1}(x_k)$ are distinct, so it follows from Condition A that there is f_0 in G_F such that $f_0(x) = x$ and $(f_0 g^{-1})(\bar{x}) \in \bar{U}$. Let $f = f_0 g^{-1}$, then $f(g(x)) = f_0(x) = x$ and $f(\bar{x}) = f_0(g^{-1}(\bar{x})) \in \bar{U}$, i.e., $f \in W$.

LEMMA 2.2. *Let g be in the open set $W = W[\bar{x}, \bar{U}]$. There is a neighborhood U of x in F such that if $y \in U \cap G_F(x)$ then for some f in G_F, $f(x) = y$ and $f \in W[\bar{x}, g^{-1}(\bar{U})]$.*

Proof. First suppose $x \neq x_i$ for all i. Let $U = F - \{x_1, \ldots, x_k\}$, thus U is open and contains x. Suppose $y \in U \cap G_F(x)$, say $y=h(x)$ for some h in G_F. Since $W[\bar{x}, g^{-1}(\bar{U})]$ is nonvoid (it contains the identity map of F) and since the points $x=h^{-1}(y), x_1, \ldots, x_k$ are distinct, Lemma 2.1 implies that there is f such that $f(h^{-1}(y))=y$ and $f \in W[\bar{x}, g^{-1}(\bar{U})]$. Since $h^{-1}(y)=x$, we are done in this case.

Now suppose $x=$ some x_i, say $x=x_1$. Let $U=F-\{x_2, \ldots, x_k\} \cap g^{-1}(U_1)$. Again U is a neighborhood of x. If $y \in U \cap G_F(x)$ and $y \neq x$ then, as above, Lemma 2.1 applies to give the desired f. If $y=x$ then we take f to be the identity map.

The next result gives a relationship between certain quotient spaces of G_F and subsets of F. It is the analogue of Theorem 2.6 of [2]. We use the following notation. If H is a subgroup of G_F then G_F/H is the collection of left cosets with the quotient topology. For each $x \in F$ the subgroup of maps leaving x fixed is denoted G_x, and θ_x is the map of G_F/G_x into F given by: $\theta_x(gG_x)=g(x)$.

THEOREM 2.3. *θ_x is a homeomorphism of G_F/G_x onto $G_F(x)$.*

Proof. It is easy to see that θ_x is well defined, one-to-one, and onto.

Let ν denote the quotient map of G_F onto G_F/G_x. The map θ_x is continuous if the composition $\theta_x\nu$ is [1, Theorem 9, p. 95] and a simple argument shows that if $\theta_x\nu$ is open then so is θ_x.

Thus, continuity of θ_x follows immediately from the fact that G_F has the point-open topology. To see that θ_x is open, let $W=W[\bar{x}, \bar{U}]$ be a basis element for the point-open topology and let $V=\theta_x\nu(W)=\{g(x) \mid g \in W\}$. We must show V is open in the relative topology for $G_F(x)$.

Suppose $y \in V$, say $y=g(x)$ for some g in W. Choose an open set U in F satisfying the requirements of Lemma 2.2. Then $\tilde{U}=U \cap G_F(x)$ is a neighborhood of x in $G_F(x)$ and $g(\tilde{U})$ is a neighborhood of y in $G_F(x)$. We show that $g(\tilde{U}) \subseteq V$. Let z be in $g(\tilde{U})$, say $z=g(w)$ for some $w \in \tilde{U}$. Choose f such that $f(x)=w$ and $f \in W[\bar{x}, g^{-1}(\bar{U})]$ and let $h=gf$. Then $h \in W[\bar{x}, \bar{U}]=W$ and $h(x)=gf(x)=g(w)=z$. Since z was arbitrary in $g(\tilde{U})$ we have shown that $g(\tilde{U}) \subseteq V$. Since y was arbitrary in V this shows V is open in $G_F(x)$ and completes the proof of the theorem.

The final result of this section is crucial in the proof of the main theorem. Points $\bar{x}$ and $\bar{y}$ of F^k are *independent* provided no component of $\bar{x}$ is a component of $\bar{y}$.

LEMMA 2.4. *Let H be a subgroup of G_F and suppose $W[\bar{x}, \bar{U}]$ is a nonvoid open set missing H where $\bar{x} \in F^k$, $k \geqq 1$. Suppose there is $\bar{y}$ in F^k such that $\bar{y}$ is in $H(\bar{x})$ and $\bar{x}$ and $\bar{y}$ are independent. Then $\bar{x}$ lies in infinitely many distinct sets of the form $gH(\bar{x})$.*

Proof. $\bar{x}$ is certainly in $H(\bar{x})$. Suppose we have n distinct sets $g_1H(\bar{x}), g_2H(\bar{x}), \ldots, g_nH(\bar{x})$ $(n \geqq 1)$ containing $\bar{x}$; we show there is $g_{n+1}H(\bar{x})$, distinct from the previous sets, containing $\bar{x}$.

Case 1. Suppose $H(\bar{x})$ contains $n+1$ independent elements, say $\bar{x}, \bar{x}_1, \ldots, \bar{x}_n$. For each $i=1, \ldots, n$ write $\bar{x}_i=(y_1^i, \ldots, y_k^i)$. For fixed i and any j $(1 \leqq j \leqq k)$, $G_F(y_j^i)=G_F(x_j)$ meets U_j and hence meets $g_i(U_j)$. By Condition A there is a function

g_{n+1} such that $g_{n+1}(x_i)=x_i$ for $i=1,\ldots,k$ and $g_{n+1}(y_j^i)\in g_i(U_j)$ for $i=1,\ldots,n$ and $j=1,\ldots,k$. Thus $g_{n+1}(\bar{x})=\bar{x}$ and for $i=1,\ldots,n$, $g_{n+1}(\bar{x}_i)\in g_i(\bar{U})$. The first condition on g_{n+1} implies that $\bar{x}\in g_{n+1}H(\bar{x})$. The second condition implies that $g_{n+1}H(\bar{x})$ is distinct from $g_iH(\bar{x})$ for $1\leqq i\leqq n$, because $g_{n+1}(\bar{x}_i)$ meets $g_i(\bar{U})$ while $g_i(\bar{x}_i)$ does not (H misses $W[\bar{x},\bar{U}]$).

Case 2. Suppose $H(\bar{x})$ contains at most n independent elements. The same is true of the sets $g_iH(\bar{x})$, $i=1,\ldots,n$. It follows that there is a finite set $Q\subset F$ such that if $\bar{w}=(w_1,\ldots,w_k)$ is in F^k and no w_i is in Q then $\bar{w}$ is in no $g_iH(\bar{x})$. We have independent points $\bar{x}$ and $\bar{y}$ in $H(\bar{x})$; apply Condition A to get a map g_{n+1} such that $g_{n+1}(\bar{x})=\bar{x}$ and $g_{n+1}(y_i)\in F-Q$ for $i=1,\ldots,k$ (where $\bar{y}=(y_1,\ldots,y_k)$). Since $g_{n+1}(\bar{y})$ is no $g_iH(\bar{x})$, for $1\leqq i\leqq n$, $g_{n+1}H(\bar{x})$ is distinct from $g_1H(\bar{x}),\ldots,g_nH(\bar{x})$ and we are done.

3. The main theorem.

THEOREM 3.1. *Let F' and F be spaces satisfying Condition* A *and let Φ be an isomorphism of $G_{F'}$ onto G_F which is also a homeomorphism. For each x in F', $\Phi(G_x)$ is the subgroup G_y of a point y of F. The induced function $x\to y$ from F' into F is one-to-one and onto.*

Proof. Fix $x\in F'$ and let $H=\Phi(G_x)$; thus H is a proper closed subgroup of G_F and, for some n, there exist $\bar{x}\in F^n$ and $\bar{U}\subset F^n$ such that the nonvoid basic open set $W=W[\bar{x},\bar{U}]$ in G_F misses H. We assume n is the smallest integer with this property.

We first show $n=1$ and $H(\bar{x})$ is a single point of $F^1=F$. Suppose not; then $H(\bar{x})$ contains two independent points $\bar{x}$ and $\bar{y}$. (This is trivial if $n=1$ while, if $n>1$, it follows from Lemma 3.14 of [2].) By our Lemma 2.4 there exist infinitely many distinct sets $\{g_iH(\bar{x})\mid i=1,2,\ldots\}$ each containing $\bar{x}$. In particular, the cosets $\{g_iH\mid i=1,2,\ldots\}$ are distinct.

Let $V=\Phi^{-1}(W)$; since V is open in $G_{F'}$ it contains a nonvoid basic open set, $W[y_i,U_i]$, $i=1,\ldots,m$. Let $h_i=\Phi^{-1}(g_i)$ for $i=1,2,\ldots$. Since the sets

$$\{h_iG_x\mid i=1,2,\ldots\}$$

are distinct, we may choose k so that $h_k(x),y_1,\ldots,y_m$ are distinct. By Lemma 2.1 there is $g\in G_{F'}$ such that $g\in W[y_i,U_i]\subset V$ and $gh_k(x)=x$, i.e., $gh_kG_x=G_x$. Thus $f=\Phi(g)$ is a map in G_F such that $f\in\Phi(V)=W$ and $fg_kH=H$. Since $\bar{x}\in g_kH(\bar{x})$, we have from the last condition that $f(\bar{x})\in H(\bar{x})$. But $H(\bar{x})\cap\bar{U}=\varnothing$ and $W=W[\bar{x},\bar{U}]$; so we have a contradiction.

To summarize, we have shown that $H(\bar{x})$ is a single point of F, say $H(\bar{x})=\{y\}$. It follows that $H\subseteq G_y$ and we next show that $H=G_y$.

Suppose there is $g\in G_y-H$, then putting $f=\Phi^{-1}(g)$ we have $f\in G_{F'}-G_x$. We show that G_xfG_x is dense in $G_{F'}$. Let $W=W[x_i,U_i]$, $i=1,\ldots,n$, be nonvoid and open in $G_{F'}$. Since $f(x)\neq x$ there is, by Condition A, a map h_1 such that $h_1(x)=x$ and $h_1(x_i)\neq f^{-1}(x)$ for $i=1,\ldots,n$; i.e., $h_1\in G_x$ and $fh_1(x_i)\neq x$ for $i=1,\ldots,n$. Now again by Condition A, there is h_2 such that $h_2(x)=x$ and $h_2fh_1(x_i)\in U_i$, $i=1,\ldots,n$.

Thus $h_2fh_1 \in G_xfG_x \cap W$ and G_xfG_x is dense. But then $\Phi(G_xfG_x)$ is dense in G_F which is absurd since $\Phi(G_xfG_x) \subseteq HG_yH = G_y$ which is a proper closed subgroup of G_F.

Thus we have: $\Phi(G_x) = G_y$. Obviously a dual result holds for Φ^{-1} so that the induced correspondence $x \to y$ from F' into F is one-to-one and onto.

THEOREM 3.2. *Let F', F, and Φ be as in Theorem* 3.1, *and let ϕ be the map of F' onto F given by*: $\phi(x) = y$ *where* $\Phi(G_x) = G_y$. *Then ϕ is a homeomorphism if* (i) *both spaces satisfy Condition* B, *or*, (ii) *both spaces are* 1-*homogeneous.*

REMARK. If both spaces are assumed to satisfy the first axiom of countability then Condition B can be replaced by the corresponding statement for sequences.

Proof. Suppose Condition B holds in F and let $\{x_\lambda\}$ be a net in F' converging to x. If $\{\phi(x_\lambda)\}$ does not converge to $\phi(x)$ in F then there is a subnet $\{y_\mu\}$ of $\{\phi(x_\lambda)\}$ and a function g in $\bigcap_\mu G_{y_\mu} - G_{\phi(x)}$. Put $z_\mu = \phi^{-1}(y_\mu)$, then $\{z_\mu\}$ is a subnet of $\{x_\lambda\}$ and $\Phi^{-1}(g)$ is in $\bigcap_\mu G_{z_\mu} - G_x$. Since $\{z_\mu\}$ converges to x this is impossible. Thus ϕ is continuous. Dually, if F' satisfies Condition B then ϕ^{-1} is continuous.

For the second part of the proof, assume both spaces are 1-homogeneous. Fix a point x of F'; then we have:

$$F' = G_{F'}(x) \approx G_{F'}/G_x \approx G_F/G_{\phi(x)} \approx G_F(\phi(x)) = F,$$

where $\approx$ denotes a homeomorphism. The first and last equality signs follow from 1-homogeneity, the first and last homeomorphisms from Theorem 2.3, and the middle homeomorphism is induced by Φ. The composition of these maps yields a homeomorphism ψ and we assert that $\psi = \phi$. It is enough to prove that for each g in $G_{F'}$, $\psi(g(x)) = \phi(g(x))$. One easily verifies that $\psi(g(x)) = \Phi(g)(\phi(x))$. This last expression is $\phi(g(x))$ if and only if $\Phi(G_{g(x)}) = G_{\Phi(g)(\phi(x))}$. But

$$\Phi(G_{g(x)}) = \Phi(gG_xg^{-1}) = \Phi(g)G_{\phi(x)}\Phi(g)^{-1} = G_{\Phi(g)(\phi(x))}, \qquad \text{Q.E.D.}$$

4. **Condition** A. Most of this section is devoted to exhibiting spaces to which Theorem 3.1 applies, i.e., spaces in which Condition A holds. At the end we give a condition which is necessary in order that a space satisfy Condition A.

We remark that Theorem 2.3 applies to all spaces mentioned below. In all these spaces Condition B holds although the second part of the theorem applies directly in the ω-homogeneous case.

To begin with, any ω-homogeneous space satisfies Condition A; this includes all connected manifolds without boundary of dimension at least 2, the Hilbert cube, and some totally disconnected spaces such as the Cantor set or the set of points with rational coordinates in E^n.

More generally let M be a connected manifold whose boundary ∂M has dimension at least 2 and suppose no two components of ∂M are homeomorphic, then M satisfies Condition A.

QUESTION. Let M be a connected manifold of dimension at least 3 and let K be a subset of a component of ∂M. If K satisfies Condition A then does $\mathring{M} \cup K$ satisfy Condition A?

Spaces satisfying the condition can be manufactured by patching together manifolds of different dimensions. As an example, let $F=\tilde{B}^4 \cup T$ where $\tilde{B}^4$ is the standard 4-ball in E^4 with the north pole removed and T is a 3-cell in E^4 such that $T \cap B^4=\partial T$ is the 2-sphere $\{(x_1, \ldots, x_4) \in \partial B^4 \mid x_4=0\}$.

It should be pointed out that a space need not be connected or totally disconnected in order that it satisfy Condition A. Let F be a space each component of which is open in F and satisfies Condition A; then if no two components of F are homeomorphic, F satisfies A.

We end this section by giving a necessary condition that a space satisfy Condition A. This is a take-off on the fact that a circle is not 4-homogeneous. Let us say that a subset S of a space F contains a *free arc* if it contains an arc A with endpoints a and b such that, with the relativized topology, $A-\{a, b\}$ is open in S. Our condition can be stated as follows: *In order that F satisfy Condition* A *it is necessary that, for each x in F, $G_F(x)$ contains no free arc.*

We sketch a proof of this fact. Suppose γ is a homeomorphism of $I=[0, 1]$ into F such that $\gamma(I) \subseteq G_F(x)$ for some x and $\gamma((0, 1))$ is open relative to $G_F(x)$. Let $y=\gamma(0)$, $x_1=\gamma(1/4)$, $x_2=\gamma(1/2)$, $x_3=\gamma(3/4)$ and let U be open in F such that $U \cap G_F(x)=\gamma((1/4, 1/2))$. Since $G_F(x)=G_F(y)$ we have $G_F(y) \cap U \neq \varnothing$. One then shows by an order argument that no member of G_F (indeed, no self-homeomorphism of $G_F(x)$) leaves x_1, x_2, and x_3 fixed and takes y into U; thus Condition A fails.

5. **A better theorem for manifolds.** In the preceding sections we have not used the hypothesis that G_F consists of all self-homeomorphisms of F. This fact is important; we shall show that for manifolds a sharpening of the previous results is obtained by using fewer homeomorphisms.

A subgroup H of G_F is a *determining group* if Conditions A and B hold with G_F replaced by H. Clearly what is proved in §§2 and 3 is that *if H' and H are determining groups for spaces F' and F then a map Φ of H' onto H which is an isomorphism and a homeomorphism induces a homeomorphism of F' onto F.*

Using this notion we circumvent Conditions A and B as hypotheses to obtain the following very general theorem.

THEOREM 5.1. *Let M and N be manifolds of dimension at least* 3 (*with or without boundary, connected or disconnected*) *and suppose Φ is an isomorphism and homeomorphism of G_M onto G_N. Then Φ induces a homeomorphism of M onto N.*

Proof. Let C_M denote the component of the space G_M containing the identity map e_M of M and define C_N similarly. Then $\Phi(C_M)=C_N$. We shall show below that C_M and C_N are determining subgroups for M and N, hence, by the above remarks, there is an induced homeomorphism as asserted.

Let H_M denote the subgroup of G_M consisting of all homeomorphisms leaving each component of M and each component of ∂M fixed. (If M or ∂M is connected delete the corresponding restriction.) We prove that $H_M = C_M$. It is easy to see that H_M is a closed subgroup of G_M containing e_M. If $g \in G_M - H_M$ then there exist distinct components K and K' of either M or ∂M such that $g(K) = K'$. Pick $x \in K$ and let U be an open set containing K' and missing every component of M or ∂M which misses K'. Then $W[x, U]$ is an open and closed set in G_M containing g and missing H_M. This implies that $C_M \subseteq H_M$. To complete the proof it suffices to show H_M is connected.

Let $W = W[x_i, U_i]$ be open in G_M and suppose $W \cap H_M$ is nonvoid. Thus, each U_i meets the component of M containing x_i and, if $x_i \in \partial M$, then U_i meets the component of ∂M containing x_i. From these facts and the fact that M has dimension at least 3 we can obtain an isotopy $\{g_t \mid 0 \leqq t \leqq 1\}$ with the following properties: $g_0 = e_M$, $g_t \in H_M$ for all t and $g_1(x_i) \in U_i$ for each i. This isotopy is a connected subset of G_M, lying in H_M, joining e_M to W. It follows that H_M is connected, thus $H_M = C_M$.

Clearly the dimension requirements insure that H_M is a determining group. A dual result holds for N and the proof is complete.

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