CAPACITIES OF SETS AND HARMONIC ANALYSIS ON THE GROUP 2^ω

BY

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1. **Introduction.** 1.1. The broad question toward which the results of this paper are directed is the same question which has been investigated in a colorful line of preceding papers (e.g., Fine [5], [6] and Crittenden-Shapiro [4]): How do the theories of trigonometric and Walsh series compare? The main theorems here are modeled after those in classical papers by Beurling [3], Salem-Zygmund [11], and Broman [2]. The success which I had in transplanting these results seems to depend upon two similarities between trigonometric and Walsh functions; that they are both essentially the characters of compact, metric groups and that the $2n$th Dirichlet kernel for Walsh series has many of the properties of the Abel kernel for trigonometric series.

The problem treated in §3 is roughly the following: Describe the sets upon which certain classes of Walsh series must converge. In the general theory of orthogonal series we find the fundamental theorem of Rademacher-Menchoff (Alexits [1, p. 80]), "Let $\phi_n(x)$ be an orthonormal system (on a finite interval). The orthogonal series

$$\sum_{n=0}^{\infty} c_n\phi_n(x) \tag{1}$$

is convergent almost everywhere if the condition

$$\sum_{n=1}^{\infty} c_n^2 \log^2 n < \infty \tag{2}$$

is fulfilled." For trigonometric series we have the slightly stronger result of Kolmogoroff-Seliverstoff, (Zygmund [14, Vol. 2, p. 163]) which says that if

$$\sum_{n=1}^{\infty} (a_n^2+b_n^2) \log n < \infty \tag{3}$$

then the Fourier series

$$a_0/2+\sum_{n=1}^{\infty} a_n \cos nx+b_n \sin nx \tag{4}$$

converges at all points except possibly for a set of measure zero. But how large are these sets of measure zero? In answer to that we have the very satisfying results

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in Chapter 4 of Kahane-Salem [8] which are summarized as follows: In order that there exist a series

$$a_0/2+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \tag{5}$$

satisfying the condition

$$\sum_{n=1}^{\infty}(a_n^2+b_n^2)n^{\beta} < \infty, \qquad 0<\beta\leqq 1, \tag{6}$$

and divergent on some closed set $E\subset[0, 2\pi]$, it is necessary and sufficient that E be of $(1-\beta)$-capacity zero for $0<\beta<1$, or logarithmic capacity zero if $\beta=1$. Our Theorem 3.1 is a variant of this last result.

2. **Potential theory.** 2.1. This section is primarily a collection of well-known results in potential theory which are needed in the next section.

In the sequel we shall be working almost entirely on the dyadic group, 2^{ω}, i.e., all sequences of zeroes and ones with addition (mod 2) defined pointwise. The topology is the product topology which is the same as that given by the invariant metric, $\delta(x, y)$, where if $x=(x_1, x_2, \ldots)$ and $y=(y_1, y_2, \ldots)$ are in 2^{ω} then

$$\delta(x, y) = \sum_{n=1}^{\infty}|x_n-y_n|/2^n. \tag{1}$$

Note that the diameter of 2^{ω} is one. After this we shall write $|x-y|$ instead of $\delta(x, y)$.

By Tychonoff's theorem 2^{ω} is compact. Since it is compact and metric, the dual group of 2^{ω} is countable and discrete (see Rudin [10]). N. J. Fine in his classical paper on the Walsh functions, [5], shows that the natural map $\lambda\colon 2^{\omega}\to[0, 1]$ defined by

$$\lambda(x) = \sum_{n=1}^{\infty}x_n/2^n \tag{2}$$

is continuous, one-to-one except for a countable set, preserves Haar measure and carries the characters of 2^{ω} onto the Walsh functions.

Fine [5] noted the above mapping, λ, and its properties but did not make any real use of it in the sense that he viewed the Walsh functions in the classical way, as being defined on the unit interval. For him this was the sensible thing to do since his results were variants of topological and measure theoretical theorems from the classical theory of trigonometric series. However, the results emulated here, in which the possible closed sets of divergence of trigonometric series are characterized by their capacities, seem to depend more fundamentally on the group and its metric (in that case, the circle). In carrying them over to the Walsh functions it seems necessary to retain the group and the metric which are intrinsic in the Walsh functions. 2^{ω} then is the setting for what follows.

2.2. DEFINITION. If E is a closed subset of 2^ω (in the sequel, E will always denote a closed set), then $\mathfrak{M}(E)$ is the set of all nonnegative, Borel measures of norm one on 2^ω supported on E.

2.3. Fix $0 \leqq \alpha < 1$. With our previous notation, let $\nu \in \mathfrak{M}(E)$ and form the so-called *energy integral*

$$I(\nu) = \iint \Phi(x-y)\, d\nu(x)\, d\nu(y) \tag{1}$$

where $\Phi(x) = |x|^{-\alpha}$ if $0<\alpha<1$ or $\log_2 1/|x|$ if $\alpha=0$. Since the integrand is nonnegative, the integral always exists, but may be $+\infty$.

There are then two cases: Either $I(\nu) = +\infty$ for all ν in $\mathfrak{M}(E)$ or

$$V = \inf I(\nu) < \infty, \qquad \nu \in \mathfrak{M}(E). \tag{2}$$

2.4. DEFINITION. E is said to be of *capacity zero* if $I(\nu) = +\infty$ for all ν in $\mathfrak{M}(E)$, or the *capacity of* E is

$$C \equiv V^{-1/\alpha} \quad \text{if } 0 < \alpha < 1 \qquad \text{or} \qquad C \equiv 2^{-V} \quad \text{if } \alpha = 0, \tag{1}$$

if $V < \infty$.

$$V(x;\nu) = \int \Phi(x-y)\, d\nu(y) = \Phi * d\nu(x) \tag{2}$$

is the *potential function* associated with ν. $U(x;\nu)$ is nonnegative. It is also easy to see that $U(x;\nu)$ is lower semicontinuous on 2^ω and continuous in the complement of E.

2.5. In order to facilitate progress we shall have to change our kernel slightly: For $x \in 2^\omega$ let $\{x\} = 2^{-n}$, where n is the number of zeroes in x preceding the first one ($\{0\}=0$). Note that the range of this function is just the integral powers of two between zero and one. From the definition of $\{x\}$ the inverse image of 2^{-n} is $\{x \in 2^\omega : x_1 = \cdots = x_n = 0,\ x_{n+1} = 1\}$. This is a basic open set and so the function $x \to \{x\}$ is continuous. Then let

$$K(x) = \{x\}^{-\alpha} \quad \text{if } 0 < \alpha < 1 \qquad \text{or} \qquad \log 1/\{x\} \quad \text{if } \alpha = 0. \tag{1}$$

(All logarithms shall be taken to the base two.) K has all the properties of Φ that we have used so far (K is continuous except at zero and nonnegative) so that a potential theory with respect to K is equally valid. In fact, since

$$\{x\} = 2 \sum_{k=1}^{n+1} x_k 2^{-k} \tag{2}$$

when $\{x\} = 2^{-n}$, we have

$$|x| = \sum_{k=1}^{\infty} x_k 2^{-k} \leqq \{x\} \leqq 2|x|. \tag{3}$$

So

$$\Phi(x) \leqq 2^\alpha K(x) \leqq 2^\alpha \Phi(x) \quad \text{if } 0 < \alpha < 1 \qquad \text{or}$$
$$\Phi(x) \leqq K(x) + 1 \leqq \Phi(x) + 1 \quad \text{if } \alpha = 0. \tag{4}$$

Thus a set is of capacity zero with respect to K if and only if it is of capacity zero with respect to Φ. For our purposes then, these kernels are the same.

Our first gain in making this switch is that it is now easy to compute the Walsh-Fourier series of the kernel function.

Let

$$G_n = \{x \in 2^\omega : x_1 = \cdots = x_n = 0\};$$

then

$$\{x\} = 2 \sum_{k=1}^{\infty} 2^{-k} \chi_{G_k}(\delta^k + x) \tag{5}$$

where

$$\delta_j^k = 1 \quad \text{if } j = k, \qquad \delta^k = (\delta_1^k, \delta_2^k, \ldots)$$
$$= 0 \quad \text{if } j = k,$$

now

$$K(x) = 2^{-\alpha} \sum_{k=1}^{\infty} 2^{\alpha k} \chi_{G_k}(\delta^k + x), \qquad 0 < \alpha < 1 \tag{6}$$

so

$$\begin{aligned} \gamma_n &= \int K(x)\psi_n(x)\,dx \\ &= 2^{-\alpha} \sum_{k=1}^{\infty} 2^{\alpha k} \int \chi_{G_k}(\delta^k + x)\psi_n(x)\,dx, \end{aligned} \tag{7}$$

$$\int \chi_{G_k}(\delta^k + x)\psi_n(x)\,dx = \int \chi_{G_k}(x)\psi_n(\delta^k + x)\,dx \tag{8}$$

since Haar measure is translation invariant

$$= \psi_n(\delta^k) \int \chi_{G_k}(x)\psi_n(x)\,dx. \tag{9}$$

We know from the general theory of Fourier transforms since G_k is a closed subgroup of 2^ω,$\hat{G}_k$, the dual group of G_k is isomorphic to $2^\omega/(0; G_k)$ under the natural (restriction) mapping. $(0, G_k)$, the annihilator of G_k in the Walsh functions, is $\{\psi_n : n < 2^k\}$. Therefore

$$\int \chi_{G_k}(x)\psi_n(x)\,dx = 2^{-k} \Delta_n^k \tag{10}$$

where

$$\Delta_n^k = 1 \quad \text{if } n < 2^k,$$
$$= 0 \quad \text{if } n \geqq 2^k$$

another way to see (11) is to transform the integral by λ, the natural mapping from 2^ω to $[0, 1]$. Haar measure is preserved and the integrand becomes a product of simple functions.

Putting (7), (10), and (11) together we have

$$\gamma_n = 2^{-\alpha}\sum_{k=1}^{\infty} 2^{(\alpha-1)k}\psi_n(\delta^k)\,\Delta_n^k \tag{11}$$

$$= 2^{-\alpha}\sum_{k=1+\log[n]}^{\infty} 2^{(\alpha-1)k}\psi_n(\delta^k) \tag{12}$$

where $[n]$ is the greatest power of 2 in n, e.g., $[3]=2$.

$$= 2^{-\alpha}\Big(\sum_{k=2+\log[n]} 2^{(\alpha-1)k} - 2^{(\alpha-1)(1+\log[n])}\Big) \tag{13}$$

$$= 2^{-\alpha}(2^{1-\alpha}[n]^{1-\alpha})^{-1}((2^{1-\alpha}-1)^{-1}-1) \tag{14}$$

$$= A/[n]^{1-\alpha},\ A \text{ a positive constant} \qquad \text{if } n > 0, \tag{15}$$

$$\begin{aligned}\gamma_0 &= \int K(x)\,dx = 2^{-\alpha}\sum_{k=1}^{\infty} 2^{\alpha k}2^{-k}\\ &= 2^{-1}(1-2^{\alpha-1})^{-1}.\end{aligned} \tag{16}$$

If $\alpha=0$ we proceed in the same manner

$$\begin{aligned}K(x) &= \log 1/\{x\}\\ &= -1+\sum_{k=1}^{\infty} k\chi(\delta^k_{G_k}+x) \quad \text{by (5).}\end{aligned} \tag{17}$$

Therefore

$$\gamma_n = \int K(x)\psi_n(x)\,dx$$

$$= \sum_{k=1}^{\infty}(k-1)\int \chi_{G_k}(\delta^k+x)\psi_n(x)\,dx \tag{18}$$

$$= \sum_{k=1}^{\infty}(k-1)\psi_n(\delta^k)2^{-k}\,\Delta_{nk} \qquad \text{by (9) and (10)} \tag{19}$$

$$= \sum_{k=1+\log[n]}^{\infty}(k-1)\psi_n(\delta^k)2^{-k} \qquad \text{if } n > 0 \tag{20}$$

$$= \sum_{k=2+\log[n]}^{\infty}(k-1)2^{-k} - \log[n]2^{-(1+\log[n])} \tag{21}$$

$$= 2^{-1}\sum_{k=1-\log[n]} k2^{-k} - 2^{-1}[n]^{-1}\log[n] \tag{22}$$

$$= [n]^{-1}, \tag{23}$$

$$\gamma_0 = \sum_{k=1}^{\infty}(k-1)2^{-k} \qquad \text{from (21)} \tag{24}$$

$$= 2^{-2}\sum_{k=0}^{\infty} k2^{-(k-1)} = 1. \tag{25}$$

Therefore γ_n is positive for all n and α, $0 \leqq \alpha < 1$ and

$$\gamma_0 - A \geqq 0. \tag{26}$$

2.6. LEMMA. (i) *$s_n(x; k) \to k(x)$ for all $x \in 2^\omega$, $x \neq 0$,*
(ii) *$|s_n(x; k)| \leqq Bk(x)$, B a positive constant depending only on α.*

Proof. (i)

$$\begin{aligned}
s_n(x; k) &= \gamma_0 + \sum_{k=1}^{n-1} A/[k]^{1-\alpha}\psi_k(x) \\
&= \gamma_0 + A\sum_{k=1}^{n-2} (1/[k]^{1-\alpha} - 1/[k+1]^{1-\alpha}) \sum_{j=1}^{k} \psi_j(x) \\
&\quad + A/[n-1]^{1-\alpha} \sum_{j=1}^{n-1} \psi_j(x) \\
(1) \quad &= \gamma_0 + (2^{1-\alpha} - 1) \sum_{k=1}^{\log[n-1]} 1/2^{k(1-\alpha)} \sum_{j=1}^{2^k - 1} \psi_j(x) \\
&\quad + A/[n-1]^{1-\alpha} \sum_{j=1}^{n-1} \psi_j(x) \\
&= \gamma_0 + A(2^{1-\alpha} - 1)\left(\sum_{k=1}^{\log[n-1]} 1/2^{k(1-\alpha)} \sum_{j=0}^{2^k-1} \psi_j(x) - \sum_{k=1}^{\log[n-1]} 1/2^{k(1-\alpha)}\right) \\
(2) \quad &\quad + A/[n-1]^{1-\alpha}\left(\sum_{j=0}^{n-1} \psi_j(x) - 1\right) \\
&= \gamma_0 + A(2^{1-\alpha} - 1) \sum_{k=1}^{\log[n-1]} 1/2^{k(1-\alpha)} D_{2^k}(x) + A/[n-1]^{1-\alpha} D_n(x) \\
&\quad - A
\end{aligned}$$

where $D_n(x) = \sum_{j=0}^{n-1} \psi_j(x)$ is the nth Dirichlet kernel for the Walsh functions.
It is well known (See Fine [5]) that

$$D_{2^k}(x) = 2^k \chi_{G_k}(x). \tag{3}$$

From this we can show that

$$|D_n(x)| \leqq 1/|x|. \tag{4}$$

It is true for $n=0$, and the rest follows from

$$\begin{aligned}
|D_n(x)| &= \left|\sum_{k=0}^{n-1} \psi_k(x)\right| \\
(5) \quad &= \left|\sum_{k=0}^{[n]-1} \psi_k(x) + \psi_{[n]}(x) \sum_{k=0}^{n-[n]1} \psi_k(x)\right| \\
&\leqq [n]\chi_{G_{\log[n]}}(x) + |D_{n-[n]}(x)|
\end{aligned}$$

by induction when we consider the three cases

$$x \in G_{1+\log[n]}, \quad x \in G_{\log[n]} - G_{1+\log[n]} \quad \text{and} \quad x \in 2^\omega - G_{\log[n]}.$$

Thus if $x \neq 0$

$$\lim_{n\to\infty} s_n(x; k) = (\gamma_0 - A) + \lim_{n\to\infty} A(2^{1-\alpha}-1) \sum_{k=1}^{n} 1/2^{k(1-\alpha)} D_{2^k}(x). \tag{6}$$

This latter sequence of functions is increasing with integrals bounded by

$$A(2^{1-\alpha}-1) \sum_{k=1}^{\infty} 1/2^{k(1-\alpha)} = A. \tag{7}$$

So by Beppo Levi the limit exists everywhere (but may be $+\infty$) and is in L_1. But K is continuous on 2^ω except at zero, so that if $x \neq 0$

$$\lim_{n\to\infty} s_n(x; K) = \lim_{n\to\infty} s_{2^n}(x; K)$$

$$= \lim_{n\to\infty} \int D_{2^n}(x+t) K(t)\, dt \tag{8}$$

$$= \lim_{n\to\infty} 2^n \int \chi_{G_n}(x+t) K(t)\, dt \tag{9}$$

by (3)

$$= K(x). \tag{10}$$

(ii) Since by the above

$$(\gamma_0 - A) + \lim_{n\to\infty} A(2^{1-\alpha}-1) \sum_{k=1}^{n} 1/2^{k(1-\alpha)} D_{2^k}(x) = K(x) \tag{11}$$

and the fact that these summands are nonnegative by 2.5 (27) and increasing we have

$$0 \leqq (\gamma_0 - A) + A(2^{1-\alpha}-1) \sum_{k=1}^{n} 1/2^{k(1-\alpha)} D_{2^k}(x) \leqq K(x). \tag{12}$$

Looking at (2) and (12) we see that it remains only to bound $A/[n-1]^{1-\alpha}|D_n(x)|$ by a constant multiple of $K(x)$:

$$1/[n-1]^{1-\alpha}|D_n(x)| \leqq 2^{1-\alpha}/[n]^{1-\alpha}|D_n(x)| \tag{13}$$

now verify that

$$1/[k]^{1-\alpha}|D_k(x)| \leqq 2K(x) \tag{14}$$

for $k=1$ and assume that it holds for $k<n$. Then

$$1/[n]^{1-\alpha}|D_n(x)| \leqq 1/[n]^{1-\alpha}([n]\chi_{G_{\log[n]}} + |D_{n-[n]}(x)|) \tag{15}$$

by (5)

$$\begin{aligned} &\leqq 2[n]^\alpha \leqq 2K(x) \qquad \text{if } x \in G_{\log[n]} \qquad \text{or} \\ &\leqq 2K(x) \qquad \text{if } x \in G_{\log[n]} \end{aligned} \tag{16}$$

by the inductive hypothesis. Thus

(17) $$1/[n-1]^{1-\alpha}|D_n(x)| \leqq 2^{2-\alpha}K(x)$$

and we have

(18) $$|s_n(x; K)| \leqq (2^{2-\alpha}A+1)K(x).$$

2.7. NOTE. If ν is in $\mathfrak{M}(E)$ and $d\nu \sim \sum_{n=0}^{\infty} c_n\psi_n$, then by the general theory of Fourier transforms

(1) $$U(x; \nu) = k * d\nu(x) \sim \sum_{n=0}^{\infty} \gamma_n c_n \psi_n(x)$$

and $I(\nu) = \iint k(x+y)\, d\nu(x)\, d\nu(y)$

(2) $$= \iint \lim_{n\to\infty} s_n(x+y; K)\, d\nu(x)\, d\nu(y) \quad \text{by 2.6 (i)}$$

(3) $$= \lim_{n\to\infty} \iint s_n(x+y; K)\, d\nu(x)\, d\nu(y) \quad \text{by 2.6 (ii)}$$

and the Lebesgue Dominated Convergence Theorem if $I(\nu)$ is finite

(4) $$= \sum_{n=0}^{\infty} \gamma_n c_n^2.$$

If $I(\nu) = \infty$ then $\sum_{n=0}^{\infty} \gamma_n c_n^2 = \infty$ by Fatou's Theorem. In either case (4) holds. And lastly, note that since the kernel is integrable with respect to Haar measure, any set of capacity zero is also of Haar measure zero.

2.8. The following two theorems are standard results in potential theory. Consult [7] and [8] for proofs.

THEOREM. *If E is of positive capacity, then there exists a unique μ in (E) such that $I(\mu) = V$.*

2.9. THEOREM. *The potential function, $U(x; \mu)$, of the equilibrium distribution has the following properties*:

(i) *$U(x; \mu) \geqq V$ nearly everywhere in E, i.e., except for a set which is of measure zero with respect to every measure of finite energy.*

(ii) *$U(x; \mu) \leqq V$ for all x in the support of μ.*

(iii) *$U(x; \mu)$ is bounded on 2^ω.*

3. Sets of divergence. We now arrive at the main theorem.

3.1. THEOREM. *Let*

(1) $$f(x) \sim \sum_{n=0}^{\infty} a_n \psi_n(x)$$

be such that

(2) $$\sum_{n=0}^{\infty} a_n^2 [n]^{1-\alpha} < \infty, \qquad 0 \leqq \alpha < 1,$$

where we define $[0] = \gamma_0/A$ for convenience. Then if $s_n(x; f)$ diverges on a closed set E, the α-capacity of E is zero.

Proof of theorem. Suppose the capacity of E is not zero. Then we have an equilibrium distribution μ for E such that $U(x;\mu)\leqq M$ on 2^ω.

From the hypothesis it is easy to construct $w(n)$ *nondecreasing and tending* to infinity so that

$$\sum_{n=0}^{\infty}[n]^{1-\alpha}w(n)a_n^2<\infty. \tag{4}$$

Let

$$A_n=(w)^{1/2}(n)a_n. \tag{5}$$

Then the partial sums

$$S_n(x)=\sum_{k=0}^{n-1}A_k\psi_k(x) \tag{6}$$

are unbounded on E. For, if not,

$$\begin{aligned} s_n(x;f) &= \sum_{k=0}^{n-1}((w)^{1/2}(k)a_k\psi_k(x))/(w(k))^{1/2} \\ &= \sum_{k=0}^{n-2}\left(\sum_{j=0}^{k}A_j\psi_j(x)\right)(1/(w)^{1/2}(j)-1/(w)^{1/2}(j+1)) \\ &\quad +1/w^{1/2}(n-1)\sum_{j=0}^{n-1}A_j\psi_j(x) \qquad \text{by Abel} \end{aligned} \tag{7}$$

and so $s_n(x;f)$ would converge. Define

$$E^{\pm}=\{x: S_n(x)\to\pm\infty\}. \tag{8}$$

Either $\mu(E^+)>0$ or $\mu(E^-)>0$, so without loss of generality we assume the former. Also for $n=1,2,\ldots$ let $n(x)\equiv$ the least $k\leqq n$ such that

$$S_k(x)=\max_{1\leqq j\leqq n}S_j(x). \tag{9}$$

Then $S_{n(x)}(x)=\max_{1\leqq j\leqq n}S_j(x)$ is a Borel measurable function, $S_{n(x)}(x)\geqq a_0$ and goes to $+\infty$ for all x in E^+. The upshot of all this then is that

$$I=\int S_{n(x)}(x)\,d\mu(x)\to\infty. \tag{10}$$

However, by the hypothesis and the Riesz-Fischer theorem there exists a function F in $L_2(2^\omega)$ such that

$$F(x)\sim\sum_{n=0}^{\infty}w^{1/2}(n)[n]^{(1-\alpha)/2}a_n\psi_n(x)=\sum_{n=0}^{\infty}[n]^{(1-\alpha)/2}A_n\psi_n(x). \tag{11}$$

Then

$$[k]^{(1-\alpha)/2}A_k\psi_k(x)=\int F(t)\psi_k(t)\,dt\cdot\psi_k(x)$$

so

$$A_k\psi_k(x)=\int\frac{F(t)\psi_k(t)}{[k]^{(1-\alpha)/2}}\,dt\psi_k(x). \tag{12}$$

Now we have

$$I = \int S_{n(x)}(x)\, d\mu(x) = \int \sum_{k=0}^{n(x)-1} A_k \psi_k(x)\, d\mu(x)$$

$$= \int \sum_{k=0}^{n(x)-1} \int \frac{F(t)\psi_k(x+t)}{[k]^{(1-\alpha)/2}}\, dt\, d\mu(x) \tag{13}$$

$$= \int F(t) \left[\int G_{n(x)}(x+t)\, d\mu(x) \right] dt \tag{14}$$

where

$$G_n(x) = \sum_{k=0}^{n-1} \psi_k(x)/[k]^{(1-\alpha)/2}. \tag{15}$$

Therefore

$$I^2 \leqq \int F^2(t)\, dt \cdot \int \left(\iint G_{n(x)}(x+s) G_{n(y)}(y+s)\, d\mu(x)\, d\mu(y) \right) ds \tag{16}$$

by Schwartz

$$= \|F\|_2^2 \iint \sum_{k=0}^{n(x,y)-1} (\psi_k(x+y)/[k]^{(1-\alpha)})\, d\mu(x)\, d\mu(y) \tag{17}$$

by the orthogonality of the Walsh functions.

Here

$$n(x, y) = \min \{n(x), n(y)\}$$

$$= \|F\|_2^2 \iint (1/AK_{n(x,y)}(x+y))\, d\mu(x)\, d\mu(y) \tag{18}$$

$$\leqq \|F\|_2^2 \iint (B/AK(x+y))\, d\mu(x)\, d\mu(y) \quad \text{by 2.7 (ii)} \tag{19}$$

$$= (B/A)\|F\|_2^2 \int U(y; \mu)\, d\mu(y)$$

$$\leqq BM\|F_2\|^2/A \quad \text{by 2.9 (iii).} \tag{20}$$

But this contradicts (10) so that the assumption that E is of positive capacity must be false.

We now begin maneuvering to prove a converse for the preceding theorem.

3.2. LEMMA. *Let $\{E_n\}_{n=1}^{\infty}$ be a nest of closed sets such that $\bigcap_{n=1}^{\infty} E_n = E$. Then if V_n does not go to infinity as $n \to \infty$, E is of positive capacity.*

Proof. Suppose V_n does not go to ∞, then there exists a subsequence of the E_n's, which we relabel $\{E_n\}_{n=1}^{\infty}$ such that $V(E_n) \leqq M < \infty$. By a well-known result the sequence of product measures derived from the equilibrium distributions on the E_n's has a weakly convergent subsequence whose weak limit we call ν. The corresponding subsequence of the μ_n's itself has a subsequence, $\{u_{n_j}\}_{j=1}^{\infty}$ say, which converges weakly to a measure $\mu \in \mathfrak{M}(E)$. By the same computation used to show uniqueness of the equilibrium potential distribution, $\nu = \mu \times \mu$, so

$$I(\mu) = \int K(x-y)\, d(\mu \times \mu)(x, y)$$

(1)
$$= \int K(x-y)\, d\nu(x, y)$$

$$= \lim_{j \to \infty} \int K(x-y)\, d\mu_{n_j}(x)\, d\mu_{n_j}(y)$$

$$\leqq M.$$

3.3. THEOREM. *If E is of capacity zero, then there is a Walsh-Fourier series*

(1)
$$\sum_{n=0}^{\infty} a_n \psi_n(x)$$

such that

(2)
$$\sum a_n^2 [n]^{1-\alpha} < \infty$$

and $s_n(x)$ goes to $+\infty$ for all x in E.

Proof. Let

(3)
$$E_N = \{x \in 2^\omega : |x - E| < 1/N\}.$$

Then E_N is of positive measure and so of positive capacity. By the results of §2 there exists an equilibrium distribution, μ_N, on E_N,

(4)
$$d\mu_N \sim \sum_{n=0}^{\infty} c_{nN} \psi_n$$

and a corresponding potential function

(5)
$$U_N(x) \sim A \sum_{n=0}^{\infty} c_{nN}/[n]1-\alpha \psi_n(x)$$

and potential

(6)
$$V_N = I(\mu_N) = A \sum_{n=0}^{\infty} c_{nN}^2/[n]1-\alpha < \infty$$

such that $U_N(x) \geqq V_N$ for almost all x in E_N (and therefore in E). However $\{E_N\}_{N=1}^{\infty}$ fulfills the hypothesis of Lemma 3.2 and so $V_N \to \infty$ since E is of capacity zero.

If we denote the Walsh-Fourier series of $U_N(x)$ by $\sum_{n=0}^{\infty} d_{nN} \psi_n(x)$, then

(7)
$$V_N = \sum_{n=0}^{\infty} d_{nN}^2 [n]^{1-\alpha}.$$

Looking at this situation from a slightly different angle, we assign to the function $U \in L_1$, $U(x) \sim \sum_{n=0}^{\infty} d_n \psi_n(x)$, such that $\sum_{n=0}^{\infty} d_n^2 [n]^{1-\alpha} < \infty$, the norm

$$\|U\| = \left(\sum_{n=0}^{\infty} d_n^2 [n]^{1-\alpha} \right)^{1/2}. \tag{8}$$

It is easily seen that these functions form a Hilbert space with this norm and that $\|U(x; \nu)\| = (I(\nu))^{1/2}$. This we call the *energy norm.*

Since $U_N(x) \geqq 0$ throughout 2^ω, the $2n$th partial sums of its Walsh-Fourier series are nonnegative also. And as $U_N(x) \geqq V_N$ almost everywhere on E_N, we have that $s_{2n}(x; U_N)$ is eventually greater than $V_N - 1$ at each interior point of E_N and in particular at each point of E.

Choose a sequence $N(j)$ going to ∞ sufficiently rapidly so that

$$(V_{N(j)})^{1/2} j^2 \to \infty \tag{9}$$

and consider the series

$$\sum_{n=1}^{\infty} (U_{N(j)}(x)/j^2 (V_{N(j)})^{1/2}). \tag{10}$$

It converges in the energy norm, so let

$$U(x) \sim \sum_{n=0}^{\infty} d_n \psi_n(x) \tag{11}$$

denote its sum. Then

$$\begin{aligned} s_{2^n}(x; U) &= \int D_{2^n}(x-y) \sum_{j=1}^{\infty} (U_{N(j)}(x)/j^2 (V_{N(j)})^{1/2})\, dy \\ &= \sum_{j=1}^{\infty} (1/j^2 (V_{N(j)})^{1/2}) \int D_{2^n}(x-y) U_{N(j)}(y)\, dy \qquad \text{by monotone convergence} \\ &= \sum_{j=1}^{\infty} (1/j^2 (V_{N(j)})^{1/2}) s_{2^n}(x; U_{N(j)}) \\ &\to \infty \qquad \text{for } x \in E. \end{aligned} \tag{12}$$

Q.E.D.

4. **Conclusion.** In a sense our main result, Theorem 3.1 has begged the original question, which was to determine if the Salem-Zygmund theorem held for Walsh series as defined in the classical way on [0, 1]. The question has been reduced then to: How does the natural mapping $\lambda: 2^\omega \to [0, 1]$ treat capacity? In particular does it preserve sets of capacity zero?

If x and y are in 2^ω, then $|\lambda(x) - \lambda(y)| \leqq |x - y|$, i.e., the natural mapping from 2^ω to $[0, 1]$ is a contraction. Thus if a set is of zero capacity in 2^ω its image under λ is also of zero capacity. Whether the reverse implication is true is not known.

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