SIMPLE-CONNECTIVITY AND THE BROWDER-NOVIKOV THEOREM

BY

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In this note we construct a family of odd-dimensional, closed, combinatorial manifolds, none of which has the homotopy type of a closed differentiable manifold. These manifolds all have an infinite cyclic fundamental group.

W. Browder [1] and S. P. Novikov [9] have proved that a simply-connected finite complex, $K$, satisfying Poincaré duality with respect to an odd-dimensional fundamental class $u \in H_{2n+1}(K)$, $(n \geq 2)$ has the homotopy type of a closed $(2n+1)$-dimensional differential manifold provided there exists a vector bundle, $\xi$, over $K$ whose Thom space $T(\xi)$ has a spherical top homology class (i.e., $\pi_q(T(\xi)) \rightarrow H_q(T(\xi))$ is surjective for $q = 2n + 1 + \dim \xi$).

Denoting one of our combinatorial manifolds by $M$, we prove that $SM$, the suspension of $M$, has a spherical top homology class, so that the trivial real line bundle over $M$ satisfies the Browder-Novikov hypothesis. The manifolds, $M$, show that the Browder-Novikov theorem cannot be extended to the nonsimply-connected case, at least not without additional hypotheses on $K$. (Recall that the Thom space of the trivial real line bundle over a space $X$ has the homotopy type of $S^1 \vee SX$.)

The manifolds $M^{2n+1}$ are constructed from certain knotted homotopy $(2n-1)$-spheres in $S^{2n+1}$ ($n$ is odd). Let $A$ be (the space of) the tangent unit disk bundle over $S^n$, and $W$ the differential manifold with boundary obtained by plumbing together two copies $A_1$, $A_2$ of $A$. (See [4], [5], or [8].) The images $S_1$, $S_2$ in $W$ of the zero cross-sections in $A_1$ and $A_2$ respectively have a single (transversal) intersection point. Denote by $\Sigma^{2n-1}$ the boundary of $W$. It was proved in [8], or more generally follows from Smale theory, that $\Sigma^{2n-1}$ is combinatorially equivalent to $S^{2n-1}$.

We now imbed $W$ into $S^{2n+1}$. It is well known, and easy to see, that $W$ can be differentiably imbedded into $S^{2n+1}$ so that if $v$ denotes a normal vector-field on $W$ in $S^{2n+1}$, and if $S_1'$, $S_2'$ are the translates of $S_1$, $S_2$ by a small positive amount $\varepsilon$ along $v$, then $\lambda_i = L(S_i', S_i')$ for $i = 1, 2$, are odd integers, where $L(\ , \ )$ denotes the linking coefficient in $S^{2n+1}$. (For further remarks on this see the lemma on normal bundles at the end of this paper.)

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Let $\phi: \Sigma^{2n-1} \times D^2 \to S^{2n+1}$ be a tubular neighborhood of $bW$, the boundary of $W$, and let $h: S^{2n-1} \to \Sigma^{2n-1}$ be a combinatorial equivalence. Then

$$\psi: S^{2n-1} \times D^2 \to S^{2n+1}$$

given by $\phi \circ (h \times \text{id})$ is a combinatorial imbedding. Let $M^{2n+1} = \chi(S^{2n+1}, \psi)$ be the combinatorial manifold obtained from $S^{2n+1}$ by spherical modification:

$$M^{2n+1} = (S^{2n+1} - \psi(S^{2n-1} \times B^2)) \cup D^{2n} \times S^1,$$

where $B^2 = \text{int } D^2$, and $(x, y) \in S^{2n-1} \times S^1 \subset D^{2n} \times S^1$ is identified with $\psi(x, y)$. As usual we denote by $\psi': D^{2n} \times S^1 \to M$ the imbedding induced by the inclusion.

**Proposition.** The manifolds $M^{2n+1}$ constructed above for odd $n$ have the following properties for $n > 1$:

1. $\pi_i M \cong \pi_i(S^1)$ for $i < n$;
2. $\pi_n M$ as a $\mathbb{Z}[J]$-module, where $J = \pi_1(S^1)$ generated by $t$, is isomorphic to the $\mathbb{Z}[J]$-module with presentation

$$\{x_1, x_2; \lambda_1(t-1)x_1 + (\lambda(t-1) + 1)x_2, \lambda(t-1) - t)x_1 + \lambda_2(t-1)x_2\}.$$

(Here $\lambda$ is an integer depending on the imbedding $W \subset S^{2n+1}$, and can be chosen arbitrarily.)

3. $H_\ast(M) \cong H_\ast(S^1 \times S^{2n})$;
4. $\pi_{2n+2}(SM) \to H_{2n+2}(SM)$ is surjective, where $SM$ denotes the suspension of $M$.
5. For $n = 3$, $7$ the combinatorial equivalence $h: \Sigma^{2n-1} \to S^{2n-1}$ can be taken to be a diffeomorphism so that the manifold $M^{2n+1}$ has a differential structure. For $n = 4k + 1$, $M^{2n+1}$ does not have the homotopy type of any closed differential manifold.

**Remark.** Since isomorphic $\mathbb{Z}[J]$-modules have identical elementary ideals, and the 0th elementary ideal of the module under (2), i.e., the ideal generated by the determinant of the relation matrix, is $(\lambda_1 \lambda_2 - \lambda^2 + \lambda(t-1)^2 + t$ it follows that by varying the coefficients $\lambda_1, \lambda_2, \lambda$ we can get infinitely many distinct homotopy types for the manifold $M^{2n+1}$.

The proofs of (1) and (2) rely on the method for calculating the homotopy groups of the complement of a knot, exposed in [4]. It is clear that

$$M - \psi'(D^{2n} \times S^1) = S^{2n+1} - \psi(S^{2n-1} \times D^2).$$

Hence $\pi_i M = \pi_i(S^{2n+1} - bW)$ for $i < 2n - 1$. Now, a homomorphism

$$I: \pi_1(S^{2n+1} - bW) \to \mathbb{Z}$$

is given by assigning to $\alpha \in \pi_1(S^{2n+1} - bW)$ the intersection coefficient with $W$ of a representative curve $f: S^1 \to S^{2n+1} - bW$. Clearly, $I$ is surjective. To prove that $I$ is injective, let $U$ be a tubular neighborhood of $W$ and let $Y = S^{2n+1} - U$. Then first, $\pi_1 Y = \{1\}$ by the van Kampen theorem, since $W$ and hence $bY$ are simply connected. Secondly, $\pi_1(Y, bY) = \{1\}$ by the homotopy exact sequence of $(Y, bY)$. 


An element \( \alpha \in \text{Ker } I \) can be represented by a differentiably imbedded curve \([0, 1] \to S^{2n+1} - bW\) which intersects \( W \) transversally in a finite number of interior points. Since \( I(\alpha) = 0 \), there will be a pair of consecutive intersection points with \( W \) having opposite intersection coefficients. The arc joining these points represents an element of \( \pi_1(Y, bY) \) and thus is homotopic to an arc in \( U \) which can then be pushed away from \( W \), reducing the number of intersection points by 2. Eventually, we get a representative of the given element \( \alpha \) whose image is contained in \( Y \). Since \( Y \) is simply connected, \( \alpha = 1 \).

To calculate \( \pi_1 M \) for \( i \leq n \), denote by \( W_+ \) and \( W_- \) the two copies of \( W \) in \( bU = bY \). Let \( X_1 = S^{2n+1} - bW \). Following Seifert, we construct the universal covering \( X \) of \( X_1 \) as the union of countably many copies of \( Y \) which we denote by \( Y^k, k \in \mathbb{Z} \), with \( W^k \) identified with \( W^{k+1} \) for every \( k \). Then \( \pi_i(S^{2n+1} - bW) = 0 \) for \( 2 \leq i \leq n \), follows immediately from \( \pi_i = 0 \) for \( i < n \). The module \( \pi_n(S^{2n+1} - bW) \) is isomorphic to \( H_n(X) \) for which we get a presentation by the Mayer-Vietoris theorem. If we denote by \( \xi_1, \xi_2 \) the generators of \( H_nW \) represented by \( S_1, S_2 \) respectively, \( \nu_+ \) and \( \nu_- \) the obvious mappings of \( W \) onto \( W_+ \) and \( W_- \), then a presentation for \( H_n(X) \) is obtained from a presentation of \( \pi_nY \) (as an abelian group) by adjoining the relations \( \nu_+(\xi_i) = \nu_-(\xi_i) \). Now, it is easy to see that \( H_nY \) is free abelian on 2 generators \( x_1, x_2 \), and if \( S \) is a sphere in \( Y \), the class of \( S \) is given by

\[
(S) = L(S, x_1) + L(S, x_2).
\]

The presentation for \( H_n(X) \cong \pi_n M \) claimed in (2) follows readily, with \( \lambda = L(S_1, S_2) \).

The proof of (3) is immediate.

To prove (4) we first observe that \( M \) has the homotopy type of a cell complex of the form

\[
K = (S^1 \vee S^1_1 \vee \cdots \vee S^1_n) \cup e_1^{n+1} \cup \cdots \cup e_n^{n+1} \cup e_2^n \cup e_2^{n+1}.
\]

(In fact, it can be proved using Smale theory that \( M \) has a handle decomposition inducing the above cell structure. See [4].) Since \( H_nM = 0 \), we have, up to homotopy type

\[
SM = (S^2 \cup f e^{2n+1}) \cup g e^{2n+2}.
\]

Now, since \( H^1 M = \mathbb{Z} \), there is a map \( M \to S^1 \) whose composition with the inclusion \( S^1 \to M \) is homotopic to the identity \( S^1 \to S^1 \). Taking the suspension of these maps we see that \( S^2 \) is a retract of \( M \). It follows that the attaching map \( f \) is trivial, and thus

\[
SM = (S^2 \vee S^{2n+1}) \cup g e^{2n+2},
\]

up to homotopy type. Since \( H_{2n+2} M = \mathbb{Z} \), it follows that \( g \) must have degree 0 (on \( S^{2n+1} \)), and since \( \pi_{2n+1}(S^2 \vee S^{2n+1}) = \pi_{2n+1}(S^2) + \pi_{2n+1}(S^{2n+1}) \), \( g \) is homotopic to a mapping \( h \) into \( S^2 \). Thus \( SM \) has the homotopy type of

\[
S^{2n+1} \vee (S^2 \cup g e^{2n+2}).
\]
Using again the retraction $SM \to S^2$, we see that $h$ is trivial. So $SM$ has the homotopy type of

$$S^2 \vee S^{2n+1} \vee S^{2n+2}.$$  

(4) is now obvious.

The proof of (5) will rely essentially on recent results of E. Brown and F. Peterson [2]. Suppose $M^{2n+1}$ is a closed differential manifold ($n$ odd, $\neq 3, 7$) with the properties (1), (2), and (3). We construct a knot as follows. Let $\phi: S^1 \times D^{2n} \to M$ be a differentiable embedding representing the generator $t \in \pi_1 M$. Performing a spherical modification we obtain a manifold $\Sigma^{2n+1} = \chi(M, \phi)$ which is easily seen to be a differential homotopy sphere. Replacing $M^{2n+1}$ by the connected sum $(M \# (-\Sigma))$ if necessary, we may assume that $\Sigma^{2n+1}$ is diffeomorphic to $S^{2n+1}$. (This operation is not in fact really necessary for what follows.) Since

$$S^{2n+1} = (M - \phi(S^1 \times B^{2n})) \cup D^2 \times S^{2n-1},$$

we have an imbedding $f: S^{2n-1} \to S^{2n+1}$. (It is however essential that $S^{2n-1}$ here be the sphere with the usual differential structure.) An argument similar to the one used in the proof of (1) and (2) shows that $\pi_i(S^{2n+1} - f(S^{2n-1})) = \pi_i(S^1)$ for $i < n$, and $\pi_n(S^{2n+1} - f(S^{2n-1}))$ is the $Z[J]$-module whose presentation is given in (3). It is well known that $f(S^{2n-1})$ is the boundary of an orientable submanifold $V$ of $S^{2n+1}$. Moreover J. Levine has proved that the manifold $V$ can be taken to be $(n-1)$-connected as a consequence of $\pi_i(S^{2n+1} - f(S^{2n-1})) = \pi_i(S^1)$ for $i < n$. It follows from Poincaré duality (for $V$) that we may find a basis, $\{\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n\}$, for $H_n(V)$ having the intersection numbers $\delta_{ij} = I(\xi_i, \eta_j)$, $O = I(\xi_i, \xi_j) = I(\eta_i, \eta_j)$. Using a normal vector field along $V$ we may imbed $V \times [\xi, \xi]$ in $S^{2n+1}$ in such a way that $V$ corresponds to $V \times \{0\}$. If $\alpha \in H_n(V)$ we denote by $\alpha^\pm$ the elements of $H_n(S^{2n+1} - V)$ represented by the corresponding class in $V \times \{-\varepsilon, +\varepsilon\} \subseteq S^{2n+1} - V$. Since any $\alpha \in H_n(V)$ can be represented by an imbedded sphere and any two such can be put in general position it is clear that $L(\alpha^+, \beta) - L(\alpha^-, \beta) = L(\alpha^+ - \alpha^-, \beta) = \delta_{ij} = I(\alpha, \beta)$. It now follows from the methods used to establish (3) that there is a presentation of $\pi_n(S^{2n+1} - \partial V)$ with $2s$ generators and the relation matrix

$$R = \begin{bmatrix}
(t-1)A & (t-1)B + E \\
(t-1)B' - tE & (t-1)C
\end{bmatrix}$$

where $E$ is the $s \times s$ identity matrix, $B'$ is the transpose of $B$ and $A = \|L(\xi^+_i, \xi_j)\|$, $B = \|L(\xi^+_i, \eta_j)\|$ and $C = \|C_{ij}\| = \|L(\eta^+_i, \eta_j)\|$. (We remark that the matrices $A$ and $C$ are symmetric (because $n$ is odd).) Using the lemma below we recognize $a_{ii} \mod 2$ as the obstruction to trivializing the normal bundle in $V$ to an imbedded sphere representing $\xi_i$. A similar relation holds between $c_{ii}$ and $\eta_i$ and so

$$c(V) = \sum_{i=1}^s a_{ii}c_{ii} \mod 2$$

is the Arf invariant of the quadratic form of $V$ as defined in [6].
Now, det \( R \) is the 0th elementary ideal of \( \pi_\ast(M) = \pi_\ast(S^{2n+1} - f(S^{2n-1})) \) and hence det \( R \) and \( (\lambda_1 \lambda_2 - \lambda^2 + \lambda)(t-1)^2 + t \) must generate the same ideal in \( \mathbb{Z}[J] \). An argument of Robertello's [10], sketched below for the convenience of the reader, shows that det \( R = t^s(1 + t^2)c(V) \) modulo the ideal generated by 2 and \( (t-1)^4 \).

Since \( (\lambda_1 \lambda_2 - \lambda^2 + \lambda)(t-1)^2 + t = (t^2 + 1) + t \mod 2 \), it follows that \( C(V) = 1 \mod 2 \) but since \( \partial V \) is diffeomorphic to \( S^{2n-1} \) this contradicts the Brown-Peterson result [2] when \( n = 4k + 1, \ k \geq 1 \).

Robertello's argument in brief is this. Let \( R = (x_{ij}) \)—thus \( x_{ij} \) is divisible by \( t-1 \) except if \( 1 \leq i \leq s \) and \( j = i+s \) or \( 1 \leq j \leq s \) and \( i = j+s \). Let \( S_{a,b} \) be the set of permutations \( (i_1, \ldots, i_{2s}) \) for which \( i_k \neq s+k \) for exactly \( \alpha \) values of \( k \in [1, s] \) and \( i_{s+k} \neq k \) for exactly \( \beta \) values of \( k \in [1, s] \). Thus

\[
\det R = \sum_{0 \leq a, b \leq s} \sum_{S_{a,b}} x_{1,i_1}x_{2,i_2} \cdots x_{2s,i_{2s}}.
\]

Now the individual terms in \( S_{a,b} \) for \( \alpha, \beta \in [2, s] \) are divisible by \( (t-1)^4 \) so that we need only consider the first few \( S_{a,b} \)'s. \( S_{0,0} \) contains only the permutation \( (s+1, \ldots, 2s, 1, 2, \ldots, s-1, s) \) which gives rise to the term

\[
\prod_{i=1}^{s} [(t-1)b_{ii}t][(t-1)b_{ii} + 1] = t^s \mod 2.
\]

The sets \( S_{0,1} \) and \( S_{1,0} \) are empty. The sum \( \sum_{S_{1,1}} \) is

\[
\sum_{i,k} \left( \prod_{j=1; j \neq i}^{s} ((t-1)b_{jj} - t) \right) \left( \prod_{j=1; j \neq k}^{s} ((t-1)b_{jj} + 1) \right) = t^s \mod 2.
\]

Further similar calculation shows that \( \sum_{S_{2,1}} + \sum_{S_{1,2}} = 0 \). (Essential use is made of the fact that \( A \) and \( C \) are symmetric.)

Before stating the lemma on normal bundles, recall that an \( n \)-dimensional bundle, \( \xi^n \), over an \( n \)-sphere determines an element \( [\xi^n] \in \pi_\ast(BSOn) \)—where \( BSOn \) is a classifying space for the group \( SOn \). We denote by \( T_n \) the tangent bundle of \( S^n \).

**Lemma on normal bundles.** Let \( S^n \subset S^{2n+1} \) be a differentiable imbedding, \( \nu \) a never vanishing normal field, \( \bar{S}^n \subset S^{2n+1} \) a disjointly imbedded sphere obtained by "pushing" \( S^n \) along \( \nu \) and finally \( \eta \), the complementary normal bundle—i.e., \( \eta(x) = \) the vectors normal to \( S^n \) at \( x \) but perpendicular to \( \nu(x) \). Then

\[
[\eta] = L(S^n, \bar{S}^n)[T_n] \in \pi_\ast(BSOn).
\]

**Proof.** There is no loss in generality (see [3]) in assuming that the imbedding is the usual one—to wit \( S^n \subset S^n \times R \subset R^{n+1} = R^{n+1} \times 0 \subset R^{n+1} \times R^n = S^{2n+1} - \infty \).

Thus we may refer the normal vector field, \( \nu \), to the *standard* framing of this normal bundle—thus \( \nu \) and \( \eta \) are described completely by a function \( f_\nu : S^n \to R^{n+1} - 0 \).

Since the entities involved in our assertion are unchanged if we vary \( \nu \) (through never-zero normal fields) we may assume that \( f_\nu \) is a differentiable map to \( S^n \) having the south pole as a regular value. Then it is clear that \( \bar{S}^n \) intersects \( D^{n+1} \times 0 \).
transversally once for each inverse image of the south pole and in fact \( L(S^n, \mathbb{S}^n) \) = the (algebraic) number of such inverse images = the degree of \( f_\nu \). On the other hand, we clearly now have \( \eta = \{(x, v) \in S^n \times R^{n+1} \mid f_\nu(x) = v \} \) and this is obviously the “pull-back” under \( f_\nu \) of the tangent bundle of \( S^n \). Thus \( [\eta] = \deg f_\nu [T_n] \).

**Bibliography**


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