

EXTREMAL LENGTH AND CONFORMAL CAPACITY⁽¹⁾

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1. **Introduction.** The theory of extremal length has been successfully applied to conformal mappings, analytic functions of a complex variable, and in recent years it has found application in the study of quasiconformal mappings in space. Its application in quasiconformal mapping theory begins essentially with a theorem proved by Gehring [G3], that the conformal capacity of a space ring R is directly related to the extremal length of the family of curves that join the boundary components of R . He has also shown that the conformal capacity is related to the extremal length of a family of surfaces that separate the boundary components of R . Gehring assumes that the separating surfaces are sufficiently smooth; Krivov [K] establishes a similar result under the assumption that the extremal metric is well-behaved. Under similar assumptions, other authors have dealt with the extremal length of separating surfaces, cf. [FU, Theorem 9], [SA], and [H].

The purpose of this paper is to eliminate the need for these assumptions. Thus, we consider the case of two disjoint compact sets C_0, C_1 contained in the closure of a bounded, open, connected set G . It is proved that the conformal capacity C of C_0, C_1 , relative to G is related to the $n/n-1$ -dimensional module M of all closed sets that separate C_0 from C_1 in the closure of G by

$$C^{-1/(n-1)} = M.$$

This is accomplished by using a technique of Gehring's [G1, Lemma 3] which eliminates all assumptions concerning the behavior of the extremal metric. Then, a surface-theoretic approximation theorem, first developed in [FF, 8.23], permits the consideration of arbitrary closed separating sets.

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2. Preliminaries.

2.1. E^n will denote Euclidean n -space and L_n is n -dimensional *Lebesgue measure*. *Hausdorff k -dimensional measure* in E^n will be denoted by H^k (for its definition and properties see [F2]) and in this paper, only H^1 and H^{n-1} will be used. If $A \subset E^n$, then *bdry A* will mean the boundary of A and for $x \in E^n$, $\delta(x, A)$ will be the *distance from x to A* . More generally, $\delta(A, B)$ will denote the *distance between*

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the sets A and B . $S(x, r)$ will stand for the open n -ball of radius r and centered at x . If A is an L_n -measurable subset of E^n , then $\mathcal{L}^p(A)$ will denote those functions f for which $|f|^p$ is L_n -integrable over A and $\|f\|_p$ will be its \mathcal{L}^p norm.

2.2. A real valued function u defined on an open set $U \subset E^n$ is said to be *absolutely continuous in the sense of Tonelli* (ACT) on U if it is ACT on every interval $I \subset U$, cf. [SS, p. 169]. Thus, the gradient of u , ∇u , exists at L_n -almost every point in U . The following co-area formula, which was proved in [Z2], will be used frequently throughout this paper.

2.2.1. THEOREM. *If A is an L_n -measurable subset of U and u is ACT on U , then*

$$\int_A |\nabla u(x)| dL_n(x) = \int_{-\infty}^{\infty} H^{n-1}[u^{-1}(s) \cap A] dL_1(s).$$

Therefore,

$$\int_U f(x) |\nabla u(x)| dL_n(x) = \int_{-\infty}^{\infty} \int_{u^{-1}(s)} f(x) dH^{n-1}(x) dL_1(s)$$

whenever $f \in \mathcal{L}^1(U)$.

2.3. *Conformal capacity.* The following will be standard notation throughout. G is an open, bounded, connected set in E^n and C_0, C_1 , are disjoint compact sets in the closure of G . We will let $R = G - (C_0 \cup C_1)$ and $R^* = R \cup C_0 \cup C_1$. The *conformal capacity of C_0, C_1 relative to the closure of G* is defined as

$$(1) \quad C[G, C_0, C_1] = \inf \int_R |\nabla u|^n dL_n$$

where the infimum is taken over all functions u which are continuous on R^* , ACT on R , and assume boundary values 1 on C_1 and 0 on C_0 . Such functions are called *admissible* for $C[G, C_0, C_1]$. Sometimes we will write C for $C[G, C_0, C_1]$.

If $C_0 \cup C_1 \subset G$ and if $C_0 \cup C_1$ consists of only a finite number of nondegenerate components, then the arguments of [G2, §§3-7] can be applied with only slight modifications to prove that the infimum in (1) is attained by a unique admissible function u which is ACT in G . (By using the methods of Chapter III in [FU], one can prove the existence of an extremal for more general situations.) This extremal function u satisfies the variational condition

$$(2) \quad \int_R |\nabla u|^{n-2} \nabla u \cdot \nabla w dL_n = 0$$

for any function w which is ACT on G , assumes boundary value 0 on $C_0 \cup C_1$, and for which $|\nabla w| \in \mathcal{L}^n(R)$.

2.4. *Integral currents.* The following notion of exterior normal, which was first introduced in [F1, p. 48], will be used later.

An L_n -measurable set $E \subset E^n$ has the unit vector $n(x)$ as exterior normal to E at x if, letting

$$\begin{aligned} S(x, r) &= \{y : |y-x| < r\}, \\ S_+(x, r) &= S(x, r) \cap \{y : (y-x) \cdot n(x) \geq 0\} \\ S_-(x, r) &= S(x, r) \cap \{y : (y-x) \cdot n(x) \leq 0\} \\ \alpha &= L_n[S(x, 1)] \end{aligned}$$

we have

$$2 \lim_{r \rightarrow 0} L_n[S_-(x, r) \cap E] / \alpha r^n = 1, \quad 2 \lim_{r \rightarrow 0} L_n[S_+(x, r) \cap E] / \alpha r^n = 0.$$

The set of points x for which $n(x)$ exists is called the *reduced boundary of E* and will be denoted by $\beta(E)$. Obviously, $\beta(E) \subset \text{bdry } E$. The importance of the exterior normal is seen in the following general version of the Gauss-Green Theorem [F3], [DG].

2.4.1. THEOREM. *If E is a bounded L_n -measurable set with $H^{n-1}[\beta(E)] < \infty$, then*

$$\int_E \text{div } f(x) \, dL_n(x) = \int_{\beta(E)} f(x) \cdot n(x) \, dH^{n-1}(x)$$

whenever $f: E^n \rightarrow E^n$ is continuously differentiable.

This theorem enables us to regard a bounded, open set $U \subset E^n$ with

$$H^{n-1}(\text{bdry } U) < \infty$$

as an *integral current mod 2* (or integral class), i.e., an integral current T with coefficients in the group of integers mod 2, [Z1, §3.6] or [FL]. Thus, if ϕ is a differential n -form of class C^∞ , then

$$T(\phi) = \int_U \phi \, dL_n.$$

The *boundary of T* , ∂T , is defined as $\partial T(\omega) = T(d\omega)$ whenever ω is an $n-1$ form and $d\omega$ is its exterior derivative. Now $\beta(U)$ is a Hausdorff $(n-1)$ -rectifiable set and therefore, 2.4.1 allows us to identify $\beta(U)$ with ∂T , [Z1, §3.6]. Thus, the *support of ∂T* is $\text{cl } (\beta(U)) \subset \text{bdry } U$, the *mass of T* is $M(T) = L_n(U)$, and $M(\partial T) = H^{n-1}[\beta(U)]$ [Z1, §3.6].

In view of this identification the following theorem is an immediate consequence of [Z1, 6.2] although the original version was given in [FF, 8.23]. An open set is called a *convex cell* if it can be expressed as the finite intersection of open half-spaces and an *n -dimensional polyhedral set* is the union of a finite number of convex cells.

2.4.2. THEOREM. *If $U \subset E^n$ is a bounded, open set with $H^{n-1}(\text{bdry } U) < \infty$, then there is a sequence of n -dimensional polyhedral sets P_i such that*

- (i) $P_i \subset \{x : \delta(x, U) < i^{-1}\}$,
- (ii) $L_n(P_i) \rightarrow L_n(U)$ as $i \rightarrow \infty$,
- (iii) $H^{n-1}(\text{bdry } P_i) \rightarrow H^{n-1}[\beta(U)]$ as $i \rightarrow \infty$.

Moreover, by employing an argument similar to that of [FL, (5.6) and (5.7)], it can be arranged that

$$(iv) \text{ bdry } P_i \subset \{x : \delta[x, \beta(U)] < i^{-1}\}.$$

Of course, one could apply [FL, 5.6 and p. 170] directly after verifying that the boundary of U in Fleming's sense is $\beta(U)$. Finally, defining the measure μ_i as the restriction of H^{n-1} to bdry P_i and μ as the restriction of H^{n-1} to $\beta(U)$, (iii) above and [Z1, 5.7] or [FL, 4.2] imply that

$$(v) \mu_i \rightarrow \mu \text{ weakly as } i \rightarrow \infty,$$

that is,

$$\int f d\mu_i \rightarrow \int f d\mu \text{ for every continuous } f.$$

2.5. *The module of a system of measures.* Instead of dealing with extremal length, we prefer to work with the module as developed in [FU].

Let \mathcal{M} be the class of all Borel measures on E^n . With an arbitrary system $E \subset \mathcal{M}$ of measures is associated a class of nonnegative Borel functions f subject to the condition

$$(3) \int_{E^n} f d\mu \geq 1 \quad \text{for every } \mu \in E.$$

We will write $f \wedge \mu$ if (3) is satisfied and $f \wedge E$ if (3) is satisfied for every $\mu \in E$. For $0 < p < \infty$, the *module of E* , $M_p(E)$, is defined as

$$(4) M_p(E) = \inf \left\{ \int_{E^n} f^p dL_n : f \wedge E \right\}.$$

A statement concerning measures $\mu \in \mathcal{M}$ is said to hold for M_p -a.e. μ if the statement fails to hold for only a set of measures E_0 where $M_p(E_0) = 0$. The proofs of the following statements can be found in [FU, Chapter 1].

- (i) $M_p(E) \leq M_p(E')$ if $E \subset E'$,
- (ii) $M_p(E) \leq \sum_{i=1}^{\infty} M_p(E_i)$ if $E = \bigcup_{i=1}^{\infty} E_i$.
- (iii) If $\bar{\mu}$ is the completion of μ and $L_n(A) = 0$, then $\bar{\mu}(A) = 0$ for M_p -a.e. $\mu \in \mathcal{M}$.
- (iv) If $f \in \mathcal{L}^p(E^n)$, then f is $\bar{\mu}$ integrable for M_p -a.e. $\mu \in \mathcal{M}$.
- (v) If $\|f_i - f\|_p \rightarrow 0$, then there is a subsequence f_{i_j} such that

$$\int_{E^n} |f_{i_j} - f| d\bar{\mu} \rightarrow 0 \quad \text{for } M_p\text{-a.e. } \mu \in \mathcal{M}.$$

(vi) If $E \subset \mathcal{M}$, then $M_p(E) = 0$ if and only if there is a nonnegative Borel function $f \in \mathcal{L}^p$ such that

$$\int_{E^n} f d\mu = \infty \quad \text{for every } \mu \in E.$$

(vii) If $p > 1$ and $E \subset \mathcal{M}$, then there is a nonnegative Borel function f such that

$$\int_{E^n} f^p dL_n = M_p(E)$$

and $\int f d\mu \geq 1$ for M_p -a.e. $\mu \in E$.

For the applications in this paper, the measures μ are complete (in fact, they are the restrictions of Hausdorff measure to compact sets) and for such measures, we have the following

2.5.1 THEOREM. *If $p \geq 2$, $E_1 \subset E_2 \subset \dots$ are sets of complete measures, and $E = \bigcup E_i$, then*

$$M_p(E) = \lim_{i \rightarrow \infty} M_p(E_i).$$

Proof. In view of (i) above, observe that the limit exists and is dominated by $M_p(E)$. Therefore, only the case where the limit is finite needs to be considered.

For each i , let f_i be the Borel function associated with E_i as in (vii) above. Clarkson's inequality [C] states, for any i and j ,

$$(5) \quad \int_{E^n} \left| \frac{f_i + f_j}{2} \right|^p dL_n + \int_{E^n} \left| \frac{f_i - f_j}{2} \right|^p dL_n \leq 2^{-1} \int_{E^n} |f_i|^p dL_n + 2^{-1} \int_{E^n} |f_j|^p dL_n.$$

If $i > j$, then $2^{-1} \cdot (f_i + f_j) \wedge \mu$ for M_p -a.e. $\mu \in E_j$. Therefore, because of (5),

$$(6) \quad \int_{E^n} \left| \frac{f_i - f_j}{2} \right|^p dL_n \leq 2^{-1} M_p(E_i) - 2^{-1} M_p(E_j).$$

The right side of (6) tends to zero as $i, j \rightarrow \infty$ with $i > j$ and therefore, there is a nonnegative function f such that $\|f_i - f\|_p \rightarrow 0$. Thus, from the above properties of module (especially (v)), we have that $f \wedge \mu$ for M_p -a.e. $\mu \in E$. This implies

$$M^p(E) \leq \int_{E^n} f^p dL_n = \lim_{i \rightarrow \infty} M_p(E_i)$$

which is all that is required to prove.

3. Module of separating sets and conformal capacity. In this section we will establish the relationship between conformal capacity and arbitrary closed separating sets. G, R^*, R, C_1 , and C_0 will be as defined in §2.3.

We will say that a set $\sigma \subset E^n$ separates C_0 from C_1 in R^* if $\sigma \cap R$ is closed in R and if there are disjoint sets A and B which are open in $R^* - \sigma$ such that $R^* - \sigma = A \cup B$, $C_0 \subset A$, and $C_1 \subset B$. Let Σ denote the class of all sets that separate C_0 from C_1 in R^* . With every $\sigma \in \Sigma$, associate a complete measure μ in the following way: for every H^{n-1} -measurable set $A \subset E^n$, define

$$\mu(A) = H^{n-1}(A \cap \sigma \cap R).$$

From the properties of Hausdorff measure, it is clear that the Borel sets of E^n are μ -measurable and therefore the module of Σ can be as defined as in 2.5. Thus, for $n' = n/n - 1$,

$$M_{n'}(\Sigma) = \inf \left\{ \int_{E^n} f^{n'} dL_n : f \wedge \Sigma \right\}$$

where $f \wedge \Sigma$ means that f is a nonnegative Borel function on E^n such that

$$\int_{\sigma \cap R} f dH^{n-1} \geq 1 \quad \text{for every } \sigma \in \Sigma.$$

3.1. As in [V1, 2.3] it can be shown that if Σ' denotes those $\sigma \in \Sigma$ for which $H^{n-1}(\sigma \cap R) = \infty$, then $M_{n'}(\Sigma') = 0$.

3.2. LEMMA. *Let u be an admissible function for $C[G, C_0, C_1]$ (see 2.3) and let $S \subset [0, 1]$ be an L_1 -measurable set. If $\Sigma(S) = \{u^{-1}(s) : s \in S\}$ and $M_{n'}[\Sigma(S)] = 0$, then $L_1(S) = 0$.*

Proof. Since $M_{n'}[\Sigma(S)] = 0$, (vi) of 2.5 provides a nonnegative Borel function $f \in \mathcal{L}^{n'}$ such that

$$\int_{u^{-1}(s) \cap R} f dH^{n-1} = \infty \quad \text{for every } s \in S.$$

However, Hölder's inequality and the co-area formula (2.2.1) imply

$$\infty > \int_R f |\nabla u| dL_n \geq \int_0^1 \int_{u^{-1}(s) \cap R} f(x) dH^{n-1}(x) dL_1(s)$$

and therefore $L_1(S) = 0$.

3.3. THEOREM. $M_{n'}(\Sigma) \geq C[G, C_0, C_1]^{-1/n-1}$.

Proof. Choose $\varepsilon > 0$ and let f be any function for which $f \wedge \Sigma$. Let u be an admissible function for $C = C[G, C_0, C_1]$ such that

$$\int_R |\nabla u|^n dL_n < C + \varepsilon.$$

R^* is connected since G is and therefore it is clear that $u^{-1}(s) \in \Sigma$ for all $0 < s < 1$. Hence, by Hölder's inequality and 2.2.1, we have

$$\begin{aligned} \left(\int_{E^n} f^{n'} dL_n \right)^{1/n'} (C + \varepsilon)^{1/n} &\geq \left(\int_{E^n} f^{n'} dL_n \right)^{1/n'} \left(\int_R |\nabla u|^n dL_n \right)^{1/n} \\ &\geq \int_R f |\nabla u| dL_n \\ &\geq \int_0^1 \int_{u^{-1}(s)} f dH^{n-1}(x) dL_1(s) \geq 1. \end{aligned}$$

Since ε is arbitrary,

$$\int_R f^{n'} dL_n \geq C^{-1/n-1}$$

which is also true for the infimum over all $f \wedge \Sigma$, and thus the result follows.

3.4. The opposite inequality will be established by means of a sequence of approximations and we will begin by first assuming that $C_0 \cup C_1$ consists only of a finite number of nondegenerate continua and that $C_0 \cup C_1 \subset G$. We will also assume initially that $H^{n-1}(\text{bdry } G) < \infty$.

Let V be an open connected set such that $G \supset \text{cl } V \supset V \supset C_0 \cup C_1$ and let v be the extremal function for $C[V, C_0, C_1]$ (see 2.3). Since v satisfies the variational condition (2), the proof of the following lemma is very similar to that of Lemma 3 in [G3] and will not be given here. It is possible to obtain a stronger result, but the following is sufficient for our purposes. Recall from 2.4 the definition of an n -dimensional polyhedral set.

3.5. LEMMA. *Let π be the boundary of an n -dimensional polyhedral set P (P not necessarily contained in V) such that $C_0 \subset P$ and $C_1 \subset E^n - \text{cl } P$. Then π separates C_0 from C_1 in V and*

$$\int_{\pi(b)} |\nabla v|^{n-1} dL_n \geq 2bC[V, C_0, C_1]$$

whenever $0 < b < \delta(\pi, C_0 \cup C_1)$ and where $\pi(b) = \{x : \delta(x, \pi) < b\}$.

3.6. REMARK. In Lemmas 3.7 and 3.8, the integral average, f_r , of $|\nabla v|^{n-1}$ will be used. Thus, defining $\nabla v = 0$ on the complement of V , for each $r > 0$ we have

$$f_r(x) = \alpha(r)^{-1} \int_{S(x,r)} |\nabla v(y)|^{n-1} dL_n(y)$$

where $\alpha(r) = L_n[S(x, r)]$. It is well known that f_r is continuous and that

$$f_r \rightarrow |\nabla v|^{n-1} L_n\text{-a.e. as } r \rightarrow 0.$$

Also, by a result of K. T. Smith [S] and Lebesgue's dominated convergence theorem, $\|f_r\|_n \rightarrow \|\nabla v|^{n-1}\|_n$ as $r \rightarrow 0$ and consequently, $\|f_r - |\nabla v|^{n-1}\|_n \rightarrow 0$.

3.7. LEMMA. *With π as in 3.5,*

$$\int_{\pi} f_r dH^{n-1} \geq C[V, C_0, C_1]$$

whenever $r < \delta(\pi, C_0 \cup C_1)$.

Proof. Choose $b > 0, r > 0$ so that $b + r < \delta(\pi, C_0 \cup C_1)$. If π_y denotes the translation of π through the vector y , then Fubini's Theorem and 3.5 imply

$$\begin{aligned} \int_{\pi(b)} f_r(x) dL_n(x) &= \alpha(r)^{-1} \int_{S(0,r)} \int_{\pi(b)} |\nabla v(x+y)|^{n-1} dL_n(x) dL_n(y) \\ (7) \qquad \qquad \qquad &= \alpha(r)^{-1} \int_{S(0,r)} \int_{\pi_y(b)} |\nabla v(x)|^{n-1} dL_n(x) dL_n(y) \\ &\geq 2bC[V, C_0, C_1] \end{aligned}$$

since π_y satisfies the conditions of 3.5. In addition to this, if $d(x) = \delta(x, \pi)$, then $|\nabla d| = 1$ L_n -a.e. on $E^n - \pi$ [F4, (3) of 4.8] and therefore 2.2.1 gives

$$(8) \qquad \int_{\pi(b)} f_r(x) dL_n(x) = \int_0^b \int_{d^{-1}(s)} f_r(x) dH^{n-1}(x) dL_1(s).$$

Let $F(s)$ denote the inner integral on the right. Since f_r is continuous on E^n and π is the boundary of a polyhedral set, it is clear that

$$\lim_{s \rightarrow 0} F(s) = 2 \int_{\pi} f_r dH^{n-1}.$$

Hence, from (7) and (8) we have

$$C[V, C_0, C_1] \leq \lim_{b \rightarrow 0} (2b)^{-1} \int_0^b F(s) dL_1(s) = \int_{\pi} f_r dH^{n-1}.$$

3.8. LEMMA. *If Σ is the class of sets that separate C_0 from C_1 in G , then*

$$\int_{\sigma \cap R} |\nabla v|^{n-1} dH^{n-1} \geq C[V, C_0, C_1]$$

for M_n -a.e. $\sigma \in \Sigma$.

Proof. Select $\sigma \in \Sigma$ and let U be that part of a partition of $G - \sigma$ that contains C_0 . Since $H^{n-1}(\text{bdry } G) < \infty$ by assumption, by appealing to 3.1 we can take $H^{n-1}(\text{bdry } U) < \infty$. Recall that $G \supset \text{cl } V$ and that $\nabla v = 0$ on the complement of V . Hence, we can choose r_0 so small that the support of f_{r_0} is contained in G (and therefore for all $r \leq r_0$) and $r_0 < \delta(\text{bdry } U, C_0 \cup C_1)$.

By applying 2.4.2 to the set U , we obtain a sequence of n -dimensional polyhedral sets P_i . Let $\pi_i = \text{bdry } P_i$. From properties (i), (ii), and (iv) of 2.4.2 it is clear that eventually $C_0 \subset P_i$ and $C_1 \subset E^n - \text{cl } P_i$. Thus Lemma 3.7 applies to π_i for all large i . Now, $\beta(U) \subset \text{bdry } U \subset (\text{bdry } G) \cup \sigma$ and since the support of f_r is contained in G for all $r \leq r_0$, it is clear that

$$(9) \quad \int_{\sigma} f_r dH^{n-1} \geq \int_{\beta(U)} f_r dH^{n-1} \quad \text{for all } r \leq r_0.$$

Since f_r is continuous, (v) of 2.4.2 and 3.7 imply

$$\int_{\beta(U)} f_r dH^{n-1} = \lim_{i \rightarrow \infty} \int_{\pi_i} f_r dH^{n-1} \geq C[V, C_0, C_1] \quad \text{for } r \leq r_0.$$

Thus, from (9), we have

$$(10) \quad \int_{\sigma} f_r dH^{n-1} \geq C[V, C_0, C_1] \quad \text{for } r \leq r_0.$$

In light of the fact that $\|f_r - |\nabla v|^{n-1}\|_n \rightarrow 0$ as $r \rightarrow 0$ (3.6), the result follows from (iv) and (v) of 2.5 and (10).

3.9. LEMMA. *Let u be the extremal function for $C[G, C_0, C_1]$ (see 3.4 and 2.3). Then for M_n -a.e. $\sigma \in \Sigma$,*

$$\int_{\sigma \cap R} |\nabla u|^{n-1} dH^{n-1} \geq C[G, C_0, C_1] = C.$$

Proof. Let V_i be a sequence of open connected sets such that

$$G \supset \text{cl } V_{i+1} \supset V_{i+1} \supset \text{cl } V_i \supset V_i \supset C_0 \cup C_1$$

and $G = \bigcup V_i$. Let v_i be the extremal function for $C[V_i, C_0, C_1] = C_i$.

We will first show that $C_i \rightarrow C$ as $i \rightarrow \infty$. Recall that $C < \infty$. If $i > j$, then the restriction of v_i to V_j is an admissible function for V_j and thus, so is $2^{-1} \cdot (v_i + v_j)$. As in the proof of 2.5.1, an application of Clarkson's inequality [C] gives

$$(11) \quad \int_R \left| \frac{\nabla v_i - \nabla v_j}{2} \right|^n dL_n \leq \frac{1}{2} C_i - \frac{1}{2} C_j \quad \text{for } i > j.$$

Since C_i is a monotonically increasing sequence bounded above by C , (11) implies the existence of a vector-valued function f such that

$$(12) \quad \int_R |\nabla v_i - f|^n dL_n \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

In fact, since $C_0 \cup C_1$ consists only of a finite number of nondegenerate continua, an argument similar to that of [G2, p. 362] shows that there is an admissible function u' such that $\nabla u' = f$ L_n -a.e. on R . Thus, (12) shows that

$$(13) \quad \lim_{i \rightarrow \infty} C_i = C.$$

This also implies that u' is the extremal function for C , i.e., $u' = u$.

Since $\| |\nabla v_i| - |\nabla u| \|_n \rightarrow 0$ as $i \rightarrow \infty$, there is a subsequence of $|\nabla v_i|$ (which will still be denoted by $|\nabla v_i|$) such that $|\nabla v_i| \rightarrow |\nabla u|$ L_n -a.e. and therefore that $|\nabla v_i|^{n-1} \rightarrow |\nabla u|^{n-1}$ L_n -a.e. on R . This fact, along with

$$\| |\nabla v_i|^{n-1} \|_{n'} \rightarrow \| |\nabla u|^{n-1} \|_{n'}$$

leads to

$$\| |\nabla v_i|^{n-1} - |\nabla u|^{n-1} \|_{n'} \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

Thus, with (iv) and (v) of 2.5, we have (for another subsequence)

$$(14) \quad \lim_{i \rightarrow \infty} \int_{\sigma \cap R} |\nabla v_i|^{n-1} dH^{n-1} = \int_{\sigma \cap R} |\nabla u|^{n-1} dH^{n-1} \quad \text{for } M_n\text{-a.e. } \sigma \in \Sigma.$$

Lemma 3.8 states that for each i ,

$$\int_{\sigma \cap R} |\nabla v_i|^{n-1} dH^{n-1} \leq C_i \quad \text{for } M_n\text{-a.e. } \sigma \in \Sigma,$$

and therefore, the result follows from (13), (14), and (ii) of 2.5.

3.10. THEOREM. *If G is a bounded, open, connected set, if $C_0 \cup C_1$ consists of only a finite number of nondegenerate continua, and if $C_0 \cup C_1 \subset G$, then*

$$M_n(\Sigma) = C[G, C_0, C_1]^{-1/n-1}.$$

Proof. If it were the case that $H^{n-1}(\text{bdry } G) < \infty$, then the result would follow immediately from 3.3 and 3.9. For, if we let $f = |\nabla u|^{n-1} \cdot C^{-1}$, where u is the extremal for C , then by 3.9 there is $\Sigma_0 \subset \Sigma$ such that $f \wedge \Sigma_0$ and $M_n(\Sigma_0) = M_n(\Sigma)$. Hence, by 3.3,

$$C^{-1/n-1} \leq M_n(\Sigma) \leq \int_R f^{n'} dL_n = C^{-1/n-1}.$$

In order to eliminate the assumption $H^{n-1}(\text{bdry } G) < \infty$, select a sequence of open, connected sets $V_1 \subset V_2 \subset \dots$ such that $C_0 \cup C_1 \subset V_1$, $H^{n-1}(\text{bdry } V_i) < \infty$, and $G = \bigcup V_i$. As in the proof of 3.9, let v_i be the extremal for $C[V_i, C_0, C_1] = C_i$ and again we have

$$(15) \quad C_i \rightarrow C, \quad \| |\nabla v_i|^{n-1} - |\nabla u|^{n-1} \|_{n'} \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

Thus, for a subsequence

$$(16) \quad \lim_{i \rightarrow \infty} \int_{\sigma \cap R} |\nabla v_i|^{n-1} dH^{n-1} = \int_{\sigma \cap R} |\nabla u|^{n-1} dH^{n-1} \quad \text{for } M_n\text{-a.e. } \sigma \in \Sigma.$$

For each i , every $\sigma \in \Sigma$ separates C_0 from C_1 in V_i , and thus applying 3.9 with V_i replacing G , we have

$$\int_{\sigma \cap R} |\nabla v_i|^{n-1} dH^{n-1} \geq \int_{\sigma \cap V_i} |\nabla v_i|^{n-1} dH^{n-1} \geq C_i \quad \text{for } M_n\text{-a.e. } \sigma \in \Sigma.$$

(Observe that 3.9, with V_i replacing G , applies only to Σ_i , which are those sets that separate C_0 from C_1 in V_i . However, a class in Σ which is M_n -zero relative to Σ_i is M_n -zero relative to Σ .) Hence, in view of (15) and (16),

$$(17) \quad \int_{\sigma \cap R} |\nabla u|^{n-1} dH^{n-1} \geq C \quad \text{for } M_n\text{-a.e. } \sigma \in \Sigma$$

which, as we have seen from above, is sufficient to establish the theorem.

3.11. COROLLARY. *With the hypotheses of 3.10, and if u is the extremal for $C[G, C_0, C_1]$, then*

- (i) $0 \leq u(x) \leq 1$ for all $x \in G$,
- (ii) $\int_{\sigma \cap R} |\nabla u|^{n-1} dH^{n-1} \geq C[G, C_0, C_1]$ for M_n -a.e. $\sigma \in \Sigma$,
- (iii) $\int_{u^{-1}(s)} |\nabla u|^{n-1} dH^{n-1} = C[G, C_0, C_1]$, for L_1 -a.e. $s \in [0, 1]$.

Proof. By truncating u at levels 1 and 0 if necessary, a new admissible function would be formed whose gradient would be bounded above by the gradient of u . However, the extremal is unique and thus (i) follows. (ii) is just a restatement of (17).

In order to prove (iii), let

$$F(s) = \int_{u^{-1}(s)} |\nabla u|^{n-1} dH^{n-1} \quad \text{for } L_1\text{-a.e. } s,$$

and observe that since G is connected, $u^{-1}(s) \in \Sigma$ whenever $0 < s < 1$. Thus, (ii) above and 3.2 imply that $F(s) \geq C$ for L_1 -a.e. $s \in [0, 1]$. However, (i) and an application of 2.2.1 give

$$C = \int_R |\nabla u|^n dL_n = \int_0^1 F(s) dL_1(s)$$

which implies that $F(s) = C$ for L_1 -a.e. $s \in [0, 1]$.

3.12. **REMARK.** The following observation has some interest in view of Theorem 2 of [G2].

In addition to the hypotheses of 3.11, assume that $H^{n-1}(C_0) = 0$. Then there is a point $x_0 \in C_0$ such that

$$\limsup_{x \rightarrow x_0} |\nabla u(x)| = \infty.$$

For if this were not the case, then, since C_0 is compact, there would be a constant $K > 0$ and an open set U such that $G - C_1 \supset \text{cl } U \supset U \supset C_0$ and $|\nabla u|^{n-1} < K$ L_n -a.e. on U . Choose $\varepsilon > 0$. Since $H^{n-1}(C_0) = 0$, C_0 can be covered by a countable number of open n -balls B_i such that

$$(18) \quad \bigcup B_i \subset U \quad \text{and} \quad \Sigma H^{n-1}(\text{bdry } B_i) < \varepsilon K^{-1}.$$

Since C_0 is compact, a finite number of the B_i will cover C_0 , say B_1, B_2, \dots, B_k . According to (vi) of 2.5, there is a nonnegative Borel function $f \in \mathcal{L}^n(R)$ such that (ii) of 3.11 fails to hold for only those $\sigma \in \Sigma$ for which

$$\int_{\sigma \cap R} f dH^{n-1} = \infty.$$

By employing 2.2.1, we can replace each n -ball B_i , $i = 1, 2, \dots, k$, by a slightly larger one B'_i such that

$$\int_{S'_i} f dH^{n-1} < \infty \quad \text{where } S'_i = \text{bdry } B'_i,$$

$|\nabla u|^{n-1} < K$ H^{n-1} -a.e. on S'_i , and (18) still holds. Now let $\sigma = \text{bdry}(\bigcup B'_i)$. Then $\sigma \in \Sigma$ and (ii) of 3.11 implies that $C[G, C_0, C_1] < \varepsilon$, which means that it is zero since ε is arbitrary. This would mean that $\nabla u = 0$ L_n -a.e. on G . That is, since G is connected and u is ACT on G , u would be constant, a contradiction.

In the following theorem, we will consider the general case of two disjoint compact sets C_0 and C_1 which are contained in the closure of an open, bounded, connected set G .

$$3.13. \text{ THEOREM. } M_n(\Sigma) = C[G, C_0, C_1]^{-1/n-1}.$$

Proof. In view of 3.3, we may assume that $C = C[G, C_0, C_1] \neq 0$. For each positive integer i let

$$K_0(i) = \text{cl} \{x : \delta(x, C_0) < (2i)^{-1}\} \text{ and}$$

$$H_0(i) = \{x : \delta(x, C_0) < i^{-1}\}.$$

Define $K_1(i)$ and $H_1(i)$ similarly and let $G_i = G \cup H_0(i) \cup H_1(i)$. Since G is connected, it is clear that G_i is open and connected and notice that $K_0(i) \cup K_1(i)$ consists only of a finite number of nondegenerate continua. We will consider only those i for which $K_0(i)$ and $K_1(i)$ are disjoint. Thus, let v_i be the extremal function for $C_i = C[G_i, K_0(i), K_1(i)]$ and observe that if $i > j$, then the restriction of v_j to G_i is an admissible function for C_i . Finally, let Σ_i be those sets σ that separate $K_0(i)$ from $K_1(i)$ in G_i and subject to the condition that $\sigma \cap [H_0(i) \cup H_1(i)] = \emptyset$. The purpose for this requirement is that now an M_n -null class in Σ_i is also M_n -null in Σ . It is clear that $\Sigma_1 \subset \Sigma_2 \subset \dots, \Sigma_i \subset \Sigma$ for all i , and

$$(19) \quad \bigcup_{i=1}^{\infty} \Sigma_i = \Sigma.$$

Since C_i is a monotonically decreasing sequence bounded below by C , we can employ again part of the argument of 3.9 to find a vector-valued function f such that

$$\lim_{i \rightarrow \infty} \int_{E^n} |\nabla v_i - f|^n dL_n = 0$$

and therefore, for a subsequence,

$$(20) \quad \lim_{i \rightarrow \infty} \|\nabla v_i|^{n-1} - |f|^{n-1}\|_{n'} = 0.$$

Hence, if $L = \lim_{i \rightarrow \infty} C_i$, $g_i = |\nabla v_i|^{n-1} \cdot C_i^{-1}$, and $g = |f|^{n-1} \cdot L^{-1}$ then (v) of 2.5 provides another subsequence such that

$$(21) \quad \lim_{i \rightarrow \infty} \int_{\sigma \cap R} g_i dH^{n-1} = \int_{\sigma \cap R} g dH^{n-1} \quad \text{for } M_n\text{-a.e. } \sigma \in \Sigma.$$

Now by employing 3.11 with G, C_0, C_1 replaced by $G_i, K_0(i), K_1(i)$ respectively, we have for each i ,

$$(22) \quad \int_{\sigma \cap R} g_i dH^{n-1} \geq 1 \quad \text{for } M_n\text{-a.e. } \sigma \in \Sigma_i.$$

Therefore, (ii) of 2.5, (19), and (21) imply

$$(23) \quad \int_{\sigma \cap R} g dH^{n-1} \geq 1 \quad \text{for } M_n\text{-a.e. } \sigma \in \Sigma.$$

Since v_i is the extremal for C_i , (ii) of 3.11 and 3.3 show that, for each i ,

$$\int_{E^n} (g_i)^{n'} dL_n = C_i^{-1/n-1}.$$

Thus, with (20), (22), and (23), we have

$$C^{-1/n-1} \geq \lim_{i \rightarrow \infty} C_i^{-1/n-1} = \lim_{i \rightarrow \infty} \int_{E^n} (g_i)^{n'} dL_n = \int_{E^n} g^{n'} dL_n \geq M(\Sigma).$$

Theorem 3.3 gives the opposite inequality, and thus the proof is complete.

We will conclude with a result concerning null sets for conformal capacity.

3.14. THEOREM. *Suppose C_0 and C_1 are disjoint compact sets in the closure of G . If $E \subset G - (C_0 \cup C_1)$ is a compact set with $H^{n-1}(E) = 0$, then*

$$C[G, C_0, C_1] = C[G - E, C_0, C_1].$$

Proof. The topological dimension of E is no more than $n - 2$ since $H^{n-1}(E) = 0$ and therefore, $G - E$ is connected. Thus, the right hand side of the equality has meaning. Clearly,

$$(24) \quad C[G, C_0, C_1] \geq C[G - E, C_0, C_1].$$

The opposite inequality will be established by considering separating sets. Let Σ^* be those sets that separate C_0 from C_1 in

$$[(G - E) - (C_0 \cup C_1)] \cup [C_0 \cup C_1]$$

and let Σ be those that separate C_0 from C_1 in R^* . In light of (24) and 3.13 it is sufficient to prove

$$(25) \quad M_n(\Sigma) \geq M_n(\Sigma^*).$$

To this end, let f be a function for which $f \wedge \Sigma$. In order to establish (25), we need only show $f \wedge \Sigma^*$. Choose $\sigma^* \in \Sigma^*$ and notice that $\sigma^* \cup E \in \Sigma$. Thus,

$$\int_{\sigma^* \cup E} f dH^{n-1} \geq 1$$

and since $H^{n-1}(E) = 0$,

$$\int_{\sigma^*} f dH^{n-1} \geq 1.$$

This shows that $f \wedge \Sigma^*$ and consequently, proves the theorem.

If G is compactified E^n , Bagby has shown that $C[G, C_0, C_1] = M_n(\Gamma)$ where Γ is the family of all arcs that meet both C_0 and C_1 (Ph.D. thesis, Harvard Univ., Cambridge, Mass., 1966). By using 3.13, [FU], and [W] one can show that this result is valid when G is an open, bounded, connected set and $C_0 \cup C_1 \subset G$. (Moreover, if $C_0 \cup C_1 \subset \text{cl } G$, the result is also valid if certain conditions are imposed on the tangential behavior of $(\text{bdry } G) \cap (C_0 \cup C_1)$.) Thus, if Γ^* is the family of arcs that join C_0 to C_1 in $G - E$, then 3.14 implies

$$M_n(\Gamma^*) = M_n(\Gamma).$$

This result was obtained by Väisälä [V2] in the case where C_0 and C_1 are non-degenerate continua.

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