TWO CHARACTERIZATIONS OF COMPACT LOCAL TREES(1)

BY
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1. **Introduction.** In the introduction to [7], Ward discusses the problem of characterizing "one-dimensional" topological spaces by their inherent order properties. The second characterization of compact local trees appearing in this paper is just such a theorem. The first characterization is necessary for the proof of the second and is stated as

THEOREM A. *A necessary and sufficient condition that a locally connected continuum be a local tree is that it have at most finitely many prime chains, each of which is composed of finitely many arcs meeting only at their endpoints.*

To state our second main theorem we will use the following conventional notation: if $\leq$ is a partial order relation (reflexive, anti-symmetric, transitive) on a set P, then for $x \in P$

$$L(x) = \{y \in P : y \leq x\},$$

$$M(x) = \{y \in P : x \leq y\}.$$

THEOREM B. *If X is a compact, Hausdorff space, then a necessary and sufficient condition that X be a local tree is that X admit a partial order relation $\leq$ such that*

(i) $\leq$ *is semicontinuous*;

(ii) $\leq$ *is order dense*;

(iii) *for $x \in X$ and $y \in X$, $L(x) \cap L(y)$ is the nonvoid union of finitely many chains*;

(iv) *the set $\{x \in X$: there exists $b < x$ such that x lies in the boundary of $M(b)\}$ is finite.*

The author wishes to express his debt and appreciation to L. E. Ward, Jr. for his encouragement and suggestions.

2. **Preliminary remarks.** Let A, B, C be subsets of a topological space X. If $\bar{B} \cap C = \phi = B \cap \bar{C}$, then we write $B|C$. If

$$A = B \cup C, \qquad B|C, \qquad B \neq \phi \neq C,$$

Received by the editors April 18, 1966.

(1) Research for this paper was supported in part by the National Science Foundation under Grant Number GP-1457.

then we say $A = B \cup C$ is *a separation of* A. If X is connected, then a subset D of X is said *to separate two subsets* M and N of X in case

$$X - D = B \cup C, \qquad B|C, \qquad M \subset B, \qquad N \subset C.$$

By a *continuum* we mean a compact, connected, Hausdorff space. If points p and q of a connected space X cannot be separated by any point of X, then we write $p \sim q$. A *prime chain* of X is a nondegenerate subcontinuum E containing distinct elements a and b with $a \sim b$, and representable as

$$E = \{x \in X : a \sim x \text{ and } x \sim b\}.$$

In §2 of [3], the reader will find essentially the following

PROPOSITION 2.1. *Let E be a nondegenerate subset of a continuum X. E is a prime chain of X if and only if E is maximal in X with respect to the property that for each pair p, q of elements of E, $p \sim q$.*

If A is a subset of a topological space X, then we let $F(A)$ denote the *boundary* of A; that is, $F(A)$ is the set of points x of X such that each neighborhood of x (= open set containing x) meets both A and $X - A$.

A point x of a connected space X is a *cutpoint* of X if x separates two points of X. A point x of X is an *endpoint* if for each neighborhood U of x there exists a neighborhood V of x such that $\overline{V} \subset U$ and $F(V) = \overline{V} - V$ contains exactly a single point. In [3], Wallace proves the following theorem in item (2.10):

THEOREM 2.2. *For each point x of a continuum exactly one of the following is valid: x is a cutpoint; x is an endpoint; or x is contained in a unique prime chain.*

If C is a subset of a partially ordered set P, and for each x and y in C if $x \leq y$ or $y \leq x$, then $\leq$ is said to be a *linear order* on C, and C is called a *chain in P*. A *maximal chain* is a chain properly contained in no other chain. If $A \subset P$, then an element x of A is said to be *minimal* (*maximal*) *in A* in case $L(x) \cap A = \{x\}$ ($M(x) \cap A = \{x\}$). A point 0 in a partially ordered set P is called a *zero* in case $0 \leq x$ for all x in P. Finally, P is said to be *order dense* (and the partial order on P is also called order dense) in case for $x < y$ in P, there exists z in P such that $x < z < y$.

A partial order relation on a topological space X is said to be *semicontinuous* if $M(x)$ and $L(x)$ are closed subsets of X for each x in X. Let X be a compact, Hausdorff space with a semicontinuous partial order. In [4] Wallace shows every closed subset of X has maximal and minimal elements, so that each closed chain in X has a unique maximal element and a unique minimal element. He also shows that maximal chains are closed. If we require further that X be order dense, then by Theorem 4 of [5] each maximal chain is connected. Thus in X each of the sets $M(x)$, $L(x)$, and $M(x) \cap L(y)$ are connected for every x and y in X. All these results can be found in [5].

If X is a locally connected continuum, the *cutpoint order on X with zero p* is defined as follows: let $p \in X$ be arbitrary but fixed, and for $x \in X$ and $y \in X$ define $x \leq y$ in case $x=p$, $x=y$, or x separates p and y in X. By Lemma 12 of [5], the cutpoint order is a semicontinuous partial order on X.

A *tree* is a continuum in which every pair of distinct points can be separated by a third point. A *local tree* is a connected space every point of which lies in a neighborhood whose closure is a tree. In [6], Ward has the following characterization of trees which we shall use repeatedly throughout this paper:

THEOREM 2.3. *Let X be a compact Hausdorff space. A necessary and sufficient condition that X be a tree is that X admit a partial order, $\leq$, satisfying*

(i) *$\leq$ is semicontinuous;*

(ii) *$\leq$ is order dense;*

(iii) *for $x \in X$, $y \in X$, $L(x) \cap L(y)$ is a nonvoid chain;*

(iv) *$M(x)-x$ is an open set, for each $x \in X$.*

In proving the necessity of the condition, Ward proves that for each $0 \in X$, the cutpoint order on the tree X with zero 0 satisfies (i)–(iv), and he observes further in [7] that this partial order is the unique partial order on X with zero 0, satisfying (i)–(iv). Also, in the sequel we shall frequently use the fact (shown in [6]) that a tree is locally connected.

3. **Theorem A.** In this section we present the lemmas and theorems leading to the proof of Theorem A.

LEMMA 3.1. *Let T be a tree with cutpoint order $\leq$ and zero 0. A point $m \in T$ is a maximal element in T if and only if m is an endpoint of T and $m \neq 0$.*

Proof. Suppose m is not an endpoint of T, and $0 \neq m$. Then m is a cutpoint by Theorem 2.2, and

$$T-m = A \cup B, \qquad A|B, \qquad A \neq \phi \neq B,$$

where we may assume $0 \in A$. Then there exists $x \in B$, and by the definition of $\leq$, $m < x$, so m is not maximal.

On the other hand, if m is an endpoint it cannot separate any pair of points. Thus if $m \neq 0$, then m is a maximal element of T.

A continuum is called *cyclic* if it is a prime chain. We note that a compact local tree is a locally connected continuum.

LEMMA 3.2. *Let E be a compact, cyclic local tree, let R be a nonvoid open subset of E such that $\bar{R}$ is a tree, and let $\leq$ be the cutpoint order on $\bar{R}$ with some x in $\bar{R}$ as the zero. Then every maximal element of $\bar{R}$ lies in $F(R)$.*

Proof. Suppose on the contrary, that m is a maximal element of $\bar{R}$ and that $m \in R$. By Lemma 3.1, m is an endpoint of $\bar{R}$, so there exists an open set V such that $m \in V \subset R$ and $F(V)=\{a\}$. Now

$$E-a = V \cup (E-\bar{V}), \quad V \mid (E-\bar{V}),$$

and $E-\bar{V}\neq\phi$ since E is cyclic and $\bar{V}\subset\bar{R}$, a tree in E. This contradicts that E is cyclic. Thus all maximal elements of $\bar{R}$ lie in $F(R)$.

LEMMA 3.3. *Let E be a compact, cyclic local tree, let R be a nonvoid open subset of E such that $\bar{R}$ is a tree, and let $\leq$ be the cutpoint order on $\bar{R}$ with some x in R as the zero. If M is a closed, connected chain with $x\in M\subset R$, then $\bar{R}-M$ has at most finitely many components, say $C_1,\ldots,C_k$, and $\bar{C}_i\cap M$ is a single point $p_i=\inf\bar{C}_i$, for each $i=1,2,\ldots,k$.*

Proof. Suppose that $\bar{R}-M$ has infinitely many components, $\{C_\alpha : \alpha\in I\}$. For each $\alpha\in I$, let t_α be a maximal element of $\leq$ in C_α. (The point t_α exists by Lemma 3.2 and by the fact that E is cyclic.) By Lemma 3.2, each t_α lies in $F(R)$. The set $\{t_\alpha : \alpha\in I\}$ is infinite, so there exists $y\in F(R)$ such that y is an accumulation point of $\{t_\alpha : \alpha\in I\}$. Since $y\in\bar{R}-M$, and since $\bar{R}$ is locally connected there exists a connected subset V of $\bar{R}$ such that $y\in V$, V is open in the relative topology of $\bar{R}$, and $V\subset\bar{R}-M$. Since V is connected, there exists $\beta\in I$ such that $V\subset C_\beta$, but this means V contains at most one t_α, contradicting that y is an accumulation point of $\{t_\alpha : \alpha\in I\}$. Thus $\bar{R}-M$ has at most finitely many components, $C_1,\ldots,C_k$.

Suppose for some integer i, $1\leq i\leq k$, there exist $a,b\in M\cap\bar{C}_i$, and say $a<b$. Since $\leq$ is order dense, there exists $c\in M$ such that $a<c<b$. This implies $\bar{R}-c=A\cup B$, where $A|B$, x, $a\in A$ and $b\in B$. Since C_i is connected, $C_i\subset A$ or $C_i\subset B$, but this is impossible since $A\cap\bar{C}_i\neq\phi\neq B\cap\bar{C}_i$. Thus $\bar{C}_i\cap M$ is a singleton for all $i=1,2,\ldots,k$. But if $p_i=\bar{C}_i\cap M$, then p_i separates x and C_i in $\bar{R}$, so p_i is the zero of $\bar{C}_i$.

Let X be a topological space and let n be an integer. A point x of X is said to be *of order $\leq n$ in X* if for every neighborhood U of x there exists a neighborhood V of x such that $\bar{V}\subset U$ and $F(V)$ contains at most n points. A point x in X is said to *have order n* if x is of order $\leq n$ but x is not of order $\leq m$ for each $m<n$. We shall say that x *has order at least n* if x is not of order $\leq m$ for $m<n$.

THEOREM 3.4. *If E is a compact, cyclic local tree, then there are at most finitely many points of E which have order at least* 3.

Proof. Assume otherwise that E contains an infinite subset A each of whose points has order at least 3. By compactness, A has an accumulation point x. Let R be a neighborhood of x such that $\bar{R}$ is a tree. By Lemma 3.3, $\bar{R}-x$ has finitely many components, so at least one such component C has the property that x is an accumulation point of $C\cap A$. The set $\bar{C}=C\cup\{x\}$ is a subtree of $\bar{R}$. The point x is not a cutpoint of $\bar{C}$, so by Theorem 2.2 x is an endpoint of $\bar{C}$. Let $\leq$ be the cutpoint order on $\bar{C}$ with x as the zero, and let M be a nondegenerate, closed, connected chain in $\bar{C}$, containing x and no maximal elements of $\bar{C}$. By Lemma 3.3,

$$\bar{C}-M=\bigcup\{C_i : i=1,2,\ldots,n\},$$

where the C_i are the components of $\bar{C}-M$. The set $\bar{C}_i$ is a subtree of $\bar{C}$ and has the zero element m_i relative to $\leq$, where $m_i=\bar{C}_i \cap M$ by Lemma 3.3. Let $m=\inf\{m_1,\ldots,m_n\}$, say $m_k=m$. Then $\bar{C}-m=C_k \cup (\bar{C}-\bar{C}_k)$, where $C_k \mid \bar{C}-\bar{C}_k$; so m is a cutpoint, therefore $m\neq x$. Now,

$$\bigcup\{C_i : i = 1, 2, \ldots, n\} \subset M(m),$$

so that $L(m)-m=\bar{C}-M(m)$ is open in the relative topology of $\bar{C}$ and contains infinitely many points of $A \cap C$. Further,

$$L(m)-\{x, m\} = C-M(m) \subset R,$$

so $L(m)-\{x, m\}$ is open in E. By the order density of $\leq$, there exists

$$y \in (L(m)-\{x, m\}) \cap A.$$

Let U be any connected neighborhood of y such that $\bar{U}\subset L(m)-\{x, m\}$. Suppose there exist distinct points a, b, and c in $F(U)$. Recalling from Theorem 2.3 (iii) that $L(m)$ is a chain, we may assume without loss of generality that $a<b<c$, which implies that there exist sets P and Q such that

$$\bar{C}-b = P \cup Q, \qquad P|Q, \qquad a\in P \quad \text{and} \quad c \in Q.$$

Since $b\notin U$ and U is connected, $U\subset P$ or $U\subset Q$, but this is impossible since $a\in \bar{U}$ and $c\in \bar{U}$. Thus $F(U)$ has at most two elements, which contradicts $y\in A$, and the proof is complete.

A well-known theorem of R. L. Moore states that every nondegenerate continuum contains at least two noncutpoints. By an *arc* in a topological space X, we mean a subspace of X which is a continuum and has exactly two noncutpoints. In case X is a metric space, an arc in X is a homeomorph of the closed unit interval of the real line.

In the proof of the next theorem and in §4, we will have occasion to use the following conventional notation: if $\leq$ and $\leqq$ are two partial order relations on a set X, then for $x\in X$,

$$L(x, \leq) = \{y\in X : y \leq x\}, \qquad L(x, \leqq) = \{y\in X : y \leqq x\},$$

and similarly for $M(x, \leq)$ and $M(x, \leqq)$.

THEOREM 3.5. *A cyclic continuum is a local tree if and only if it is the union of finitely many arcs, which may meet only at their endpoints.*

Proof. Let E be a compact, cyclic local tree. By Theorem 3.4, there are points $x_1, x_2, \ldots, x_n$ of E, where n is a positive integer, such that every point of $E-\{x_1,\ldots,x_n\}$ has order ≤ 2. Since E is cyclic, each point of $E-\{x_1,\ldots,x_n\}$ has order 2. We shall choose $n>1$. Let C be a component of $E-\{x_1,\ldots,x_n\}$. We wish to show $\bar{C}$ is an arc. We note first that by the local connectivity of E, C is open in E and that $\phi\neq F(C)\subset\{x_1,\ldots,x_n\}$.

Let $y\in C$. There exists a neighborhood R of y such that $\bar{R}$ is a tree, $\bar{R}\subset C$, and $F(R)=\{a, b\}$. Let $\leq$ be the cutpoint order on $\bar{R}$ with zero a. By Lemma 3.2,

b is the only maximal element in $\bar{R}$. Thus, for each $z \in R$, $a < z < b$, so z is a cutpoint, and therefore $\bar{R}$ is an arc.

At this point, we need to prove the following lemma:

If R and S are arcs contained in $\bar{C}$ with endpoints r_1, r_2 and s_1, s_2, respectively, and if both

$$(R-\{r_1, r_2\}) \cap (S-\{s_1, s_2\}) \neq \phi,$$

$$(R-\{r_1, r_2\}) \cup (S-\{s_1, s_2\}) \subset C,$$

then $R \cup S$ is an arc.

Let $z \in (R-\{r_1, r_2\}) \cap (S-\{s_1, s_2\})$, let $\leq$ be the cutpoint order on R with zero z, and let $\leqq$ be the cutpoint order on S with zero z. As a consequence of Lemma 3.1, $L(r_1, \leq)$ and $L(r_2, \leq)$ are the maximal chains in R, their union being R and their intersection being $\{z\}$. Similar statements hold for $L(s_1, \leqq)$ and $L(s_2, \leqq)$ with respect to S. Suppose that $L(r_1, \leq)$ does not contain either $L(s_1, \leqq)$ or $L(s_2, \leqq)$, and that neither $L(s_1, \leqq)$ nor $L(s_2, \leqq)$ contains $L(r_1, \leq)$. If

$$L(r_1, \leq) \subset L(s_1, \leqq) \cup L(s_2, \leqq),$$

then there exist u and v distinct from z such that

$$u \in L(r_1, \leq) \cap L(s_1, \leqq), \qquad v \in L(r_1, \leq) \cap L(s_2, \leqq),$$

but this is impossible since z would separate u and v in $L(r_1, \leq)$. Then there exist $w \in L(r_1, \leq)$, $a \in L(s_1, \leqq)$, and $b \in L(s_2, \leqq)$ such that

$$w \notin L(s_1, \leqq) \cup L(s_2, \leqq) = S \quad \text{and} \quad a, b \notin L(r_1, \leq).$$

By the closure and order-density of the maximal chains of R and S, we may assume $s_1 \neq a$, $s_2 \neq b$, and $r_1 \neq w$. We note that

$$\{z\} = L(a, \leqq) \cap L(b, \leqq) \cap L(w, \leq)$$

and that $L(a, \leqq) \cap L(w, \leq)$ is a closed chain in both R and S. Let

$$d = \sup_{\leqq} [L(a, \leqq) \cap L(w, \leq)].$$

Clearly, $a \neq d \neq w$, $z \leqq d \leqq a$, $z \leq d \leq w$, $d \in C$, and

$$L(a, \leqq) \cap M(d, \leqq) \cap L(w, \leq) = \{d\}.$$

Suppose $d \neq z$. Let U be any neighborhood of d for which

$$\bar{U} \subset C-[M(a, \leqq) \cup M(w, \leq) \cup L(r_2, \leq)].$$

By remarks in §2, each of the chains $L(a, \leqq) \cap M(d, \leqq)$, $L(w, \leq) \cap M(d, \leq)$, and $L(d, \leq)$ is connected. Furthermore, each has a point in U and a point in $\bar{C}-\bar{U}$, so that each of the three chains meets $F(U)$. By the preceding paragraph, the only point any pair of the three chains have in common is $d \in U$. Thus $F(U)$

contains at least three points, contradicting that d has order 2. We obtain a similar contradiction if $e \neq z$, where

$$e = \sup_{\leqq} [L(b, \leqq) \cap L(w, \leq)].$$

We conclude from the foregoing that

$$z = \sup_{\leqq} [L(a, \leqq) \cap L(w, \leq)]$$

and

$$z = \sup_{\leqq} [L(b, \leqq) \cap L(w, \leq)].$$

For any neighborhood U of z for which

$$\bar{U} \subset C - [M(a, \leqq) \cup M(w, \leq) \cup M(b, \leqq)]$$

we have that $F(U)$ meets each of $L(a, \leqq)$, $L(w, \leq)$, and $L(b, \leqq)$. But the intersection of any pair of these chains is $z \in U$, which again contradicts that $z \in C$ is of order 2. It follows that $R \cup S$ is an arc.

We may cover C with open sets R such that $\bar{R} \subset C$ and $\bar{R}$ is an arc. By the connectivity of C, for any pair of points p, q in C, there is a finite family $R_1, R_2, \ldots, R_n$ of these open sets such that $p \in R_1$, $q \in R_n$, and

$$R_i \cap R_j \neq \phi \quad \text{if and only if} \quad |i-j| < 2.$$

By induction on the preceding lemma, $\bar{R}_1 \cup \bar{R}_2 \cup \cdots \cup \bar{R}_n$ is an arc. Thus any pair of points of C lie in an open, connected subset of C whose closure is an arc and is contained in C.

Since E is cyclic, $F(C)$ is nondegenerate. Let $x \neq y$ be elements of $F(C)$. There exist neighborhoods U and V of x and y, respectively, such that

$$\bar{U} \cap \{x_1, x_2, \ldots, x_n\} = \{x\}, \qquad \bar{V} \cap \{x_1, \ldots, x_n\} = \{y\}, \qquad \bar{U} \cap \bar{V} = \phi,$$

and such that $\bar{U}$ and $\bar{V}$ are trees. If $\leq$ is the cutpoint order on $\bar{U}$ with zero x, and if $a \in U \cap C$, then $L(a)$ is a closed, connected chain and hence is an arc. Since $\bar{U} \cap \{x_1, \ldots, x_n\} = \{x\}$, and since $L(a) - x$ is connected, then $L(a) - x$ lies in a component of $E - \{x_1, \ldots, x_n\}$, namely C. Thus $L(a) = A$ is an arc in $\bar{C}$ from x to a. Similarly, we obtain an arc B in $\bar{C}$ from y to some $b \in V \cap C$. From the preceding paragraph there is an arc T in $\bar{C}$ from a to b, so by the lemma $A \cup T \cup B$ is an arc in $\bar{C}$ from x to y. If $d \in C$, then there exists an open subset R of C such that $a, d \in R$ and $\bar{R}$ is an arc contained in C. By the lemma, $\bar{R} \cup (A \cup T \cup B)$ is an arc. Since the endpoints of $\bar{R}$ lie in C, $\bar{R} \subset A \cup T \cup B$. Thus $C \subset A \cup T \cup B$, but since $A \cup T \cup B$ is closed, $\bar{C} = A \cup T \cup B$, and $F(C) = \{x, y\}$.

It remains to show that the components of $E - \{x_1, x_2, \ldots, x_n\}$ are finite in number. We may choose neighborhoods $R_1, R_2, \ldots, R_n$ of $x_1, x_2, \ldots, x_n$ respectively, such that $\bar{R}_i$ is a tree $(i = 1, 2, \ldots, n)$ and $\bar{R}_i \cap \bar{R}_j = \phi$ for $i \neq j$. By Lemma 3.3

for each $i=1, 2, \ldots, m$, $\bar{R}_i - x_i$ has finitely many components, each of which is contained in a component of $E-\{x_1, x_2, \ldots, x_n\}$. It is clear that each component of $E-\{x_1, x_2, \ldots, x_n\}$ must contain points of at least two of the sets $\bar{R}_i - x_i$. This proves that a compact, cyclic local tree has the stated properties of the theorem. The converse is obvious.

We shall call the decomposition of a compact, cyclic local tree E into arcs whose endpoints are $\{x_1, \ldots, x_n\}$ $(n \geq 2)$, the *arc-decomposition* of E. In case E has no points of order at least 3, the foregoing shows that E is the union of two arcs with endpoints x_1 and x_2, where x_1 and x_2 are arbitrarily chosen in E, and therefore E is a "simple closed curve." Otherwise, $x_1, \ldots, x_n$ are the points of order at least 3 in E, and the arc-decomposition is unique.

We shall say a space X is *regular* in case if $x \in U \subset X$, and if U is open, then there exists a neighborhood V of x such that $\bar{V} \subset U$ and $F(V)$ is finite. By [1, p. 140], a tree is regular, and consequently a local tree is also regular.

THEOREM 3.6. *A necessary and sufficient condition that a locally connected continuum be a local tree is that it have at most finitely many prime chains, each of which is the union of finitely many arcs meeting only at their endpoints.*

Proof. For the necessity, let X be a compact local tree, and let $x \in X$. From the remarks preceding the statement of the theorem, there exists a neighborhood V of x such that $\bar{V}$ is a tree and such that $F(V)$ is a finite set. Then

$$X-F(V) = V \cup (X-\bar{V}), \qquad V \mid X-\bar{V}.$$

If E is a prime chain of X, then $E \cap (X-\bar{V}) \neq \phi$. If $E \cap V \neq \phi$, then since E is cyclic and connected, $E \cap F(V)$ must contain at least two points, and no other prime chain of X can contain these same two points. Since $F(V)$ is finite, V meets at most finitely many prime chains of X.

We may cover X with such neighborhoods, each of which meets at most finitely many prime chains. This cover admits a finite subcover, so there can be at most finitely many prime chains in X. The remainder of the proof of the necessity of the condition follows from Theorem 3.5.

Let X be a locally connected continuum with at most finitely many prime chains satisfying the condition of the theorem. If X has no prime chains, then X is a tree. Suppose X has prime chains $E_1, \ldots, E_n$. If $x \in X-(E_1 \cup \cdots \cup E_n)$, then let U be a connected neighborhood of x with

$$\bar{U} \subset X-(E_1 \cup \cdots \cup E_n).$$

For any pair of points of $\bar{U}$, there exists $z \in X$ such that z separates the pair in X. By connectivity of $\bar{U}$, $z \in \bar{U}$, so $\bar{U}$ is a tree. Suppose then that x lies in one or more prime chains of X, say $x \in E_1 \cap \cdots \cap E_k$, $k \leq n$. As a consequence of Theorem 3.5, there exists a connected neighborhood U of x such that $\bar{U} \cap E_i$ is a tree for each $i=1, 2, \ldots, k$. If a pair of points y and z in $\bar{U}$ do not lie in the same prime chain of X, then some w in X separates y and z in X, and by the connectivity of $\bar{U}$, $w \in \bar{U}$; thus $\bar{U}$ is a tree.

4. **Theorem B.** A few short lemmas lead to the second characterization of local trees.

LEMMA 4.1. *Let X be a continuum which admits a semicontinuous, order dense partial order $\leq$, and let X have a zero. If E is a prime chain of X, then E has a unique minimal element.*

Proof. Since E is compact, E has a minimal element. Suppose $x \neq y$ are minimal elements of E. The zero of X lies in $L(x) \cap L(y)$, which is closed in X. Let w be a maximal element of $L(x) \cap L(y)$, let K be a maximal chain in $L(x) \cap M(w)$, and let J be a maximal chain in $L(y) \cap M(w)$. Since X is order dense, K and J are connected. Further, $K \cap J = \{w\}$. By connectivity, no pair of points of $E \cup K \cup J$ can be separated by any point of X, so $K \cup J \subset E$, which implies $x = w = y$, contradicting $x \neq y$. Thus E has a unique minimal element.

Let X be a topological space, and let $\leq$ be a semicontinuous partial order on X. For x and y in X, we write

$$x \| y \quad \text{if and only if} \quad x \not\leq y \quad \text{and} \quad y \not\leq x.$$

Throughout the sequel, N will represent the set of all points $y \in X$ such that there exists $b \in X$, $b < y$, and $y \in F(M(b))$. Note that for each $x \in X$, $F(M(x)) \subset \{x\} \cup N$. Finally, for $A \subset X$, we shall let max A denote the set of maximal elements of A and let min A denote the set of minimal elements of A.

LEMMA 4.2. *Let X be a continuum which admits a semicontinuous partial order $\leq$ such that X is order dense and has a zero, 0. If x, p and q are elements of X, $p < x$, $q < x$, and $p \| q$, then there exist connected chains P and Q, containing p and q, respectively, such that $P \cap Q = \{a, b\}$, where a is the maximal element of P and of Q, and b is the minimal element of P and of Q. Furthermore, $a \in N$, and $P \cup Q$ is contained in a prime chain of X.*

Proof. Let P' and Q' be maximal chains in $L(x)$ containing p and q, respectively. Then

$$x \in A = M(p) \cap M(q) \cap P' \cap Q',$$

$$0 \in B = L(p) \cap L(q) \cap P' \cap Q',$$

and A and B are nonvoid closed chains in X. Let $b = \max B$ and $a = \min A$. Then $b < p$, $q < a$. Letting

$$P = M(b) \cap L(a) \cap P', \qquad Q = M(b) \cap L(a) \cap Q',$$

we have $P \cap Q = \{a, b\}$, $\max P = a = \max Q$, $\min P = b = \min Q$, $p \in P$ and $q \in Q$. By the connectivity of P and Q, every neighborhood of a meets each of $P - a$ and $Q - a$, so $a \in F(M(p))$, and therefore $a \in N$. Also by connectivity, no pair of points in $P \cup Q$ can be separated by any point of X.

LEMMA 4.3. *Let X be a continuum which admits a semicontinuous, order dense partial order having a zero* 0, *such that N is a finite set. If $x \in X-N$ and $x \neq 0$, then there exists $b<x$ such that $M(b) \cap L(x)$ is a chain, and $M(b) \cap N = M(x) \cap N$.*

Proof. Let $a<x$. If $M(a) \cap L(x)$ is not a chain, then there exist $p, q \in M(a) \cap L(x)$ such that $p \| q$, and by Lemma 4.2, $M(a) \cap L(x) \cap N \neq \phi$. Let

$$n \in \max\,[M(a) \cap L(x) \cap N].$$

Then $a<n<x$, and by applying Lemma 4.2 again, we have that $M(n) \cap L(x)$ is a chain. Thus, by choosing $a \in X$ with $n<a<x$, we have $M(a) \cap L(x) \cap N = \phi$, and $M(a) \cap L(x)$ is a chain. Suppose

$$(M(a)-M(x)) \cap N = \{n_1, \ldots, n_k\}.$$

For each $i=1, 2, \ldots, k$, $a \in L(n_i) \cap M(a) \cap L(x)$. Thus

$$[L(n_1) \cup \cdots \cup L(n_k)] \cap M(a) \cap L(x)$$

is a nonvoid, closed chain in $M(a) \cap L(x)$. Let

$$p = \max\,[(L(n_1) \cup \cdots \cup L(n_k)) \cap M(a) \cap L(x)].$$

For some $1 \leq j \leq k$, $p \in L(n_j) \cap M(a) \cap L(x)$, so $n_j \in M(p)-M(x)$, and $a<p<x$. Now letting $b \in X$, $p<b<x$, we have $M(b) \cap N = M(x) \cap N$, and $M(b) \cap L(x)$ is a chain.

LEMMA 4.4. *If X is a continuum with a semicontinuous, order dense partial order having a zero* 0, *and if N is finite, then every maximal element of a prime chain of X lies in N.*

Proof. Suppose m is a maximal element of a prime chain E of X, and suppose $m \notin N$. By Lemma 4.3, there exists $b \in X$, $b<m$ such that $M(m) \cap N = M(b) \cap N$, furthermore, for any $x \in X$, if $b<x<m$, then $M(x) \cap N = M(m) \cap N$.

We claim $b<e$ or $e \leq b$, where $e = \min E$. (The existence of e is assured by Lemma 4.1.) Suppose $e \| b$. By Lemma 4.2 there exist connected chains P and Q containing b and e, respectively, such that $P \cap Q = \{c, d\}$, where $c<e<d$, and such that $P \cup Q$ lies in a prime chain. This implies $P \cup Q \subset E$, which contradicts $e = \min E$. Thus our claim is established, and we may choose $d \in X$ such that $e<d<m$ and $b<d$.

We claim next that $M(m) \cap N \neq \phi$, for otherwise, $M(d) \cap N = M(m) \cap N = \phi$ implies $F(M(d)) = \{d\}$, and therefore d is a cutpoint of X, separating X into disjoint open sets $M(d)-d$ and $X-M(d)$, one of which contains e and the other m, a contradiction that E is a prime chain.

Let $M(m) \cap N = \{n_1, \ldots, n_k\}$. For each $i=1, 2, \ldots, k$, $m<n_i$ and $M(m) \cap E = \{m\}$, so there exists $x_i \in X$ such that

$$X-x_i = A_i \cup B_i, \qquad A_i | B_i, \qquad m \in A_i, \qquad n_i \in B_i.$$

By the connectivity of $M(m) \cap L(n_i)$, we have $m < x_i < n_i$. We claim $0 \in A_i$ for each i, for otherwise if $0 \in B_i$ for some i, then by the connectivity of $L(m)$, $0 < x_i < m$, which contradicts $m < x_i$. Similarly, for each $t \in B_i$, $0 \in A_i$ implies $x_i < t$, so $B_i \subset M(x_i)$. Letting

$$B = B_1 \cup B_2 \cup \cdots \cup B_k,$$

we have that B is open and

$$M(d) \cap N = M(m) \cap N \subset B \subset \bigcup \{M(x_i) : i = 1, 2, \ldots, k\} \subset M(m) \subset M(d).$$

Therefore $F(M(d)) = \{d\}$, which leads to the same contradiction of the preceding paragraph. Thus each maximal element of a prime chain of X lies in N.

COROLLARY 4.5. *With the hypotheses of Lemma* 4.4, *X has at most finitely many prime chains.*

Proof. Suppose a maximal element of a prime chain E of X is also a maximal element for a prime chain G of X. By an application of Lemma 4.2, $\min E = \min G$, so that $E = G$. Every prime chain has a maximal element; the maximal elements of prime chains lie in N; and N is a finite set.

LEMMA 4.6. *If X is a locally connected continuum, and if E is a prime chain of X, then for each point $p \in X - E$ there exists a unique point e of E such that e separates p and $E - e$ in X.*

Proof. Let $\leq$ be the cutpoint order on X with zero p, let $q \in E$, and let

$$C = \{x \in X\colon\ x \text{ separates } p \text{ and } q \text{ in } X\} \cup \{p, q\}.$$

By [8, (1.31) and (4.2) on p. 43 and p. 51], C is a compact chain in X. For each $v \in C - \{p, q\}$, we have the separation

$$X - v = A_v \cup B_v, \qquad p \in A_v, \qquad q \in B_v,$$

and we may assume A_v is a connected set. Further, since $E - v$ is connected, $E - v \subset B_v$. Let

$$A = \bigcup \{A_v\colon\ v \in C - \{p, q\}\}, \qquad B = \bigcap \{B_v\colon\ v \in C - \{p, q\}\}.$$

Then $X = A \cup B$, $E \subset B$, A is open, and B is closed. Further $A \cap B = \phi$, so since X is connected $\bar{A} \cap B \neq \phi$. Let $x \in \bar{A} \cap B$. Then every connected neighborhood of x contains infinitely many points of $C - \{p, q\}$, but this means $x \in C$. Clearly, either $x = q$ or $x = \max (C - q)$. Since C is a chain, $\{x\} = \bar{A} \cap B$, and x separates p and $E - x$ in X. If $x \in E$, then let $e = x$. Suppose $x \notin E$. Then no point of X separates x and q, so x and q lie in a prime chain $G \neq E$. It readily follows that $G \cap E = \{q\}$, and that q separates p and $E - q$ in X. Whence in the case $x \notin E$, let $e = q$. That e is unique is easily proved.

THEOREM 4.7. *A necessary and sufficient condition that a compact Hausdorff space X be a local tree is that X admit a partial order $\leq$ such that*

(i) $\leq$ *is semicontinuous*;

(ii) $\leq$ *is order dense*;

(iii) *For $x \in X$ and $y \in X$, $L(x) \cap L(y)$ is the nonvoid union of finitely many chains*;

(iv) *The set N is finite.*

Proof. (*Sufficiency*). If x and y are minimal elements of X, then $L(x) \cap L(y) \neq \phi$ implies $x=y$. Thus X has a zero, 0. By [5, Theorem 5, p. 151] X is connected, and therefore X is a continuum satisfying the hypotheses of Lemma 4.3.

By the corollary on p. 104 of [9], a locally compact, connected Hausdorff space cannot fail to be locally connected at only finitely many points; we shall show X is locally connected at each point of $X-(N \cup \{0\})$. This will be done using the theorem on p. 104 of [9], which states in part that a sufficient condition that a locally compact, connected Hausdorff space S be locally connected is that if two points x and y of S lie in a component of an open subset G of S, then x and y lie in a subcontinuum of G.

Let $a \in X-(N \cup \{0\})$. By Lemma 4.3, there exists $b \in X$ such that $b<a$, $M(b) \cap L(a)$ is a chain, and $M(b) \cap N=M(a) \cap N$. Let

$$S = M(b)-\bigcup \{M(n) : n \in M(b) \cap N\}.$$

Clearly, S is locally compact as a subspace of X, and $S-b$ is open in X (since $F(M(b)) \subset N \cup \{b\}$). Further, S is connected since for all $x \in S$, $M(b) \cap L(x)$ is connected and is contained in S. If we show S is locally connected, then $S-b$ is locally connected and is open in X, whence X is locally connected at a.

Let x and y be two points of S which lie in a component C of an open subset G of S. Either x and y are related by $\leq$ or they are not. Suppose $x<y$. Let $z \in S$ such that $x<z<y$. The sets $(M(z)-z) \cap G$ and $G-M(z)$ are open and disjoint in S, and each meets C; since their union is $G-z$ we have $z \in C$. Thus $M(x) \cap L(y) \subset C$, and $M(x) \cap L(y)$ is a subcontinuum of G. On the other hand, suppose $x \| y$. If max $[L(x) \cap L(y)]$ is nondegenerate, then by Lemma 4.2 we have $\phi \neq L(x) \cap M(b) \cap N \subset S$, which contradicts the definition of S. Let

$$t = \sup\,[L(x) \cap L(y)].$$

For $z \in S$ and $t<z<x$, we have that the sets $(M(z)-z) \cap G$ and $G-M(z)$ are open and disjoint in S, and $x \in (M(z)-z) \cap G$, $y \in G-M(z)$, so $z \in C$. Thus $(M(t) \cap L(x))-t \subset C$. Similarly, we have $(M(t) \cap L(y))-t \subset C$. If we show $t \in C$, then we have that $M(t) \cap (L(x) \cup L(y))$ is a subcontinuum of G containing x and y. Let

$$A = \bigcup \{(M(z)-z) \cap S : t < z < x\}.$$

We claim $\bar{A} \cap S=A \cup \{t\}$. Suppose on the contrary that there exists $m \in (\bar{A}-A) \cap S$ and $m \neq t$. Since $\bar{A} \subset M(t)$, $t<m$. Let $p \in S$ and $t<p<m$. We have $m \in \bar{A} \cap M(p)$,

and therefore $A \cap M(p) \neq \phi$, for otherwise $m \in F(M(p))$ so $m \in N$, which contradicts $m \in S$. Now there exists $q \in S$ with $t < q < x$ and

$$(\dagger) \qquad S \cap (M(q)-q) \cap M(p) \neq \phi.$$

Since $p < m$ and $m \notin A$, $q \not< p$. If $p \leq q$, then $t < p \leq q < x$ implies $p \in A$, and therefore $m \in A$, which is impossible. Thus $p \| q$, but (†) and Lemma 4.2 imply $S \cap N \neq \phi$, a contradiction. Whence our claim is established that $\bar{A} \cap S = A \cup \{t\}$. Now,

$$G-t = (G \cap A) \cup (G-\bar{A}),$$

$x \in G \cap A$, $y \in G-\bar{A}$, and $A \mid (G-\bar{A})$, so $t \in C$. With our previous remarks we have that X is locally connected.

By Corollary 4.5, X has at most finitely many prime chains, and we may assume X has at least one since otherwise X is a tree. If E is a prime chain of X, we shall show E is the union of finitely many arcs which may meet only at their endpoints. By Lemma 4.4, max E is finite, say max $E = \{m_1, \ldots, m_k\}$. By (iii), $L(m_i)$ is the union of finitely many chains; we may assume they are maximal chains (and hence connected) in $L(m_i)$. By Lemma 4.1, E has a unique minimal element, e, and $e \in L(m_i)$ for all $i = 1, 2, \ldots, k$. Let K be a maximal chain in $L(m_i) \cap M(e)$. We wish to show $K \subset E$. Suppose there exists $u \in K-E$. Then there exists $v \in X$ such that

$$X-v = A \cup B, \qquad A|B, \qquad u \in A, \qquad e \in B.$$

The maximal chain K is connected, so $v \in K$. Since $L(u) \cap K$ is a connected chain and contains e, $e < v < u$, and $v \neq m_i$. If $m_i \in B$, then since $M(u) \cap K$ is connected and contains m_i we have $u < v < m_i$, a contradiction. Thus $m_i \in A$, but this means v separates m_i and e, which is also a contradiction. Thus we must conclude $K \subset E$, and therefore

$$L(m_i) \cap M(e) = L(m_i) \cap E$$

is the union of finitely many connected chains, maximal in E. Now since

$$E = [L(m_1) \cup \cdots \cup L(m_k)] \cap E,$$

E is the union of finitely many connected chains, maximal in E, say $C_1, \ldots, C_n$. If $n = 1$, then our result is obtained. We shall assume then that $n > 1$.

For each $i = 1, 2, \ldots, n$, let

$$D_i = C_i - \bigcup \{C_j \colon i \neq j,\ j = 1, 2, \ldots, n\}.$$

Let P be a nondegenerate component of D_i. The closure of P lies in C_i, so in its relative topology $\bar{P}$ has at most two noncutpoints $p \leq q$. Therefore $\bar{P}$ is an arc with endpoints p and q. If $q \in \max E$, then $q \in N$ by Lemma 4.4. If $q \notin \max E$, then for some $j \neq i$, $1 \leq j \leq n$, each neighborhood of q meets C_j; thus $q \in C_j$. The set $C_j \cap (L(q)-q)$ is connected, and every neighborhood of q meets $C_j \cap (L(q)-q)$. If $z \in P$, $p<z<q$, and if we show

$$M(z) \cap C_j \cap (L(q)-q) = \phi,$$

then $q \in F(M(z))$, and therefore $q \in N$. Suppose there exists a point

$$y \in M(z) \cap C_j \cap (L(q)-q).$$

Then we have $z<y<q$. If $y \in C_i$, then by the connectivity of P, $y \in P$ since y separates z and q in C_i; but $y \notin P$ since $y \in C_j$ and $P \subset D_i$. Thus z, y, and q lie in some C_m with $m \neq i$, but this contradicts $z \in P \subset D_i$. Hence the maximal element q of each $\bar{P}$ lies in N. Since $E \cap N$ is finite, it follows that the set of all $\bar{P}$ as i runs from 1 to n is finite. Any isolated point of $\bigcap \{C_i\colon i=1, 2, \ldots, n\}$ lies in some $\bar{P}$. If Q is a nondegenerate component of $\bigcap \{C_i\colon i=1, 2, \ldots, t\}$ with endpoints $p<q$, then Q is an arc and either $p=e$ or $p \in N$ by an analysis similar to that on $\bar{P}$. Thus the number of $\bar{P}$'s and Q's is finite, they cover E, and no two have any points in common except possibly endpoints. By Theorem 3.6, the proof of the sufficiency of the condition is complete.

(*Necessity*). Let X be a compact local tree. Choose and fix some point 0 of X, and let $\leq_1$ be the cutpoint order on X with zero 0. If X has no prime chains, then X is a tree, and $\leq_1$ satisfies (i)–(iv) by Theorem 2.3. Assume then that X has prime chains. We shall define a relation $<_2$ on each prime chain, and then we shall define $\leq$ to be a "sum" of $\leq_1$ and $<_2$.

Let E be a prime chain of X. There exists a unique point e in E such that $e \leq_1 x$ for all $x \in E$ by Corollary 4.6 or by the fact that $0 \in E$. Let $\{A_1, \ldots, A_k\}$ be an arc-decomposition of E (Theorem 3.5), and let D be the set of endpoints of $A_1, \ldots, A_k$. We shall assume $e \in D$, since otherwise e lies on a unique A_i, which can be decomposed into two arcs each having e as an endpoint; then the two arcs can be added to the collection $\{A_1, \ldots, A_k\}$, A_i can be deleted, and e can be added to D. The language of graph theory [2, pp. 1–30] is most convenient for our present purposes, and we realize E as a graph with vertices D and edges $\{A_1, \ldots, A_k\}$. As is customary, if A_i has endpoints x and y, then we write $A_i=(x, y)$. Using a concept of distance between vertices [2, p. 27], let

$$G_i = \{v \in D\colon \text{dist}\,[e, v] = i\}.$$

Since D is finite and E is connected, there exists a positive integer m such that

$$D = G_0 \cup G_1 \cup \cdots \cup G_m,$$

$G_0=\{e\}$, $G_i \neq \phi$ for each $i=0, 1, \ldots, m$, and the sets $G_0, G_1, \ldots, G_m$ are pairwise disjoint. We linearly order each G_i in some arbitrary (then fixed) manner by a relation $\prec$ and then define $x<_3 y$ for $x, y \in D$ in case either

(1) $x \in G_i$ and $y \in G_i$, $x \prec y$, and there exists an edge (x, y) in E; or
(2) $x \in G_j$, $y \in G_{j+1}$ $(0 \leq j<m)$, and there exists an edge (x, y) in E.

Clearly, for each edge $A_i=(a, b)$, $a<_3 b$ or $b<_3 a$. We now define $<_2$ on E as follows: $x<_2 y$ in case

(1) x and y lie on the same edge (a, b) of E with $a<_3 b$, and $x=a$ or x separates a and y in the arc (a, b); or

(2) x and y do not lie on the same edge, and there exists a (graph) path $(a_0, a_1, \ldots, a_n)$ $(n>1)$ such that $x \in (a_0, a_1)$, $y \in (a_{n-1}, a_n)$, and $a_0<_3 a_1<_3 \cdots <_3 a_n$.

The relation $<_2$ is defined in this manner on each prime chain E of X. We note that two distinct points of X are related by $<_2$ only if they lie in the same prime chain, and that each prime chain E has a unique minimal element, namely e. Further, $<_2$ is anti-symmetric and transitive on each prime chain. Now for $x, y \in X$, we define $x \leq y$ in case $x \leq_1 y$, $x<_2 y$, or there exists $z \in X$ such that $x<_2 z \leq_1 y$.

In order to show that $\leq$ is transitive, it suffices to consider two cases: $x<_1 y<_2 z$ and $x<_2 y<_2 z$. In the first case $x<_1 z$, for if $x \neq 0$, then x separates 0 and y in X. If $x \in E$, and E is the prime chain containing y and z, then $x=\min E$, whence $x<_1 z$. If $x \notin E$, then x separates 0 and E, and $x<_1 z$. In the case $x<_2 y<_2 z$, x and y lie in a prime chain E, and y and z lie in a prime chain G. If $E=G$, then $x<_2 z$. Suppose $E \neq G$. Then $E \cap G=\{y\}$, and there exists $w \in X$ such that

$$X-w = R \cup S, \qquad R|S, \qquad x \in R, \qquad z \in S.$$

Since each of $E-w$ and $G-w$ is connected, we have $E-w \subset R$ and $G-w \subset S$; whence $y=w$. Now, $\min E<_1 y$, $\min E \in R$, and therefore $0 \in R$, which implies $y<_1 z$. Therefore $x<_2 y<_1 z$ or $x \leq z$.

The antisymmetry of $\leq$ follows readily from the following little lemma:

($*$) *if E is a prime chain containing a and b, and if $a<_2 b<_1 c$, then b is the unique point of E which separates c and $E-b$ in X. Furthermore, if*

$$X-b = A \cup B, \qquad A|B, \qquad E-b \subset A, \qquad c \in B, \qquad \textit{then } 0 \in A.$$

Considering the first assertion, $b \neq 0$, so by hypothesis

$$X-b = P \cup Q, \qquad P|Q, \qquad 0 \in P, \qquad c \in Q.$$

Since $\min E<_1 b$ and $E-b$ is connected, $E-b \subset P$. The point b is unique by Lemma 4.6. The second statement of the lemma follows from $\min E<_1 b<_1 c$. Thus the relation $\leq$ is a partial order on X.

By [5, Lemma 12, p. 156], $\leq_1$ is a semicontinuous partial order. We shall write $M(x)=M(x, \leq)$ and $L(x)=L(x, \leq)$. Let $x \in X$. Suppose $y \notin M(x)$. The point sets $M(x, \leq_1)$ and $M(x, <_2)$ are closed in X, and $M(x, \leq_1) \cup M(x, <_2) \subset M(x)$, so there exist open, connected subsets W and Y of X such that the following are satisfied: $y \in Y \cap W$; for all $v \in Y$, $x \not\leq_1 v$; and for all $w \in W$, $x \not<_2 w$. Let U be a connected neighborhood of y contained in $Y \cap W$. Suppose for some $v \in U$, there exists $z \in X$ such that $x<_2 z<_1 v$ (equality is not possible). Then

$$X-z = A \cup B, \qquad A|B, \qquad 0 \in A, \qquad v \in B.$$

Since U is connected and $z \notin W \cap Y$, $y \in U \subset B$. This means $z <_1 y$, but $x <_2 z <_1 y$ implies $y \in M(x)$, a contradiction. Hence $U \subset X - M(x)$, and $M(x)$ is a closed set in X. $L(x)$ is closed since $L(x, \leq_1)$ is closed, and since $L(x) - L(x, \leq_1)$ is void or the finite union of closed subsets of prime chains.

To verify the order-density of $\leq$, it suffices to consider two cases: $x <_1 y$ and $x <_2 y$. If $x <_2 y$, then the existence of z such that $x <_2 z <_2 y$ is immediate from the definition of $<_2$. Suppose $x <_1 y$. If x and y lie in a prime chain E, then $x = \min E$ and $x <_2 y$. Otherwise if x and y do not lie in a prime chain there exists $z \in X$ which separates x and y in X, from which we obtain $x <_1 z <_1 y$, see [8, (1.31) p. 43].

We shall show (iv) that N is finite by showing that each element of N is a vertex in some prime chain of X. Let $x \in N$, so that there exists $b \in X$ such that $b < x$ and $x \in F(M(b))$. We claim $b <_2 x$. For if $b <_1 x$, then $x \in F(M(b))$ implies $b \neq 0$, and

$$X - b = A \cup B, \qquad A|B, \qquad 0 \in A, \qquad x \in B,$$

and therefore $B \subset M(b)$. But B is open in X, which contradicts that $x \in F(M(b))$. On the other hand, if there exists $z \in X$ such that $b <_2 z <_1 x$, then by the lemma at (*)

$$X - z = A \cup B, \qquad A|B, \qquad 0 \in A \quad \text{and} \quad x \in B,$$

and again we obtain a contradiction of the fact that $x \in F(M(b))$. Thus $b < x$ and $x \in F(M(b))$ imply $b <_2 x$, and therefore b and x lie in a prime chain E of X.

Clearly, $f \in X$ and $b \leq f < x$ imply $x \in F(M(f))$ and $f \in E$. Thus we may assume without loss of generality that b and x lie on the same edge (a, d) of E, and $a < b < x \leq d$. We wish to show $x = d$. Suppose $x \neq d$. Let U be a connected neighborhood of x such that $\overline{U}$ is a tree and such that

$$A = \overline{U} \cap (a, d) = \overline{U} \cap E,$$

where $b < \min A < x < \max A < d$. Clearly, A is an arc, a chain in (a, d). We claim $\min A = \min \overline{U}$, for if $v \in \overline{U} - E$, then by Lemma 4.6 there exists a unique $z \in E$ such that

$$X - z = P \cup Q, \qquad P|Q, \qquad E - z \subset P, \qquad v \in Q.$$

By the connectivity of $\overline{U}$, $z \in \overline{U}$, and thus $z \in A$. Since $\min E \notin A$, we have $\min E \neq z$, and therefore $0 \notin Q$ by Lemma 4.6. Hence $z <_1 v$, from which we obtain $\min A < v$. Now we have

$$x \in U \subset M(\min A) \subset M(b),$$

which implies $x \notin F(M(b))$, a contradiction. Thus $x = d$, whence N is finite.

To verify (iii), it suffices to show that $L(x)$ is the union of finitely many chains for each $x \in X$. If $p, q \in L(x)$ and if $p \| q$, then by Lemma 4.2 p and q lie in a prime chain of X. Let $E_1, E_2, \ldots, E_n$ be those prime chains of X which meet $L(x)$. Each element of $L(x)-(E_1 \cup \cdots \cup E_n)$, lies in every maximal chain of $L(x)$. By construction of $\leq$, each E_i is the union of finitely many chains maximal in E_i, so it readily follows that $L(x)$ is the union of finitely many maximal chains of $L(x)$, the number of which is at most the total number of maximal chains in the prime chains $E_1, E_2, \ldots, E_n$.

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