A DECOMPOSITION OF MIXED ABELIAN GROUPS

BY

JOHN A. OPPELT

1. Introduction. For each torsion group $T$ and torsion-free group $J$, what conditions must be satisfied by $T$ and $J$ so that every extension $G$ of $T$ by $J$ is a direct sum of groups whose torsion subgroups are primary? Adapting the methods used by Baer in [1] we obtain, in the case that $J$ is a completely decomposable group, necessary and sufficient conditions which depend only on the divisibility properties of $T$ and $J$ (see Theorem 2.4).

We will name a mixed group, whose maximal torsion subgroup is $p$-primary, a $p$-mixed group. $\text{Ext}(J, T)$ consists of equivalence classes of extensions of the form

$$E: 0 \rightarrow T \rightarrow X \rightarrow J \rightarrow 0.$$ 

By $G \in \text{Ext}(J, T)$ we mean that some extension of the form $E$ with the group $G$ as the middle term belongs to an equivalence class in $\text{Ext}(J, T)$. Finally $[G]$ denotes an equivalence class of $G \in \text{Ext}(J, T)$.

2. The main theorems. We begin by establishing a sufficient condition for every $G \in \text{Ext}(J, T)$ to be a direct sum of $p$-mixed groups. Here $J$ is torsion-free and $T$ is a torsion group. So the maximal torsion subgroup of $G$, which we call $tG$, is isomorphic to $T$ and $G/tG \cong J$. We designate the primary decomposition of $T$ by $T=\sum T_p$ where the sum is taken over relevant primes. $T^p$ will be the direct sum of all the primary components of $T$ except for $T_p$. Also note that $\text{Ext}(J, T)=0$ means that every extension $G$ of $T$ by $J$ splits, i.e., $G \cong T \oplus J$.

Theorem 2.1. Let $T=\sum T_p$ be a torsion group and $J$ be torsion-free. Suppose $J=\sum J_p$ such that $\text{Ext}(J_p, T^p)=0$. Then every $G \in \text{Ext}(J, T)$ is a direct sum of $p$-mixed groups.

Proof. We carry out the proof by computing in our setting, the standard isomorphisms

$$\text{Ext}(J, T) \cong \prod_p \text{Ext}(J_p, T) \cong \prod_p \text{Ext}(J_p, T^p)$$

where the second isomorphism follows from $\text{Ext}(J_p, T) \cong \text{Ext}(J_p, T^p) \oplus \text{Ext}(J_p, T^p)$, the last term being zero by hypothesis.

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Let \((4)_p\) below be any element of \(\text{Ext}(J_p, T_p)\) and form the element \((2)\) of \(\text{Ext}(J, T)\), where \(f\) is the Cartesian product of the maps \(f_p\).

\[
\begin{array}{ccccccccc}
0 & \rightarrow & T & \rightarrow & \sum_{p} G_p & \overset{f}{\rightarrow} & J & \rightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow f_p = \text{inclusion map} \\
(3)_p & 0 & \rightarrow & T & \rightarrow & T^p \oplus G_p & \overset{0+f_p}{\rightarrow} & J_p & \rightarrow & 0 \\
& & \downarrow \text{proj.} & & & & & & & \\
(4)_p & 0 & \rightarrow & T_p & \rightarrow & G_p & \overset{f_p}{\rightarrow} & J_p & \rightarrow & 0 \\
\end{array}
\]

Commutativity of the diagram containing \((2)\) and \((3)_p\) shows that under the first isomorphism, call it \(\ast\), of \((1)\) the \(p\)th coordinate of \(i^*\) \((2)\) is \((3)_p\). Commutativity of the diagram containing \((3)_p\) and \((4)_p\) then shows that the image of \((3)_p\) under the second isomorphism of \((1)\) is \((4)_p\).

Since the extensions in \((4)_p\) were arbitrary, we conclude that every element of \(\text{Ext}(J, T)\) has the form \((2)\), as desired.

**Remark.** The author is indebted to the referee for the above less complicated proof and for an improved presentation of §3.

We now turn our attention to completely decomposable torsion-free groups. Let \(J = \sum J_{\alpha}, \alpha \in I\), where \(r(J_{\alpha}) = 1\). Each \(J_{\alpha}\) is characterized by its type (cf. [2, p. 207]), \(r(J_{\alpha}) = (k_1, k_2, \ldots)\) where \(k_\ell\) is a nonnegative integer or \(\infty\). For any group \(G\), \(p^\alpha G\) is the subgroup of elements of infinite \(p\)-height. If \(J\) is torsion-free of rank one then necessarily \(p^\alpha J = 0\) or \(p^\alpha J = J\). If \(J\) is completely decomposable, say \(J = \sum J_{\alpha}, \alpha \in I\), then \(p^\alpha J = \sum J_\beta, \beta \in I_p \subseteq I\), where \(I_p = \{\alpha \in I \mid p^\alpha J_{\alpha} = J_{\alpha}\}\). Clearly \(p^\alpha J\) is a direct summand of \(J\).

Before we state the major theorem we prove

**Proposition 2.2.** Let \(J = K \oplus L\) be a torsion-free group and let \(T\) be a torsion-group such that \(\text{Ext}(K, T) = 0\). If \(G \in \text{Ext}(J, T)\) then \(G = K \oplus G_L\) where \(G_L \in \in \text{Ext}(L, T)\).

**Proof.** Since \(J = K \oplus L\) and \(\text{Ext}(K, T) = 0\), \(\text{Ext}(L, T) \cong \text{Ext}(J, T)\) under the map \(\beta\) induced by the projection map \(J \rightarrow L\). Hence every element of \(\text{Ext}(J, T)\) has the form

\[
(0 \rightarrow T \rightarrow G_L \rightarrow L \rightarrow 0) = \beta(0 \rightarrow T \rightarrow G_L \oplus K \rightarrow J \rightarrow 0).
\]

**Corollary 2.3.** If every \(G_L \in \in \text{Ext}(L, T)\) is a direct sum of \(p\)-mixed groups then every \(G \in \in \text{Ext}(J, T)\) is also a direct sum of \(p\)-mixed groups.

**Theorem 2.4.** Let \(T = \sum T_p\) be a torsion group and \(J = \sum J_{\alpha}, \alpha \in I\), be a completely decomposable group where \(r(J_{\alpha}) = 1\) for each \(\alpha \in I\) and \(r(J_{\alpha}) = (k_1, k_2, \ldots)\). Then the following two conditions are necessary and sufficient for every \(G \in \in \text{Ext}(J, T)\) to be a direct sum of \(p\)-mixed groups:

(A) If two primes \(p \neq q\) are such that the orders of the elements in \(T_p|p^\alpha T_p\) and \(T_q|q^\alpha T_q\) are unbounded then \(p^\alpha J \cap q^\alpha J = 0\).
(B) If \( Q = \{ p \mid pT_p \neq T_p \} \) is an infinite set of primes then \( k^a_i = 0 \) for all \( a \in I \) and for all but a finite number of \( p_i \in Q \).

Before we prove the sufficiency of this theorem we will state Baer’s Theorem \([1, 8.1]\) in the case that \( r(J) = 1 \). This states that \( \text{Ext}(J, T) = 0 \) if and only if (i) whenever the orders of the elements in \( T_p/p^{aT_p} \) are unbounded then \( p^{aJ} = 0 \) and (ii) if \( Q = \{ p \mid pT_p \neq T_p \} \) is an infinite set of primes then \( k_i = 0 \) for all but a finite number of the primes \( p_i \in Q \) where \( r(J) = (k_1, k_2, \ldots) \).

Proof of the sufficiency of Theorem 2.4. Let \( P \) be the set of primes such that the orders of the elements in \( T_p/p^{aT_p} \) are unbounded. Referring to our remarks on completely decomposable groups we have \( p^aJ = \sum J_a, \beta \in I_p \subseteq I \) where \( J = \sum J_a, a \in I \). (A) tells us that \( I_0 = \bigcup I_p \) is a disjoint union so \( \sum p^aJ, p \in P \), is a direct sum. 

In fact if \( K = \sum J_a, \gamma \in I - I_0 \) then \( J = K \oplus \sum p^aJ \).

Now \( \text{Ext}(K, T) = \prod \text{Ext}(J_a, T) = 0 \) by Baer’s Theorem. So, by Corollary 2.3, we need only show that every extension of \( T \) by \( \sum p^aJ, p \in P \), is a direct sum of \( p \)-mixed groups.

Now \( p^aJ = \sum J_a, \alpha \in I_p \). (A) says that \( q^aJ_a = 0 \) for \( \alpha \in I_p \) and for \( q \neq p \). Since each of these \( J_a \) satisfies (B) for the set \( Q - \{ p \} \) we get \( \text{Ext}(J_a, T^p) = 0 \). So \( \text{Ext}(p^aJ, T^p) = \prod \text{Ext}(J_a, T^p) = 0 \). But this means that \( \sum p^aJ \) satisfies the conditions of Theorem 2.1 and so every \( G \in \text{Ext}(\sum p^aJ, T) \) is a direct sum of \( p \)-mixed groups.

3. Necessity of (A). If the \( p \)-primary component \( T_p \) of \( T \) has the property that \( T_p/p^{aT_p} \) is a group of bounded order then \( T_p = D \oplus C \) where \( D \) is divisible and \( C \) is a pure subgroup of bounded order. Hence \( T_p \) will be a direct summand of any group \( G \) containing \( T \) and we need not concern ourselves with these primary components.

The proof of the following lemma is in Baer [1].

**Lemma 3.1.** Let \( G \) be a mixed group and \( J \) a torsion-free group with \( G/tG \leq J \). Then there exists a mixed group \( G' \) satisfying:

(i) \( G \leq G' \) and \( tG = tG' \).

(ii) \( G'/tG' = J \) (i.e., there is an isomorphism between \( G'/tG' \) and \( J \) leaving invariant the elements of \( G/tG \)).

**Proposition 3.2.** Let \( J \) be a completely decomposable group such that \( q^aJ = J \) for some prime \( q \) and let \( T_q \) be a \( q \)-primary group such that the elements of \( T_q/q^aT_q \) have unbounded orders. Then there is an extension

\[
0 \to T_q \to S \to J \to 0
\]

such that \( T_q \) is not a direct summand of the complete inverse image in \( S \) of any nonzero pure subgroup of \( J \) (in particular \( T_q \) is not a direct summand of \( S \)).

**Proof.** Let \( J = \sum J_a, \alpha \in I \) be a decomposition of \( J \) into groups of rank one. Because \( J = q^aJ \) we can find, for a given \( j_\alpha \in J_\alpha \) and for each \( \alpha \in I \), elements \( f_\alpha^i \) in \( J_\alpha \) for \( i = 0, 1, 2, \ldots \) with

\[
f_\alpha^0 = j_\alpha \quad \text{and} \quad qf_\alpha^i = f_\alpha^{i-1}.
\]
Also (see page 203 of [2]) there are elements \( c_i, d_i \in T_q \) for \( i = 1, 2, \ldots \) satisfying the equations 
\[ d_i = \sum_{s=0}^{i-1} q^{s} c_{s+1} \]
with the property that the orders of the elements \( d_i \) tend to infinity with \( i \) and that they are elements of lowest order in their cosets mod \( q'T_q \).

Now form the mixed group \( K \) generated by the elements of \( T_q \) along with the elements \( \left\{ f_{\alpha}^i \mid \alpha \in I, i = 0, 1, 2, \ldots \right\} \) and which has all the relations of \( T \), commutativity relations and the additional relations \( qf_{\alpha}^i - f_{\alpha}^{i-1} = c_i \) for \( i > 0 \). Then \( K \) is a mixed group with \( tK = T_q \) and \( K/tK \) isomorphic to a subgroup of \( J \) where \( f_{\alpha}^i \rightarrow f_{\alpha}^i \).

By Lemma 3.1 we can find a mixed group \( S \) such that \( tS = tK = T_q \), \( K \subset S \) and \( S \in \text{Ext}(J, T_q) \) where the epimorphism \( \pi: S \rightarrow J \) with kernel \( T_q \) also sends \( f_{\alpha}^i \rightarrow f_{\alpha}^i \).

Now let \( x \) be a nonzero element of any pure subgroup of \( J \) and suppose \( T_q \) is a direct summand of the complete inverse image of this pure subgroup under \( \pi \). Then \( T_q \) will also be a direct summand of \( \pi^{-1}(C(x)) \) where \( C(x) \) is the minimal pure subgroup of \( J \) containing \( x \). Write \( \pi^{-1}(C(x)) = T_q \oplus H \). We shall show that this assumption gives rise to a contradiction.

Since \( x \in J = \sum J_\alpha, \alpha \in I \), we can write \( x = x_{a(1)} + \cdots + x_{a(m)} \) uniquely where \( 0 \neq x_{a(i)} \in J_{a(i)} \) for \( 1 \leq i \leq m \). Since \( r(C(x)) = 1 \), \( C(x) \) is contained in \( \sum_{i=1}^{m} J_{a(i)} \). Now the \( m \) independent elements, \( f_{a(0)}^0 \in J_{a(0)} \), which we chose above, must form, along with \( x \), a dependent set in \( \sum_{i=1}^{m} J_{a(i)} \). So we can find integers \( n, n_1, \ldots, n_m \) such that
\[ nx = n_1 f_{a(1)}^0 + \cdots + n_m f_{a(m)}^0. \]

Now, for each \( i = 0, 1, 2, \ldots \), we have
\[ q^i \left( \sum n_j f_{a(j)}^i \right) = \sum n_j f_{a(j)}^0 \in C(x), \]
where the sums are taken over \( j = 1, 2, \ldots, m \). But \( C(x) \) is a pure subgroup of the torsion-free group \( J \) so we must have \( \sum n_j f_{a(j)}^0 \in C(x) \) for all \( i = 0, 1, 2, \ldots \). Thus the elements \( \sum n_j f_{a(j)}^i \in \pi^{-1}(C(x)) = H \oplus T_q \). Therefore, for each \( i \), we get the unique expression
\[ \sum_{j=0}^{m} n_j f_{a(j)}^i = h^i + t_q^i \]
where \( h^i \in H \) and \( t_q^i \in T_q \). Now we apply the relations in (1) and get
\[ q \left( \sum n_j f_{a(j)}^i \right) = \sum n_j f_{a(j)}^{i-1} + c_i \sum n_j. \]
We will let \( \sum n_j = N \). Multiplying by \( q \) in (2) and using (3) gives
\[ \sum n_j f_{a(j)}^{i-1} = qh^i + (qt_q^i + Nc_i). \]

Since \( qt_q^i + Nc_i \) is an element of \( T_q \) we must have, because of the uniqueness of the expression (2) that
\[ t_q^{i-1} = qt_q^i - Nc_i, \]
and
\[ q^i t_q^i - t_q^i = N \sum_{s=0}^{i-1} q^s c_{s+1} = Nd_i. \]
But then $-t_i^0 \equiv Nd_i \mod q'T_q$ for all $i$. Since the $d_i$ were chosen having the property that $O(d_i)$ tend to infinity with $i$ then the same is true for $O(Nd_i)$. To produce our contradiction we want to show that $O(-t_i^0) \geq O(Nd_i)$ for infinitely many $i$.

We can assume $N=q^k$ and so we choose $i_0$ such that $O(d_i) > q^k$ for $i \geq i_0$. Thus $q^k < O(d_i) \leq O(d_i + q^it_q)$ for every $t_q \in T_q$ by the way the $d_i$ were chosen. So $O(q^it_i) \leq O(q^k(d_i + q^it_q))$.

So for each $i \geq i_0$ we need only show that $-t_i^0 = q^k(d_i + q^it_q)$ for some $t_q \in T_q$. By successive applications of (4) we get $t_i^q = q^k t_i^q + q^k t_i^q$ for some $t_i^q \in T_q$. Therefore $t_i^q = q^k t_i^q$ for $t_i^q \in T_q$. Hence by (5)

$$-t_i^0 = q^k d_i + q^k t_i^q = q^k(d_i + q^k t_i^q).$$

This completes the proof.

We can now complete the proof of the necessity of (A). Let $T$ be a torsion group such that for some two primes $p \neq q$ the elements of $T_{pa/p^aT_p}$ and of $T_{qa/q^aT_q}$ have unbounded order; and let $J$ be a completely decomposable group such that $(p^aJ) \cap (q^aJ) \neq 0$. Then, since $J$ is a direct sum of groups of rank 1,

$$J = (p^aJ \cap q^aJ) \oplus K$$

for some $K$. Let

(1) $0 \longrightarrow T = T_p \oplus T_q \oplus T' \overset{i}{\longrightarrow} S \oplus K \oplus T' \overset{v}{\longrightarrow} J = (p^aJ \cap q^aJ) \oplus K \longrightarrow 0$

be any element of Ext($J$, $T$) for which $i$ is the identity on $T'$ and $v$ is the identity on $K$ and $v(S) = p^aJ \cap q^aJ$. In any decomposition of $S \oplus K \oplus T'$ into a direct sum of groups whose torsion subgroups are the primary components of $T$, the primary components of $T'$ will be direct summands of the terms in which they occur. Thus it will be sufficient to construct our example (1) for the case $T' = 0$ (and later add $T'$).

To further reduce our construction to the case $K = 0$ it will be sufficient to show that any direct sum decomposition $S \oplus K = G_p \oplus G_q$ with $t(G_p) = T_p$ and $t(G_q) = T_q$ induces a direct sum decomposition $S = (S \cap G_p) \oplus (S \cap G_q)$. But this follows from the fact that $S$ is the set of elements $x$ of $S \oplus K$ for which $v(x) \in p^aJ \cap q^aJ$.

Now construct two extensions which fill the bill of Proposition 3.2 (the first with $p$ in place of $q$):

$$0 \longrightarrow T_p \longrightarrow S_1 \overset{\nu_1}{\longrightarrow} J \longrightarrow 0$$

$$0 \longrightarrow T_q \longrightarrow S_2 \overset{\nu_2}{\longrightarrow} J \longrightarrow 0$$

and let $S = \{(s_1, s_2) : \nu_1 s_1 = \nu_2 s_2\}$. We can identify $T_p$ with the subgroup $(T_p, 0)$ of $S$ and $T_q$ with $(0, T_q)$. $T = T_p \oplus T_q$ is then the torsion subgroup of $S$ and we can construct the following commutative diagram

(2) $0 \longrightarrow T = T_p \oplus T_q \longrightarrow S \overset{v}{\longrightarrow} J \longrightarrow 0$

$$\begin{array}{ccc}
0 & \longrightarrow & T_q \\
\downarrow \nu_2 & & \downarrow \nu_1 \\
0 & \longrightarrow & S_2 \overset{\nu_2}{\longrightarrow} J & \longrightarrow 0
\end{array}$$

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where \( \nu(s_1, s_2) = \nu_1(s_1) = \nu_2(s_2) \) and \( \pi_2 \) is the coordinate projection map. Suppose now, by way of contradiction, that \( S = G_p \oplus G_q \) with the torsion subgroups of \( G_p \) and \( G_q \) equal to \( T_p \) and \( T_q \) respectively. Then \( J = \nu(S) = \nu(G_p) \oplus \nu(G_q) \) with at least one term, say the first, nonzero. The complete inverse image of \( \nu(G_p) \) in \( S \) is \( G_p \oplus T_q \). Thus by (2) the complete inverse image (via \( \nu_1 \)) of \( \nu(G_p) \) in \( S_2 \) is \( \pi_2(G_p) \oplus T_q \), the sum being direct since \( \text{ker } \pi_2 = T_p \). This shows that \( \pi_2(G_p) \) is torsion-free. But this contradicts Proposition 3.2, and hence completes the proof of the necessity of (A).

4. The necessity of (B). Let \( T \) be a torsion group such that the set of primes \( Q = \{ p \mid pT_p \neq T_p \} \) is infinite. Take a torsion-free group of rank one, \( J \), with \( \tau(J) = (k_1, k_2, \ldots) \) where \( k_1 > 0 \) for an infinite number of \( i \) with \( p_i \in Q \). We then construct an extension of \( T \) by \( J \) which is not a direct sum of \( p \)-mixed groups. With this accomplished we will then extend to the case where \( J \) is completely decomposable.

With this \( J \) and \( T \) choose \( j' \in J, j' \neq 0 \). Since \( \tau(j') = \tau(J) \) we have \( \nu_p(j') > 0 \) for an infinite number of \( p \in Q \). Let \( W = \{ p \in Q \mid \nu_p(j') > 0 \} \).

So for each \( p \in W \) choose \( j'_p \in J \) such that \( pj'_p = j' \). Form the mixed group \( K \) generated by the elements of \( T \) and by elements \( j \) and \( j_p \) for \( p \in W \), demanding that \( K \) has all the relations of \( T \), all commutative relations along with

\[
pj'_p - j = t_p \quad \text{for each } p \in W, \quad t_p \in T_p
\]

where \( t_p \) is chosen so that \( t_p \notin pT_p \) (which is possible since \( pT_p \neq T_p \) for \( p \in Q \) and thus for \( p \in W \)). We now have \( tK = T \) and \( K/tK \) isomorphic to a subgroup of \( J \) where \( j \rightarrow j' \) and \( j_p \rightarrow j'_p \).

What we intend to show is that \( K \) can have no direct summand \( S \) which is a torsion subgroup of \( K \) containing all but a finite number of the primary components of \( T \). We do this by contradiction.

So let \( K = S \oplus H \) where \( S \subseteq T \) and contains all but a finite number of the primary components of \( T \). Let the subset \( V \subseteq W \) be defined by \( V = W = \{ p \mid t_pH \neq 0 \} \). Since \( H \) can have only a finite number of nonzero primary components it follows that \( V \) is an infinite subset of \( Q \).

Now since \( K = S \oplus H \) is a direct sum we can write

\[
j = s + h \quad \text{and} \quad j_p = s^p + h^p
\]

uniquely where \( s, s^p \in S \) and \( h, h^p \in H \). Also, for \( p \in V \), we have

\[
ph^p - h = p(j_p - s^p) - (j - s) = t_p - ps^p + s.
\]

But \( ph^p - h \in H \) and \( t_p - ps^p + s \in S \) (since \( T_p \subseteq S \) for \( p \in V \)) so \( ph^p - h = t_p - ps^p + s = 0 \).

Now the torsion element \( s \), when written in its unique fashion as a sum of elements of the primary components of \( T \), can have only a finite number of nonzero components. So we let \( V' \) be that subset of \( V \) for which \( s \) has zero \( p \)-primary components. \( V' \) is still an infinite subset of \( Q \). Most important is the fact that \( V' \) is nonempty!
We now read the equations \( t_p - p s_p + s = 0 \) in \( T_p \) for \( p \in V' \), that is we look at the \( p \)-primary component of \( t_p - p s_p + s \) for \( p \in V' \). If we let the \( p \)-primary component of \( s_p \) be \( s^*_p \) we get \( t_p - p s^*_p = 0 \) (since the \( p \)-primary component of \( s \) is zero for \( p \in V' \)). But \( t_p - p s^*_p = 0 \) gives us \( p s^*_p = t_p \), in other words \( t_p \in p T_p \) which is contrary to the way \( t_p \) was chosen. Hence \( K \) can have no decomposition such as \( K = S \oplus H \) where \( S \) has the properties mentioned.

We can now, by applying Lemma 3.1, find a mixed group \( G \) satisfying:

(i) \( K \leq G \),
(ii) \( tK = tG = T \),
(iii) \( G/tG = J \) (leaving \( K/tG \) invariant).

If \( G \) had the above direct sum decomposition, namely \( G = S \oplus H' \) where \( S \subseteq T \) and \( S \) contains all but a finite number of the primary components of \( T \) then it would happen, since \( T \subseteq K \) and \( S \subseteq T \), that

\[ K = K \cap G = (K \cap S) \oplus (K \cap H') = S \oplus (K \cap H') \]

which we have shown to be impossible. So we have proven

**Theorem 4.1.** Let \( T = \sum T_p \) be a torsion group such that the set \( Q = \{ p \mid p T_p \neq T_p \} \) is infinite. Let \( J \) be a torsion-free group of rank one with type \( \tau(J) = (k_1, k_2, \ldots) \) where \( k_i > 0 \) for an infinite number of the \( p_i \in Q \). Then there is a mixed group \( G \in \text{Ext}(J, T) \) having no decomposition of the form \( S \oplus H \) where \( S \) is torsion and contains an infinite number of the \( T_p \) with \( p \in Q \).

**Corollary 4.2.** The group \( G \) of Theorem 4.1 is not a direct sum of \( p \)-mixed groups.

**Proof.** This follows because a rank one group \( J \) is indecomposable.

We now have the necessary ammunition to complete the proof of Theorem 2.4 by showing that \((B)\) is necessary. We have \( J = \sum J_\alpha \), where \( r(J_\alpha) = 1 \) for each \( \alpha \in I \).

If it happened that for some \( \beta \in I \), \( \tau(J_\beta) = (k_1^\beta, k_2^\beta, \ldots) \) and \( k_i^\beta > 0 \) for an infinite number of the \( p_i \in Q \) then we can get \( G \in \text{Ext}(J_\beta, T) \) satisfying Theorem 4.1. Let \( K = G \oplus \sum J_\alpha, \alpha \in I, \alpha \neq \beta \). Then \( K \in \text{Ext}(J, T) \).

Suppose now that \( K \) is a direct sum of \( p \)-mixed groups, say \( K = \sum K_p \). Since \( T \subseteq G \), \( G/T \) is a subgroup of the factor group \( K/T = \sum K_p/T_p \). Further- more \( G/T \) is a group of rank one (it is isomorphic to \( J_\beta \)) so \( G/T \) “hits” only a finite number of the \( K_p \)'s, say \( G/T \subseteq K_{p(1)}' \cdots + K_{p(n)}' \). Let \( \pi_K : K \rightarrow K/T \) be the natural quotient map. Then \( G = \pi_K^{-1}(G/T) \) so

\[ G \subseteq \pi_K^{-1}(K_{p(1)}' + \cdots + K_{p(n)}'). \]

Moreover

\[ \pi_K^{-1}(K_{p(1)}' + \cdots + K_{p(n)}') = K_{p(1)}' \oplus \cdots \oplus K_{p(n)}' \oplus S \]

where \( S \subseteq T \) and \( S = \sum T_q, q \neq p(i) \) for any \( i, 1 \leq i \leq n \). Since \( T \subseteq G \) this implies \( S \) is a direct summand of \( G \) which contradicts our choice of \( G \) in accordance with Theorem 4.1.
Bibliography


University of Notre Dame, 
Notre Dame, Indiana
University of Virginia, 
Charlottesville, Virginia