

CONJUGATING REPRESENTATIONS AND RELATED RESULTS ON SEMISIMPLE LIE GROUPS⁽¹⁾

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Introduction. In applying symmetry considerations to quantum mechanics, one is often forced to consider representations T of a symmetry group G by means of Hilbert space operators which may be either unitary or conjugate-unitary. Indeed, in some situations there is a well-determined subgroup N of G (necessarily normal and of index 2 in G) such that T_x must be unitary for $x \in N$ and conjugate-unitary for $x \in G - N$. Such a representation T will be called a conjugating representation of G (relative to N). More generally one can consider projective conjugating representations T , in which the homomorphism relation $T_x T_y = T_{xy}$ is replaced by $T_x T_y = \lambda(x, y) T_{xy}$ (the $\lambda(x, y)$ being complex scalars). These also are required in the applications to quantum mechanics.

In §1, by a small modification of Mackey's analysis, we shall classify the equivalence classes of irreducible projective conjugating representations of G in terms of the irreducible projective representations of N (provided N is of Type I). As an immediate application of this, we obtain in §2 a classification of all the irreducible unitary representations of any Type I group acting in real or quaternionic Hilbert space, in terms of those acting in complex Hilbert space. In working out this classification for a given group G , the essential step is to subdivide the self-conjugate irreducible complex representations T of G into two classes—those of real type and those of quaternionic type—according as the conjugate-linear map setting up the equivalence of T with its conjugate has positive or negative square.

Exactly the same results are obtained if, instead of unitary (not necessarily finite-dimensional) representations, we consider finite-dimensional (not necessarily unitary) representations.

The remaining sections of this paper concern finite-dimensional representations of semisimple groups. Fix a connected semisimple Lie group G . The family \hat{G} of all finite-dimensional (not necessarily unitary) irreducible complex representations of G is parametrized by the well-known Cartan-Weyl method of dominant weights. Which of the self-conjugate elements of \hat{G} are of real type and which are of quaternionic type? In §§3 and 4 this question is answered in two steps: First, Theorem 5 reduces the general case to the case that G is compact. It turns out that, if G is not compact, each self-conjugate element T of \hat{G} gives rise to a self-conjugate element

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S of \hat{M} , where M is a certain compact connected semisimple subgroup of G , and that S and T are of the same type. Finally, Theorem 6 shows how to distinguish the two types when G is compact. In that case, if ϕ is the dominant weight of a self-conjugate element T of \hat{G} , we can write 2ϕ as a positive integral combination of "strongly orthogonal" roots; and it turns out that the type (real or quaternionic) of T depends on whether the sum of the integer coefficients is even or odd. Propositions 1 to 9 give the detailed results deducible from Theorem 6 for the compact simple groups.

Putting together §§2, 3, and 4, we obtain a complete description of all the irreducible finite-dimensional real, complex, and quaternionic representations of any connected semisimple Lie group. Indeed, one can say even more. Let G be any connected Lie group, \mathfrak{g}_0 its (real) Lie algebra, and \hat{G} the family of all equivalence classes of finite-dimensional irreducible complex representations of G . By the Levi-Mal'cev theorem [1, p. 89], \mathfrak{g}_0 is the semidirect product of its radical \mathfrak{r}_0 and a semisimple subalgebra \mathfrak{s}_0 . Further, under any element of \hat{G} \mathfrak{r}_0 goes into scalar operators [1, Corollary 3, p. 76, and Theorem 4, p. 82]. From this it follows that every element T of \hat{G} is obtained by a "scalar modification" of some irreducible representation T' of \mathfrak{s}_0 , and that, if T is self-conjugate of real or quaternionic type, T' will be the same. Thus the results of §§2, 3, and 4 give a complete description of all finite-dimensional irreducible real, complex, or quaternionic representations of all connected Lie groups.

Here are a few remarks on notation. Pairs will be denoted by $\langle x, y \rangle$, and inner products on Hilbert space by (x, y) . By $f|S$ we mean the restriction of a function f to a subset S of its domain. Let R , C , and Q stand for the real, complex, and quaternion fields respectively, and E for the multiplicative group of the complex numbers of absolute value 1. If $u \in C$, \bar{u} is its complex conjugate.

Professor R. Richardson has kindly called the author's attention to the fact that Propositions 1 to 9 of our §4 are all contained in A. L. Mal'cev's article *On semi-simple subgroups of Lie groups*, *Izv. Akad. Nauk SSSR Ser. Mat.* **8** (1944), 143–174 (Amer. Math. Soc. Transl. No. 33, 1950). Indeed, it is easy to see that an irreducible complex representation of a compact group is of real or quaternion type in our terminology if and only if it is orthogonal or symplectic respectively in that of Mal'cev. However, Mal'cev's derivation of these results is considerably longer than ours, and he does not obtain our unifying Theorem 6.

1. Conjugating representations. Let X and Y be complex linear spaces. A map $A: X \rightarrow Y$ is *conjugate-linear* if $A(a\xi + b\eta) = \bar{a}A(\xi) + \bar{b}A(\eta)$ ($\xi, \eta \in X$; $a, b \in C$). The conjugate-linear maps themselves form a complex linear space, scalar multiplication being of course given by $(aA)(\xi) = a(A(\xi))$. By X^- (the *conjugate space* of X) we mean the complex linear space such that the identity map $X \rightarrow X$ is conjugate-linear on X onto X^- . If A is a linear endomorphism of X , we shall sometimes write A^- for the same map considered as a linear endomorphism of X^- . Again, if $a \in C$, it sometimes saves confusion to write $S(a)$ and $\bar{S}(a)$ for the opera-

tions of scalar multiplication by a on X and X^- respectively; then $(S(a))^- = \bar{S}(\bar{a})$.

Let X be a complex Hilbert space. The *conjugate Hilbert space* X^- is the Hilbert space such that the identity map carries X in a conjugate-linear and norm-preserving manner onto X^- . A conjugate-linear norm-preserving map A of X onto itself is called *conjugate-unitary*; we then have $(A\xi, A\eta) = (\eta, \xi)$ ($\xi, \eta \in X$).

Now fix a separable (i.e., satisfying the second axiom of countability) locally compact group G , with unit e . We denote by $Z^2(G)$ the group (under pointwise multiplication) of all Borel functions $\lambda: G \times G \rightarrow E$ satisfying: (i) $\lambda(e, x) = \lambda(x, e) = 1$ ($x \in G$); (ii) $\lambda(x, y)\lambda(xy, z) = \lambda(y, z)\lambda(x, yz)$ ($x, y, z \in G$). The elements of $Z^2(G)$ are called *multipliers*. A *projective representation* of G is a map T assigning to each x in G a unitary operator T_x on a separable Hilbert space $X(T)$ (the *space* of T) such that: (i) $T_e = 1$ (the identity operator); (ii) for some function $\lambda: G \times G \rightarrow E$ we have $T_x T_y = \lambda(x, y) T_{xy}$ for all x, y in G ; (iii) T is a Borel map in the sense that $x \rightarrow (T_x \xi, \eta)$ is a Borel function on G for each pair of vectors ξ, η in $X(T)$. The function λ in (ii) automatically belongs to $Z^2(G)$; we say that λ is the *multiplier* of T , or that T is a λ -*representation* of G . If $\lambda \equiv 1$, T is an *ordinary* (unitary) representation of G .

The preceding definitions are of course taken from [6].

We now fix (for the duration of this section) a closed normal subgroup N of G such that G/N is of order 2. In terms of N we shall define a modified notion of multiplier and of representation. If $x \in G$ and $u \in E$ define $x:u$ to be u if $x \in N$ and \bar{u} if $x \in G - N$. Let $Z_c^2(G)$ be the group (under pointwise multiplication) of all Borel functions $\lambda: G \times G \rightarrow E$ satisfying: (i) $\lambda(e, x) = \lambda(x, e) = 1$ ($x \in G$); (ii) $\lambda(x, y)\lambda(xy, z) = (x: \lambda(y, z))\lambda(x, yz)$ ($x, y, z \in G$). (Property (ii) is called the multiplier identity.) The elements of $Z_c^2(G)$ will be called *conjugating multipliers*.

In the framework of general Borel group cohomology (see [8]), the multipliers and conjugating multipliers are of course the two-cocycles relative to the trivial action of G on E and the nontrivial action $\langle x, u \rangle \rightarrow x:u$, respectively.

Let X be a separable Hilbert space. We say that T is a *conjugating projective representation* of G on X (relative of course to N) if T assigns to each x in N a unitary operator T_x on X , and to each x in $G - N$ a conjugate-unitary operator T_x on X , such that (i) $T_e = 1$; (ii) for some function $\lambda: G \times G \rightarrow E$ we have $T_x T_y = \lambda(x, y) T_{xy}$ for all x, y in G ; (iii) T is a Borel map in the sense that $x \rightarrow (T_x \xi, \eta)$ is a Borel function on G for each pair of vectors ξ, η in X . X is called the *space* of T and is written $X(T)$. The λ of (ii) is easily seen to be in $Z_c^2(G)$. We say that λ is the *conjugating multiplier* of T , or that T is a *conjugating λ -representation*. If $\lambda \equiv 1$, T is an *ordinary* conjugating representation of G . Ordinary conjugating representations, like ordinary nonconjugating ones, are strongly continuous.

A projective representation T (conjugating or not) is *irreducible* if $X(T) \neq \{0\}$ and there are no closed subspaces of $X(T)$ (except $\{0\}$ and $X(T)$) which are stable under all T_x . Two projective representations T and T' (both conjugating or both not conjugating) are equivalent (in symbols $T \cong T'$) if there is a linear isometry

F of $X(T)$ onto $X(T')$ such that $T'_x \circ F = F \circ T_x$ for all x . If this is the case, then of course the multipliers of T and T' are the same.

If $\lambda \in Z^2(G)$ [resp. $Z_c^2(G)$] we shall write \tilde{G}_λ [resp. \tilde{G}_λ] for the family of all equivalence classes of irreducible λ -representations [resp. conjugating λ -representations] of G . If $\lambda \in Z_c^2(G)$, then of course $\lambda' = \lambda \mid (N \times N)$ belongs to $Z^2(N)$; we shall then write \hat{N}_λ instead of $\hat{N}_{\lambda'}$, and speak of λ -representations of N instead of λ' -representations. Our goal now is to describe \tilde{G}_λ in terms of \hat{N}_λ for each fixed λ in $Z_c^2(G)$.

Fix once and for all an element λ of $Z_c^2(G)$, and also an element σ of $G - N$. If $D \in \hat{N}_\lambda$, let $\sigma^* \cdot D$ be the element of \hat{N}_λ defined by:

$$(1) \quad \begin{aligned} X(\sigma^* \cdot D) &= (X(D))^{-}, \\ (\sigma^* \cdot D)_m &= \bar{S} \left(\frac{\lambda(m, \sigma)}{\lambda(\sigma, \sigma^{-1}m\sigma)} \right) (D_{\sigma^{-1}m\sigma})^{-} \quad (m \in N). \end{aligned}$$

[Recall that $\bar{S}(a)$ means multiplication by a in $(X(D))^{-}$; $(D_{\sigma^{-1}m\sigma})^{-}$ means $D_{\sigma^{-1}m\sigma}$ considered as acting on $(X(D))^{-}$.] By repeated application of the multiplier identity one verifies that $\sigma^* \cdot D$ is indeed in \hat{N}_λ and is independent of the particular σ , and that $D \rightarrow \sigma^* \cdot D$ is a permutation of \hat{N}_λ of order two. (Compare [6, Lemma 4.2 and its Corollary].)

DEFINITION. An element D of \hat{N}_λ is *extendible* if there exists a conjugating λ -representation D' of G such that $X(D') = X(D)$ and $D'_m = D_m$ for all m in N .

We note that a necessary and sufficient condition for extendibility is the existence of a conjugate-unitary operator A on $X(D)$ satisfying:

$$(2) \quad A^{-1}D_m A = \bar{S} \left(\frac{\lambda(m, \sigma)}{\lambda(\sigma, \sigma^{-1}m\sigma)} \right) D_{\sigma^{-1}m\sigma} \quad (m \in N)$$

and

$$(3) \quad A^2 = S(\lambda(\sigma, \sigma))D_{\sigma^2}.$$

Given such an A , the required extension D' is obtained by putting $D'_\sigma = A$. Note that (2) alone is necessary and sufficient for $D \cong \sigma^* \cdot D$. Thus extendibility of D implies that $D \cong \sigma^* \cdot D$; but the converse is false.

If D is extendible, its extension D' to a conjugating λ -representation of G is unique to within equivalence. Indeed: If D' and D'' are two such extensions, the irreducibility of D implies that $D''_\sigma = S(a)D'_\sigma$ for some a in E . Taking $b = \sqrt{a}$, one checks that D' and D'' are equivalent under $S(b)$.

Now for any λ -representation D of N , one verifies that the following formulae define a conjugating λ -representation T of G :

$$(4) \quad \begin{aligned} X(T) &= X(D) \oplus (X(D))^{-}, \\ T_m(\xi \oplus \eta) &= D_m \xi \oplus \bar{S} \left(\frac{\lambda(m, \sigma)}{\lambda(\sigma, \sigma^{-1}m\sigma)} \right) (D_{\sigma^{-1}m\sigma})^{-} \eta, \\ T_\sigma(\xi \oplus \eta) &= (S(\lambda(\sigma, \sigma))D_{\sigma^2} \eta) \oplus \xi \end{aligned}$$

($\xi, \eta \in X(D)$; $m \in N$).

LEMMA 1. *Suppose $D \in \hat{N}_\lambda$. Then the T of (4) is irreducible if and only if D is not extendible.*

Proof. (A) Assume that D is not extendible. Clearly $T|N = D \oplus \sigma^* \cdot D$. If $D \not\cong \sigma^* \cdot D$, the only nontrivial closed $(T|N)$ -stable subspaces of $X(T)$ are $X(D) \oplus 0$ and $0 \oplus (X(D))^-$, neither of which is T -stable. So in this case T is irreducible. Now assume that $D \cong \sigma^* \cdot D$. Then, to within equivalence, we can write $X(T) = X(D) \otimes Z$ and $T_m = D_m \otimes 1_Z$ ($m \in N$), where Z is a two-dimensional Hilbert space. A nontrivial T -stable subspace of $X(T)$ would have to be of the form $X(D) \otimes Z_1$, where Z_1 is a one-dimensional subspace of Z . But then D would be extendible (take D' to be the restriction of T to $X(D) \otimes Z_1$). Thus T is irreducible in this case also.

(B) Let D be extendible to a conjugating λ -representation D' of G . Then $Y = \{\xi \oplus \eta \mid \xi = D'_\sigma \eta\}$ is a closed nontrivial T -stable subspace of $X(T)$. So T is not irreducible. This completes the proof.

We shall say that N is of Type I with respect to λ if the commuting algebra of every λ -representation of N is of Type I.

THEOREM 1. *Let G, N, λ , and σ be as above; and assume that N is of Type I with respect to λ . Then there is a natural one-to-one correspondence between \tilde{G}_λ and the set Θ of all orbits in \hat{N}_λ under the two-element group consisting of the identity and the permutation σ^* . The correspondence is as follows: Let θ in Θ correspond to T in \tilde{G}_λ ; and let $D \in \theta$. If D is extendible, T is the unique (to within equivalence) conjugating λ -representation which extends D . If D is not extendible, T is the conjugating λ -representation given by (4). Conversely, $T|N$ is a direct sum of elements of \hat{N}_λ ; and θ is just the set of those elements of \hat{N}_λ which occur in the direct sum.*

Proof. In view of Lemma 1 we have only to show (A) that, for each T in \tilde{G}_λ , $T|N$ is a direct sum of elements of \hat{N}_λ , and (B) that for each D in \hat{N}_λ there is at most one element T of \tilde{G}_λ such that $T|N$ contains D .

Let $T \in \tilde{G}_\lambda$. Three cases are possible:

Case I. $T|N$ is not a factor λ -representation of N , that is, there is a closed nontrivial $(T|N)$ -stable subspace Y of $X(T)$ such that the restrictions D and D' of $T|N$ to Y and Y^\perp respectively are disjoint (see [5, Lemma 1.1]). The equation

$$(5) \quad T_m T_\sigma = T_\sigma \bar{S} \left(\frac{\lambda(m, \sigma)}{\lambda(\sigma, \sigma^{-1} m \sigma)} \right) T_{\sigma^{-1} m \sigma} \quad (m \in N),$$

shows that $T_\sigma Y$ is $(T|N)$ -stable—indeed, that

$$(6) \quad T|N \text{ restricted to } T_\sigma Y \text{ is equivalent to } \sigma^* \cdot D$$

($\sigma^* \cdot D$ being defined as in (1), though here D need not be irreducible). Now the projections p_1 and p_2 onto Y and Y^\perp respectively commute with all T_m ($m \in N$). Since $\sigma^2 \in N$, $Y \cap T_\sigma Y$ is T -stable, and so must be $\{0\}$; also $Y + T_\sigma Y$ is T -stable, and so is dense in $X(T)$. Therefore p_2 maps $T_\sigma Y$ biuniquely onto a dense subspace

of Y^\perp . It follows from Mackey's form of Schur's Lemma (see [7, Chapter I]) that $T|N$ restricted to $T_\sigma Y$ is equivalent to D' . But D and D' are disjoint. Hence, applying Mackey's form of Schur's Lemma to $p_1|(T_\sigma Y)$, we conclude that $p_1 \equiv 0$ on $T_\sigma Y$, that is, $T_\sigma Y \subset Y^\perp$. Since $Y + T_\sigma Y$ is dense in $X(T)$, we must have

$$(7) \quad T_\sigma Y = Y^\perp.$$

Now suppose D were not irreducible. Choosing a nonzero proper closed D -stable subspace Y' of Y , we would obtain by (7) a proper closed T -stable subspace $Y' + T_\sigma Y'$ of $X(T)$, contradicting the irreducibility of T . So D is irreducible. By (6) and (7) D' is irreducible and equivalent to $\sigma^* \cdot D$. So $T|N \cong D \oplus \sigma^* \cdot D$, and $\sigma^* \cdot D \not\cong D$. One verifies that the linear isometry F of $Y \oplus Y^-$ onto $X(T)$ given by $F(\xi \oplus \eta) = \xi + T_\sigma \eta$ sets up an equivalence of T with the representation T' of G constructed from D by (4).

Case II. $T|N$ is a factor λ -representation of N , but is not irreducible. Then, since N is of Type I with respect to λ , there is an irreducible sub- λ -representation D of $T|N$, acting on a proper closed subspace Y . As in Case I, we argue that $T|N \cong D \oplus \sigma^* \cdot D$ (whence $\sigma^* \cdot D \cong D$, since $T|N$ is a factor representation). Once more let $F: Y \oplus Y^- \rightarrow X(T)$ be given by $F(\xi \oplus \eta) = \xi + T_\sigma \eta$; and define T' in terms of D as in (4). Then F is one-to-one onto a dense subspace of $X(T)$, and intertwines T' and T . It is easy to see that Mackey's form of Schur's Lemma holds for conjugating as well as for nonconjugating representations. Hence, applied to F , it shows that T' and T are equivalent. (In particular, T' is irreducible, and so by Lemma 1 D is not extendible.)

Case III. $T|N$ is irreducible. Then $D = T|N$ is extendible by definition; and T is the unique conjugating λ -representation which extends D .

Thus, in all three cases, $T|N$ is the direct sum of the elements of some orbit θ in Θ . This proves (A). Now let D be any given element of \hat{N}_λ , and T any element of \hat{G}_λ such that $T|N$ contains D . By the preceding discussion, T falls under Case I, II, or III according as $\sigma^* \cdot D \not\cong D$, $\sigma^* \cdot D \cong D$ but D is not extendible, or D is extendible, respectively. In each of these cases we have seen that T is determined to within equivalence by D . This proves (B).

REMARK 1. It seems likely that the hypothesis that N is of Type I with respect to λ cannot be omitted from Theorem 1, though we do not know of a counterexample proving this.

REMARK 2. Suppose that we had defined conjugating λ -representations as being finite-dimensional but not necessarily norm-preserving. Then Theorem 1 would hold unaltered. Only the obvious modifications in the underlying definitions need be made. The values of the multipliers λ in $Z^2(G)$ and $Z_c^2(G)$ must now be taken to lie in the multiplicative group $C - \{0\}$, instead of in E . The operators T_x are merely linear (or conjugate-linear) rather than unitary (or conjugate-unitary). In the definition of equivalence no norm-condition is imposed on $F: X(T) \rightarrow X(T')$. The proof of Theorem 1 then goes through as before (without any 'Type I' hypothesis).

2. Representations in real and quaternionic Hilbert space. Fix a separable locally compact group N (with unit e) which is of Type I (with respect to the unit multiplier). We shall apply Theorem 1 to catalogue its irreducible representations in real and quaternionic Hilbert space in terms of those in complex Hilbert space.

Let K be any one of $R, C,$ or Q . A K -unitary operator on a K -Hilbert space X is, of course, a K -linear bijection $X \rightarrow X$ which preserves the K -inner product. A K -unitary representation of N is a strongly continuous homomorphism T of N into the group of K -unitary operators on some K -Hilbert space $X(T)$ (the space of T). T is (K)-irreducible if there are no nontrivial closed T -stable K -linear subspaces of $X(T)$ (and $X(T) \neq \{0\}$). Two K -unitary representations S and T of N are (K)-equivalent if there is a K -linear isometry of $X(S)$ onto $X(T)$ which intertwines S and T . We denote by $\hat{N}^{(K)}$ the collection of all K -equivalence classes of K -irreducible K -unitary representations of G .

We recall that a quaternionic Hilbert space can be identified with a pair $\langle X, J \rangle$, where X is a complex Hilbert space and J is a conjugate-unitary operator on X satisfying $J^2 = -1$. (The Q -inner product on $\langle X, J \rangle$ is given by $(x, y)_Q = (x, y)_C + (x, Jy)_C j$, where $1, i, j,$ and k are the usual quaternion units.) Under this identification, a Q -unitary operator on $\langle X, J \rangle$ is just a C -unitary operator which commutes with J .

If $T \in \hat{N}^{(C)}$, we denote by \bar{T} (the conjugate of T) the element of $\hat{N}^{(C)}$ whose space is $(X(T))^-$ and for which $(\bar{T})_x = (T_x)^- (x \in N)$. If $\bar{T} \cong T$ we say that T is self-conjugate. Suppose T is a self-conjugate element of $\hat{N}^{(C)}$. Then there exists a conjugate-unitary operator U on $X(T)$ commuting with all $T_x (x \in N)$. Since U^2 is unitary and commutes with all the T_x , we have $U^2 = \lambda \cdot 1$ for some λ in E (1 being the identity operator on $X(T)$). Now U is determined up to a multiplicative constant μ in E , and $(\mu U)(\mu U) = \mu \bar{\mu} U^2 = U^2$; thus λ is an invariant of the representation T . I claim that $\lambda = \pm 1$. Indeed: U has an inverse U^{-1} with $U \circ U^{-1} = U^{-1} \circ U = 1$. From $\lambda^{-1} U^2 = 1$ it follows that $U^{-1} = \lambda^{-1} U$, whence $1 = U \circ U^{-1} = U \circ \lambda^{-1} U = \lambda U^2$. So $\lambda^{-1} = \lambda$, or $\lambda = \pm 1$. We shall say that T is of real type or of quaternionic type according as λ is 1 or -1 .

Now let G be the direct product of N with the two-element group $\{e, \sigma\}$. (We consider N and σ as contained in G .) Let \tilde{G} stand for \tilde{G}_δ , where $\delta \equiv 1$. In addition, we shall need the conjugating multiplier λ given by:

$$\begin{aligned} \lambda(x, y) &= 1 && \text{if either } x \in N \text{ or } y \in N; \\ \lambda(x, y) &= -1 && \text{if } x \text{ and } y \text{ are both in } G - N. \end{aligned}$$

LEMMA 2. $\hat{N}^{(R)} \cong \tilde{G}$, and $\hat{N}^{(Q)} \cong \tilde{G}_\lambda$.

Proof. Let $T \in \hat{N}^{(R)}$. We denote by $X_c(T)$ the Hilbert space complexification of $X(T)$, and by T^c the (ordinary) conjugating representation of G on $X_c(T)$ such that (i) T_x^c coincides with T_x on $X(T)$ whenever $x \in N$, and (ii) $T_x^c(\xi + i\eta) = \xi - i\eta$ ($\xi, \eta \in X(T)$). It is easily verified that T^c is irreducible, hence in \tilde{G} , and that the map $T \rightarrow T^c$ is a one-to-one correspondence between $\hat{N}^{(R)}$ and \tilde{G} .

Similarly, if $T \in \hat{N}^{(Q)}$, with $X(T) = \langle X, J \rangle$ (see the remark earlier in this section), let T^c be the conjugating λ -representation of G on X such that (i) T_x^c coincides with T_x for $x \in N$, and (ii) $T_\sigma^c = J$. Then T^c is irreducible, hence in \tilde{G}_λ ; and the map $T \rightarrow T^c$ is a one-to-one correspondence between $\hat{N}^{(Q)}$ and \tilde{G}_λ .

Theorem 1 combined with Lemma 2 gives almost immediately the following classifications of $\hat{N}^{(R)}$ and $\hat{N}^{(Q)}$ in terms of $\hat{N}^{(C)}$. We assume as before that N is of Type I.

THEOREM 2. *For each S in $\hat{N}^{(C)}$ we construct an element $S^{(R)}$ of $\hat{N}^{(R)}$ as follows:*

(A) *If S is self-conjugate of real type, there is a real form Y of $X(S)$ (i.e., $Y + iY = X(S)$, $Y \cap iY = \{0\}$) which is stable under S ; we put $X(S^{(R)}) = Y$, $S_x^{(R)} = S_x|_Y$ ($x \in N$). ($S^{(R)}$ depends to within R -equivalence only on S).*

(B) *If S is not self-conjugate of real type, let $X_r(S)$ coincide with $X(S)$ except that we restrict attention to real scalars, and, for $x \in N$, let $S_x^{(R)}$ be the endomorphism of $X_r(S)$ coinciding with S_x . Then $S^{(R)} \in \hat{N}^{(R)}$.*

Every T in $\hat{N}^{(R)}$ is of the form $S^{(R)}$ for some S in $\hat{N}^{(C)}$. If S and S' are in $\hat{N}^{(C)}$, we have $S^{(R)} \cong S'^{(R)}$ if and only if either $S \cong S'$ or $\bar{S} \cong S'$.

THEOREM 3. *For each S in $\hat{N}^{(C)}$ we construct an element $S^{(Q)}$ of $\hat{N}^{(Q)}$ as follows:*

(A) *If S is self-conjugate of quaternionic type, there is a conjugate-unitary operator J on $X(S)$ commuting with all S_x ($x \in N$) and satisfying $J^2 = -1$. Let $X(S^{(Q)})$ be the quaternionic Hilbert space $\langle X(S), J \rangle$, and for $x \in N$ let $S_x^{(Q)}$ be S_x considered as acting on $X(S^{(Q)})$; then $S^{(Q)}$ belongs to $\hat{N}^{(Q)}$, and depends (to within Q -equivalence) only on S .*

(B) *If S is not self-conjugate of quaternionic type, let $X_q(S)$ be the "quaternionification" of $X(S)$; that is, $X_q(S) = \langle X(S) \oplus (X(S))^{-}, J \rangle$, where $J(\xi \oplus \eta) = (-\eta) \oplus \xi$. For $x \in N$, let $S_x^{(Q)}$ be the Q -unitary operator on the Q -Hilbert space $X_q(S)$ which coincides with S_x on $X(S)$. Then $S^{(Q)} \in \hat{N}^{(Q)}$.*

Every T in $\hat{N}^{(Q)}$ is of the form $S^{(Q)}$ for some S in $\hat{N}^{(C)}$. If S and S' are in $\hat{N}^{(C)}$, we have $S^{(Q)} \cong S'^{(Q)}$ if and only if either $S \cong S'$ or $\bar{S} \cong S'$.

REMARK. As in Remark 2 of §1, suppose that we consider representations which are finite-dimensional rather than unitary. If T is a finite-dimensional (ordinary nonconjugating) complex representation of N , we define \bar{T} as before. Suppose now that T is irreducible and that $T \cong \bar{T}$ under a conjugate-linear operator U . Then as before $U^2 = \lambda \cdot 1$; but now we can assert only that λ is real and nonzero and is determined up to a positive factor. We define T to be of *real* or *quaternionic* type according as λ is positive or negative. With this modification only, Theorems 2 and 3 remain valid (without any 'Type I' hypothesis).

3. Self-conjugate representations of semisimple groups. For the material presupposed in the next two sections we refer the reader to [4] and [9].

Let G be a connected simply connected semisimple Lie group, \mathfrak{g}_0 its (real) Lie algebra, \mathfrak{g} the complexification of \mathfrak{g}_0 , and σ the conjugation of \mathfrak{g} corresponding

to its real form \mathfrak{g}_0 ($\sigma(X+iY) = X-iY$ for $X, Y \in \mathfrak{g}_0$). Let \hat{G} (or $\hat{\mathfrak{g}}_0$ or $\hat{\mathfrak{g}}$) be the family of all equivalence classes of irreducible finite-dimensional (not necessarily unitary) continuous complex representations of G ; we identify each element of G with the corresponding irreducible representation of \mathfrak{g}_0 , and with its extension to a complex-linear irreducible representation of \mathfrak{g} . It is easy to see that, if $T \in \hat{G}$,

$$(8) \quad (\bar{T})_X = (T_{\sigma X})^- \text{ for } X \in \mathfrak{g}.$$

We denote by B the Killing form on \mathfrak{g} . Let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} which is stable under σ (that is, $\mathfrak{h} = \mathfrak{h}_0 + i\mathfrak{h}_0$, where \mathfrak{h}_0 is a Cartan subalgebra of \mathfrak{g}_0). If $\phi \in \mathfrak{h}^*$ (the adjoint space of \mathfrak{h}), let H'_ϕ be the unique element of \mathfrak{h} such that $\phi(H) = B(H'_\phi, H)$ for all H in \mathfrak{h} . Let Δ be the set of all roots of \mathfrak{h} . We write \mathfrak{h}_r and \mathfrak{h}_r^* for the real subspaces of \mathfrak{h} and \mathfrak{h}^* spanned by the H'_α and the α respectively (where α runs over Δ). \mathfrak{h}_r^* is called the *real root space*. One verifies easily that $\sigma(\alpha): H \rightarrow [\alpha(\sigma H)]^-$ is a root whenever α is a root. So σ induces real-linear automorphisms of \mathfrak{h}_r and \mathfrak{h}_r^* and permutes Δ . Since σ preserves the Killing form on \mathfrak{h}_r , we have $\sigma(H_\alpha) = H_{\sigma(\alpha)}$ for all roots α ; here we have put $H_\alpha = (B(H'_\alpha, H'_\alpha))^{-1/2} H'_\alpha$.

Introduce a lexicographic ordering into \mathfrak{h}_r^* . Let Δ^+ be the set of all positive roots (under this ordering), and Π the set of simple roots (positive roots which cannot be expressed as a sum of two positive roots). Let $C_0 = \{H \in \mathfrak{h}_r \mid \alpha(H) > 0 \text{ for all } \alpha \in \Delta^+\}$ be the canonical Weyl chamber defined by this ordering. Since σ permutes the roots, σC_0 is another Weyl chamber, and there is a unique element w of the Weyl group W of \mathfrak{h} satisfying $w(\sigma C_0) = C_0$. Then $w \circ \sigma$ preserves the Killing form and permutes Π ; that is, $w \circ \sigma$ is an automorphism of the system of simple roots. We shall call $w \circ \sigma$ the *characteristic automorphism* for \mathfrak{g}_0 , and denote it by s .

We recall the Cartan-Weyl parametrization of the elements of \hat{G} by dominant weights. A linear functional ϕ on \mathfrak{h} is said to be *integral* if $\phi(H_\alpha)$ is an integer for all α in Π (or equivalently, for all α in Δ), and to be *dominant* if $\phi(H_\alpha)$ is a non-negative integer for all α in Π . Let T be in \hat{G} . A linear functional ϕ on \mathfrak{h} is a *weight* of T if there is a nonzero vector ξ (called a *weight vector* for ϕ) in the space $X(T)$ of T such that $T_H \xi = \phi(H)\xi$ for all H in \mathfrak{h} . In that case ϕ is integral; in particular $\phi \in \mathfrak{h}_r^*$. The space of all weight vectors for ϕ (including 0) is the *weight space* X_ϕ of ϕ . We shall say that ϕ is an *extreme weight* of T if ϕ is an extreme point of the convex hull of the set of all weights of T . (See [2, p. 41].) Among all the extreme weights of T there is exactly one which is dominant. This is called the *dominant weight* ϕ_0 of T , and is the largest weight of T with respect to the given lexicographic ordering of \mathfrak{h}_r^* . The set of all extreme weights of T is exactly the orbit of ϕ_0 under the Weyl group. The weight space of each extreme weight is one-dimensional. Further, T is determined to within equivalence by its dominant weight; and every dominant functional on \mathfrak{h} is the dominant weight of some T in \hat{G} . Thus \hat{G} is parametrized by the set of all dominant functionals on \mathfrak{h} .

We shall always denote by $T^{(\phi)}$ the element of \hat{G} having dominant weight ϕ .

THEOREM 4. For each dominant functional ϕ on \mathfrak{h} , $(T^{(\phi)})^- = T^{(s\phi)}$.

Proof. Write T for $T^{(\phi)}$. By (8), β is a weight of T if and only if $\sigma(\beta)$ is a weight of \bar{T} . So $\sigma(\phi)$ is an extreme weight of \bar{T} ; hence so is $s\phi = w(\sigma\phi)$ (w being the element of W defined earlier). But $s\phi$ is dominant since ϕ is. Therefore $s\phi$ is the dominant weight of \bar{T} .

COROLLARY 1. $T^{(\phi)}$ is self-conjugate if and only if $\phi = s\phi$.

COROLLARY 2. A necessary and sufficient condition for all elements T of \hat{G} to be self-conjugate is that the characteristic automorphism for \mathfrak{g}_0 be trivial. If G is compact, this will happen if and only if the Weyl group contains multiplication by -1 . It will always be the case if the system of simple roots has no nontrivial automorphisms.

Proof. The first statement follows from Corollary 1. If G is compact, then $\mathfrak{h}_r \subset i\mathfrak{g}_0$, and so $\sigma \equiv -1$ on \mathfrak{h}_r^* . This gives the second statement. The third follows obviously from the first.

COROLLARY 3. The characteristic automorphism s is of order 1 or 2.

Proof. This follows from Theorem 4, since $(\bar{T})^- = T$.

COROLLARY 4. If ϕ is any dominant functional, $T^{(\phi)}$ is self-conjugate if and only if $\sigma(\phi)$ is a weight of $T^{(\phi)}$.

Proof. The *only if* part follows from the first sentence of the proof of Theorem 4. Conversely, suppose $\sigma(\phi)$ is a weight of $T^{(\phi)}$. Then so is $s\phi$; and therefore $s\phi = \phi - \gamma_1 - \dots - \gamma_p$, where the γ_j are positive roots. By Corollary 3, $\phi = s^2\phi = s\phi - s\gamma_1 - \dots - s\gamma_p = \phi - \sum_{j=1}^p (\gamma_j + s\gamma_j)$. Since the γ_j and $s\gamma_j$ are all positive, this implies $p=0$, or $s\phi = \phi$. Now apply Corollary 1.

Corollary 1 of Theorem 4 tells us which elements of \hat{G} are self-conjugate. Among the self-conjugate ones, how do we discriminate real and quaternionic type?

Let T be a self-conjugate element of \hat{G} ; and let U be the nonsingular conjugate-linear operator on $X(T)$ commuting with all T_x ($x \in G$), and hence satisfying

$$(9) \quad UT_x = T_{\sigma(x)}U \quad (X \in \mathfrak{g}).$$

One verifies immediately that, if ξ is a weight vector for the weight ϕ , then $U\xi$ is a weight vector for the weight $\sigma(\phi)$. Thus

$$(10) \quad U(X_\phi) = X_{\sigma(\phi)}.$$

We know that $U^2 = k \cdot 1$, where either $k > 0$ (in which case T is of real type) or $k < 0$ (in which case T is of quaternionic type). Our goal is to find conditions determining which alternative holds.

For this we make the following more special choice of \mathfrak{h} (familiar from the proof of the Iwasawa decomposition theorem). Fix once and for all a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ of \mathfrak{g}_0 . (This means that $\mathfrak{k}_0 + i\mathfrak{p}_0$ is a compact real form of \mathfrak{g} .) Let $\mathfrak{h}_{\mathfrak{p}_0}$ be a maximal Abelian subalgebra of \mathfrak{p}_0 , and let \mathfrak{m}_0 be the centralizer $\{X \in \mathfrak{k}_0 \mid [X, \mathfrak{h}_{\mathfrak{p}_0}] = \{0\}\}$ of $\mathfrak{h}_{\mathfrak{p}_0}$ in \mathfrak{k}_0 . Let $\mathfrak{h}_{\mathfrak{k}_0}$ be a maximal Abelian subalgebra of \mathfrak{m}_0 .

Then $\mathfrak{h}_0 = \mathfrak{h}_{\mathfrak{t}_0} + \mathfrak{h}_{\mathfrak{p}_0}$ is a Cartan subalgebra of \mathfrak{g}_0 , and $\mathfrak{h} = \mathfrak{h}_0 + i\mathfrak{h}_0$ is a Cartan subalgebra of \mathfrak{g} . Now it is well known (see [3, Lemma 5]) that \mathfrak{m}_0 is the Lie algebra direct sum of its center \mathfrak{m}_0^c and its commutator \mathfrak{m}_0^s , and that \mathfrak{m}_0^s is a compact semisimple Lie algebra. Further, $\mathfrak{h}_{\mathfrak{t}_0} = \mathfrak{m}_0^c + \mathfrak{h}_{\mathfrak{t}_0}^s$, where $\mathfrak{h}_{\mathfrak{t}_0}^s$ is a Cartan subalgebra of \mathfrak{m}_0^s . Thus $\mathfrak{h}_{\mathfrak{t}}^s$ is a Cartan subalgebra of \mathfrak{m}^s (removal of subscript $_0$ meaning complexification).

The \mathfrak{h}_r defined earlier in this section coincides in the present context with $i\mathfrak{h}_{\mathfrak{t}_0} + \mathfrak{h}_{\mathfrak{p}_0}$. Let us introduce into \mathfrak{h}_r the lexicographic ordering relative to a basis H_1, \dots, H_n of \mathfrak{h}_r , where H_1, \dots, H_m ($m \leq n$), is a basis of $\mathfrak{h}_{\mathfrak{p}_0}$; and transfer this ordering to the real root space \mathfrak{h}_r^* via the Killing form, which is strictly positive definite on \mathfrak{h}_r . This ordering of \mathfrak{h}_r defines by restriction a lexicographic ordering of $i\mathfrak{h}_{\mathfrak{t}_0}^s$, and hence (via the Killing form of \mathfrak{m}^s) a lexicographic ordering of the real root space of $\mathfrak{h}_{\mathfrak{t}}^s$ (considered as a Cartan subalgebra of \mathfrak{m}^s).

We now subdivide the roots of \mathfrak{h} into two disjoint classes Δ' and Δ'' :

$$\Delta' = \{\alpha \in \Delta \mid H'_\alpha \in i\mathfrak{h}_{\mathfrak{t}_0}\} = \{\alpha \in \Delta \mid \alpha(\mathfrak{h}_{\mathfrak{p}_0}) = \{0\}\},$$

$$\Delta'' = \Delta - \Delta'.$$

It is known (see [9, Expose 11]) that the root space \mathfrak{g}_α of any root α in Δ' is contained in \mathfrak{m}^s ; in fact, \mathfrak{m}^s is spanned by $\mathfrak{h}_{\mathfrak{t}}^s$ and the \mathfrak{g}_α ($\alpha \in \Delta'$). Thus:

(11) Every root of $\mathfrak{h}_{\mathfrak{t}}^s$ is the restriction to $\mathfrak{h}_{\mathfrak{t}}^s$ of some α in Δ' .

Let Δ'_+ and Δ''_+ be the sets of positive roots in Δ' and Δ'' respectively. We note that

(12)
$$\sigma(\Delta''_+) \subset \Delta'_+;$$

(13)
$$\text{if } \alpha \in \Delta', \text{ then } \sigma\alpha = -\alpha.$$

LEMMA 3. *If ϕ is a dominant functional on \mathfrak{h} , then $\phi|_{\mathfrak{h}_{\mathfrak{t}}^s}$ is a dominant functional on $\mathfrak{h}_{\mathfrak{t}}^s$ (the latter being considered as a Cartan subalgebra of \mathfrak{m}^s).*

Proof. We know that ϕ is a weight of some T in \hat{G} . Thus $\phi|_{\mathfrak{h}_{\mathfrak{t}}^s}$ is a weight of $T|_{\mathfrak{m}^s}$. In particular $\phi|_{\mathfrak{h}_{\mathfrak{t}}^s}$ is an integral functional on $\mathfrak{h}_{\mathfrak{t}}^s$. It is easy to see from (11) that it must in fact be dominant.

Let M denote the compact connected simply connected Lie group whose Lie algebra is \mathfrak{m}_0^s .

THEOREM 5. *Let ϕ be the dominant weight of a self-conjugate element T of \hat{G} . Let S be the element of \hat{M} whose dominant weight is $\psi = \phi|_{\mathfrak{h}_{\mathfrak{t}}^s}$ (see Lemma 3). Then S is self-conjugate; and T is of real or quaternionic type according as S is of real or quaternionic type.*

In particular, if \mathfrak{m}_0 is Abelian (i.e., $\mathfrak{m}_0^c = \{0\}$), then every self-conjugate element of \hat{G} is of real type.

Proof. We begin with a general observation (which is slightly stronger than a remark on p. 17-03 of [9]). Let η be a weight vector for some weight β of an element V of \hat{G} . Then the space $X(V)$ of V is the linear span of vectors of the form

$$(14) \quad V_{Z_1} \cdots V_{Z_r} \eta,$$

where Z_j is a root vector for α_j , the $\alpha_1, \dots, \alpha_r$ being roots of \mathfrak{h} which are either all positive or all negative. Indeed: Let L be the linear span of all the vectors (14). It suffices to show that L is V -stable. Obviously L is stable under $V(\mathfrak{h})$. Its stability under V_Z where Z is an arbitrary root vector follows by induction in r from the Lie product relations for root vectors.

Recall that σ is the conjugation of \mathfrak{g} with respect to the real form \mathfrak{g}_0 . Let ξ be a weight vector in $X(T)$ for the dominant weight ϕ . Since T is self-conjugate, Corollary 1 of Theorem 4 shows that $\sigma(\phi)$ is a weight of T . Since ξ is unique to within a multiplicative constant, it follows from the preceding paragraph that there are positive roots $\gamma_1, \dots, \gamma_p, \delta_1, \dots, \delta_q$ of \mathfrak{h} , and root vectors $X_1, \dots, X_p, Y_1, \dots, Y_q$ for $\gamma_1, \dots, \gamma_p, -\delta_1, -\delta_2, \dots, -\delta_q$ respectively, such that

$$(15) \quad \sigma(\phi) = \phi - \gamma_1 - \cdots - \gamma_p = \phi - \delta_1 - \cdots - \delta_q,$$

$$(16) \quad \eta = T_{Y_1} \cdots T_{Y_q} \xi \neq 0,$$

$$(17) \quad T_{X_1} \cdots T_{X_p} \eta = k\xi \quad (0 \neq k \in C).$$

Now, among the δ_j , suppose that $\delta_1, \dots, \delta_s \in \Delta'_+$ and $\delta_{s+1}, \dots, \delta_q \in \Delta''_+$. Then we have by (13) and (15)

$$\begin{aligned} \delta_1 + \cdots + \delta_s + \delta_{s+1} + \cdots + \delta_q &= \phi - \sigma(\phi) \\ &= -\sigma(\phi - \sigma\phi) = \delta_1 + \cdots + \delta_s - \sigma\delta_{s+1} - \cdots - \sigma\delta_q. \end{aligned}$$

Therefore

$$\delta_{s+1} + \cdots + \delta_q + \sigma(\delta_{s+1}) + \cdots + \sigma(\delta_q) = 0.$$

But the left side of the last equation is a sum of positive roots. This is impossible unless $s=q$. Arguing similarly for the γ_j , we conclude that the γ_j and δ_j are all in Δ'_+ , and hence that

$$(18) \quad \text{the } X_j \text{ and the } Y_j \text{ are all in } \mathfrak{m}^s.$$

Suppose now that $\mathfrak{m}^s = \{0\}$. Then (18) shows that $p=q=0$, that is, $\phi = \sigma(\phi)$. So, if U is the nonsingular conjugate-linear operator on $X(T)$ commuting with all T_x ($x \in G$), (10) and the one-dimensionality of X_ϕ show that $U\xi = r\xi$ ($r \neq 0$), whence $U^2\xi = |r|^2\xi$. It follows that T is of real type.

Assume then that $\mathfrak{m}^s \neq \{0\}$. Let V be the $(T|\mathfrak{m}^s)$ -stable subspace of $X(T)$ generated by ξ . Possibly V is not irreducible under $T|\mathfrak{m}^s$. Recalling that M is compact, split V into a direct sum of $(T|\mathfrak{m}^s)$ -stable irreducible subspaces: $V = V_1 \oplus \cdots \oplus V_t$. Let the restriction of $T|\mathfrak{m}^s$ to V_1 be called R , and let $\pi: V \rightarrow V_1$ be the idempotent operator with range V_1 which annihilates V_j for $j > 1$. Put $\xi' = \pi\xi$; then $\xi' \neq 0$.

Since by (18) all the Y_j are in \mathfrak{m}^s , (16) shows that $\eta \in V$. Put $\eta' = \pi\eta$. By (17) and (18) $T_{x_1} \cdots T_{x_p} \eta' = k\xi' \neq 0$; so $\eta' \neq 0$. Now ξ' is a weight vector of R for the weight $\phi | \mathfrak{h}_t^s = \psi$; and η' is a weight vector for $(\phi - \delta_1 - \cdots - \delta_a) | \mathfrak{h}_t^s = \sigma\phi | \mathfrak{h}_t^s = -\psi$. So both ψ and $-\psi$ are weights of R . Since ξ' is annihilated by root vectors of positive roots, ψ is the highest weight of R . Thus $R \cong S$. So, along with its dominant weight ψ , S also has $-\psi$ as a weight. Hence, by Corollary 4 of Theorem 4, S is self-conjugate.

Since V_1 was a typical irreducible subspace of V (under $T|\mathfrak{m}^s$), we have shown that, for each j , $T|\mathfrak{m}^s$ is equivalent on V_j to S , and the component of ξ in V_j is the weight vector for the dominant weight ψ . Since V is generated by ξ , it follows that $t=1$, that is, V was already irreducible under $T|\mathfrak{m}^s$. So we can identify S with the restriction of $T|\mathfrak{m}^s$ to V .

Now let U be the nonsingular conjugate-linear operator on $X(T)$ satisfying (9). I claim that U leaves V stable. This will show that the restriction of U to V satisfies (9) with T replaced by S , and hence that T and S are both of real or both of quaternionic type. The proof of the theorem will then be complete.

To prove the claim, we first note that $U\xi \in X_{\sigma(\phi)}$, (see (10)). By (16), together with the fact that $Y_j \in \mathfrak{m}^s$ (see (18)), we conclude that $0 \neq \eta \in X_{\sigma(\phi)} \cap V$. Since $X_{\sigma(\phi)}$ is one-dimensional, this implies $X_{\sigma(\phi)} \subset V$. So $U\xi \in V$. Now V is spanned by the $\zeta = T_{z_1} \cdots T_{z_r} \xi$, where the Z_j are root vectors for negative roots in Δ' . By (9),

$$U\zeta = T_{\sigma(z_1)} \cdots T_{\sigma(z_r)} U\xi.$$

Since $U\xi \in V$, $Z_j \in \mathfrak{m}^s$, and \mathfrak{m}^s is stable under σ , it follows that $U\zeta \in V$. So $U(V) \subset V$. This proves the claim, and hence the theorem.

REMARK. Theorem 5 reduces the question of the type of a self-conjugate element of \hat{G} to the same question for the compact group M . The answer to this question in the compact case will be given in the next section.

COROLLARY 1. *If \mathfrak{g}_0 is a normal real form of \mathfrak{g} (i.e., $\mathfrak{m}_0 = \{0\}$; see [4, p. 336]), then all the elements of \hat{G} are self-conjugate of real type.*

Proof. Since \mathfrak{g}_0 is a normal real form, $\mathfrak{h}_r = \mathfrak{h}_{\mathfrak{p}_0}$, and hence $\sigma \equiv 1$ on \mathfrak{h}_r . It follows that the characteristic automorphism for \mathfrak{g}_0 is trivial. Now apply Theorems 4 and 5.

COROLLARY 2. *If \mathfrak{g}_0 is the underlying real Lie algebra of a complex semisimple Lie algebra \mathfrak{g}_c , then every self-conjugate element of \hat{G} is of real type.*

Proof. Let scalar multiplication in \mathfrak{g}_c (as opposed to that in \mathfrak{g}) be denoted by $*$. If \mathfrak{g}_u is a compact real form of \mathfrak{g}_c , then $\mathfrak{g}_0 = \mathfrak{g}_u + i * \mathfrak{g}_u$, and the latter is a Cartan decomposition of \mathfrak{g}_0 . If $\mathfrak{h}_{\mathfrak{p}_0}$ is maximal Abelian in $i * \mathfrak{g}_u$, its centralizer in \mathfrak{g}_u is $i * \mathfrak{h}_{\mathfrak{p}_0}$, which is Abelian. We now apply the last statement of Theorem 5.

4. **Self-conjugate representations of compact semisimple groups.** Assume now that G is a compact connected simply connected semisimple Lie group. We shall

keep to the notation of the beginning of §3. Notice that in this case $\sigma \equiv -1$ on \mathfrak{h} , and \mathfrak{h}^* . Two roots α, β of \mathfrak{h} will be called *strongly orthogonal* if $\alpha \neq \pm\beta$ and neither $\alpha + \beta$ nor $\alpha - \beta$ are roots; this implies that H'_α and H'_β are orthogonal under the Killing form (see [4, Theorem 4.3(i), p. 143]).

The fundamental lemma of this section is the following:

LEMMA 4. *Let T be an element of \hat{G} with dominant weight ϕ . Suppose further that 2ϕ can be written in the form*

$$(19) \quad 2\phi = m_1\gamma_1 \cdots m_p\gamma_p,$$

where $\gamma_1, \dots, \gamma_p$ are pairwise strongly orthogonal positive roots of \mathfrak{h} and m_1, \dots, m_p are positive integers. Then T is self-conjugate, and is of real or quaternionic type according as $\sum_{j=1}^p m_j$ is even or odd.

Proof. Let $X_1, \dots, X_p, Y_1, \dots, Y_p$ be root vectors for $\gamma_1, \dots, \gamma_p, -\gamma_1, \dots, -\gamma_p$ respectively. Since G is compact, it is well known that we can choose the X_j and Y_j to satisfy:

$$(20) \quad Y_j = \sigma(X_j), \quad B(X_j, Y_j) < 0.$$

Let ξ be a weight vector in $X(T)$ for the dominant weight ϕ ; and put

$$(21) \quad \eta = (T_{Y_p})^{m_p} \cdots (T_{Y_1})^{m_1} \xi.$$

Writing (19) in the form $\phi - \sum_j m_j \gamma_j = -\phi$, we see that $\eta \in X_{-\phi}$. Likewise, since X_ϕ is one-dimensional,

$$(22) \quad (T_{X_p})^{m_p} \cdots (T_{X_1})^{m_1} \eta = k\xi \quad (k \in \mathbb{C}).$$

I claim that k is real and nonzero, and that

$$(23) \quad k > 0 \text{ or } k < 0 \text{ according as } \sum_j m_j \text{ is even or odd.}$$

In proving this claim, it will be convenient to use the following notation: If $\eta_1, \eta_2 \in X(T)$, we write $\eta_1 \sim \eta_2$ to mean that $\eta_2 = r\eta_1$ for some positive real number r .

If $i \neq j$, γ_i and γ_j are strictly orthogonal, so $\gamma_i - \gamma_j$ is not a root; thus $[X_i, Y_j] = 0$. It follows that T_{X_i} and T_{Y_j} commute for $i \neq j$. So, combining (21) and (22) and rearranging, we have

$$(24) \quad k\xi = (T_{X_p}^{m_p} T_{Y_p}^{m_p}) \cdots (T_{X_1}^{m_1} T_{Y_1}^{m_1}) \xi.$$

Let us evaluate $T_{X_1}^{m_1} T_{Y_1}^{m_1} \xi$. To shorten notation we write $X\xi$ instead of $T_X \xi$ ($X \in \mathfrak{g}$). Put $[X_1, Y_1] = -H_1$. By (20) $H_1 \sim H_{\gamma_1}$ (see also (3), p. 142 of [4]). We shall prove (by induction in q) that, for all integers q satisfying $0 \leq q \leq m_1$ we have

$$(25) \quad X_1^q Y_1^q \xi \sim (-1)^q \xi.$$

This is obvious for $q=0$. Assume (25) for $q-1$ ($1 \leq q \leq m_1$). Putting $b = \phi(H_1)$, $a = \gamma_1(H_1)$, we have

$$\begin{aligned} X_1^q Y_1^q \xi &= X_1^{q-1} [X_1, Y_1] Y_1^{q-1} \xi + X_1^{q-1} Y_1 X_1 Y_1^{q-1} \xi \\ &= -X_1^{q-1} H_1 Y_1^{q-1} \xi + X_1^{q-1} Y_1 X_1 Y_1^{q-1} \xi. \end{aligned}$$

Since $Y_1^{q-1} \xi \in X_{\phi - (q-1)\gamma_1}$, we have $H_1 Y_1^{q-1} \xi = [b - (q-1)a] Y_1^{q-1} \xi$. So, iterating the calculation q times, we get

$$\begin{aligned} X_1^q Y_1^q \xi &= -[b - (q-1)a] X_1^{q-1} Y_1^{q-1} \xi + X_1^{q-1} Y_1 [X_1, Y_1] Y_1^{q-2} \xi \\ &\quad + X_1^{q-1} Y_1^2 X_1 Y_1^{q-2} \xi \\ &= -[b - (q-1)a + b - (q-2)a] X_1^{q-1} Y_1^{q-1} \xi \\ &\quad + X_1^{q-1} Y_1^2 [X_1, Y_1] Y_1^{q-3} \xi + X_1^{q-1} Y_1^3 X_1 Y_1^{q-3} \xi \\ &= \dots \\ &= -[b - (q-1)a + b - (q-2)a + \dots + b] X_1^{q-1} Y_1^{q-1} \xi \\ &\quad + X_1^{q-1} Y_1^q X_1 \xi. \end{aligned}$$

But $X_1 \xi = 0$, since ξ is a weight vector for the highest weight of T . So

$$(26) \quad X_1^q Y_1^q \xi = -2^{-1}q[2b - (q-1)a] X_1^{q-1} Y_1^{q-1} \xi.$$

Now (19) gives $2b = m_1 a + m_2 \gamma_2(H_1) + \dots + m_p \gamma_p(H_1)$. But we recall that H'_j and H'_j are orthogonal under the Killing form for $j > 1$. So $\gamma_j(H_1) = 0$ for $j > 1$, and we get $2b = m_1 a$. Hence, since $a = \gamma_1(H_1) > 0$, we have

$$2b - (q-1)a = (m_1 - q + 1)a > 0.$$

Thus (26) implies that $X_1^q Y_1^q \xi \sim -X_1^{q-1} Y_1^{q-1} \xi$. This, together with the inductive hypothesis for $q-1$, gives (25).

In particular $X_1^{m_1} Y_1^{m_1} \xi \sim (-1)^{m_1} \xi$; and the same holds for each $X_j^{m_j} Y_j^{m_j} \xi$. Hence, by (24),

$$k\xi \sim (-1)^\mu \xi,$$

where $\mu = \sum_{j=1}^p m_j$. It follows that k is real and nonzero, and that (23) holds.

This and (22) show that $\eta \neq 0$. On the other hand, we recall from (21) that $\eta \in X_{-\phi}$. So $-\phi$ is a weight of T whence, by Corollary 4 of Theorem 4, T is self-conjugate.

Thus there is a nonsingular conjugate-linear operator U on $X(T)$ satisfying (9). By (10) $U\xi \in X_{-\phi}$. Since T is self-conjugate, $-\phi$ is an extreme weight of T and so $X_{-\phi}$ is one-dimensional. So $U\xi = r\eta$ for some nonzero complex number r . Consequently, by (9), (20), (21), and (22) we get

$$\begin{aligned} U^2 \xi &= \bar{r} U \eta = \bar{r} U Y_p^{m_p} \dots Y_1^{m_1} \xi = \bar{r} X_p^{m_p} \dots X_1^{m_1} U \xi \\ &= |r|^2 X_p^{m_p} \dots X_1^{m_1} \eta = |r|^2 k \xi. \end{aligned}$$

But U^2 is a scalar operator. So the preceding equation shows that $U^2 = |r|^2 k \cdot 1$. Thus, by definition, T is of real or quaternionic type according as $|r|^2 k$ is positive

or negative. By (23) this happens according as $\sum_j m_j$ is even or odd. This completes the proof of the lemma.

Lemma 4 leaves one question unanswered: Can the dominant weight ϕ of every self-conjugate element of \hat{G} be written in the form (19)? We know of no way to settle this except by examination of each simple case. We shall adopt the usual terminology for the complex simple Lie algebras $A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7, E_8$, and for their Dynkin diagrams (which are shown in each case). For each of these algebras, if the simple roots are numbered $\alpha_1, \dots, \alpha_n$, we shall denote by $[r_1, \dots, r_n]$ the linear functional ϕ on the Cartan subalgebra \mathfrak{h} given by $\phi(H_{\alpha_j}) = r_j$ ($j = 1, \dots, n$). As always, $T^{(\phi)}$ is the element of \hat{G} with dominant weight ϕ .

Case I: A_n ($n \geq 1$), the unimodular algebra in $n + 1$ dimensions.



Let $m = n + 1$. The characteristic automorphism (for the unitary subalgebra of A_n) carries α_j into α_{m-j} ($j = 1, \dots, n$). So by Theorem 4

$$(27) \quad (T^{[r_1, \dots, r_n]})^{-1} = T^{[r_n, r_{n-1}, \dots, r_1]}.$$

So $T^{[r_1, \dots, r_n]}$ is self-conjugate if and only if $r_j = r_{m-j}$ for all $j = 1, \dots, n$.

Putting $\beta_{ij} = \alpha_i + \alpha_{i+1} + \dots + \alpha_{j-1}$ for $1 \leq i < j \leq m$, we see that the roots $\beta_{i, m+1-i}$ ($1 \leq i < m-i$) are pairwise strongly orthogonal. If

$$\phi_p = [0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0] \quad (\text{'1' in the } p\text{th and } (m-p)\text{th places}),$$

we verify that

$$(28) \quad \phi_p = \beta_{1,m} + \beta_{2,m-1} + \dots + \beta_{p,m+1-p} \quad (1 \leq p < m-p).$$

Also, if m is even and $\psi = [0, \dots, 0, 2, 0, \dots, 0]$ ('2' in the $(m/2)$ th place), then

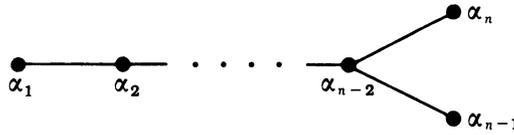
$$(29) \quad \psi = \beta_{1,m} + \beta_{2,m-1} + \beta_{m/2, (m/2)+1}.$$

Now by (27) every dominant weight of a self-conjugate element of \hat{A}_n is a non-negative integral combination of the ϕ_p and ψ ; and so by (28) and (29) is of the form (19). Lemma 4 now gives:

PROPOSITION 1. *If $n \not\equiv 1 \pmod{4}$, all the self-conjugate elements of \hat{A}_n are of real type. If $n \equiv 1 \pmod{4}$, $\phi = [r_1, \dots, r_n]$, and $T^{(\phi)}$ is self-conjugate, then $T^{(\phi)}$ is of real or quaternionic type according as $r_{(n+1)/2}$ is even or odd.*

REMARK. Consider the case $n = 1$. A compact real form of A_1 is the Lie algebra of $SU(2)$. Thus it follows from Proposition 1 that the unique p -dimensional irreducible representation of $SU(2)$ is of real or quaternionic type according as p is odd or even.

Case II: D_n ($n \geq 3$), the orthogonal algebra in $2n$ dimensions.



If n is even, the Weyl group contains multiplication by -1 , and so all T in \hat{D}_n are self-conjugate. If n is odd, the characteristic automorphism (for the compact orthogonal subalgebra) interchanges α_{n-1} and α_n , leaving the other α_j fixed; so for odd n

$$(T^{(r_1, \dots, r_n)})^- = T^{(r_1, \dots, r_{n-2}, r_n, r_{n-1})},$$

and $T^{(r_1, \dots, r_n)}$ is self-conjugate if and only if $r_{n-1} = r_n$.

For $1 \leq j \leq n-2$, let

$$\beta_j = \alpha_j + 2\alpha_{j+1} + \dots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n.$$

Then $\alpha_1, \beta_1, \alpha_3, \beta_3, \dots, \alpha_{n-2}, \beta_{n-2}$ (if n is odd) and $\alpha_1, \beta_1, \alpha_3, \beta_3, \dots, \alpha_{n-3}, \beta_{n-3}, \alpha_{n-1}, \alpha_n$ (if n is even) are pairwise strongly orthogonal roots, and we verify that

$$[0, \dots, 0, 1, 0, \dots, 0] = \beta_1 + \beta_3 + \dots + \beta_{p-1}$$

('1' in p th place, p even, $p \leq n-2$),

$$[0, \dots, 0, 2, 0, \dots, 0] = 2\beta_1 + 2\beta_3 + \dots + 2\beta_{p-2} + \beta_p + \alpha_p$$

('2' in p th place, p odd, $p \leq n-2$),

$$[0, \dots, 0, 1, 1] = \beta_1 + \beta_3 + \dots + \beta_{n-2} \quad (n \text{ odd}),$$

$$[0, \dots, 0, 2, 0] = \beta_1 + \beta_3 + \dots + \beta_{n-3} + \alpha_{n-1} \quad (n \text{ even}),$$

$$[0, \dots, 0, 0, 2] = \beta_1 + \beta_3 + \dots + \beta_{n-3} + \alpha_n \quad (n \text{ even}).$$

Applying Lemma 4 we obtain:

PROPOSITION 2. Let $n \geq 3$. If $n \not\equiv 2 \pmod{4}$, all the self-conjugate elements of \hat{D}_n are of real type. If $n \equiv 2 \pmod{4}$, then $T^{(r_1, \dots, r_n)}$ is of real or quaternionic type according as $r_{n-1} + r_n$ is even or odd.

Case III: B_n ($n \geq 2$), the orthogonal algebra in $2n+1$ dimensions.



The Weyl group contains multiplication by -1 ; so all the elements of \hat{B}_n are self-conjugate.

For $j=1, \dots, n-1$, let

$$\beta_j = \alpha_j + 2\alpha_{j+1} + \dots + 2\alpha_{n-1} + 2\alpha_n.$$

Then $\alpha_1, \beta_1, \alpha_3, \beta_3, \dots, \alpha_{n-3}, \beta_{n-3}, \alpha_{n-1}, \beta_{n-1}$ (if n is even) and $\alpha_1, \beta_1, \alpha_3, \beta_3, \dots, \alpha_{n-2}, \beta_{n-2}, \alpha_n$ (if n is odd) are pairwise strongly orthogonal roots, and we verify that

$$[0, \dots, 0, 1, 0, \dots, 0] = \beta_1 + \beta_3 + \dots + \beta_{p-1} \quad ('1' \text{ in } p\text{th place, } p \text{ even, } p < n),$$

$$[0, \dots, 0, 2, 0, \dots, 0] = 2\beta_1 + 2\beta_3 + \dots + 2\beta_{p-2} + \alpha_p + \beta_p$$

('2' in p th place, p odd, $p < n$),

$$[0, \dots, 0, 2] = \beta_1 + \beta_3 + \dots + \beta_{n-1} \quad (n \text{ even}),$$

$$[0, \dots, 0, 2] = \beta_1 + \beta_3 + \dots + \beta_{n-2} + \alpha_n \quad (n \text{ odd}).$$

Applying Lemma 4, we obtain:

PROPOSITION 3. *Let $n \geq 2$. If $n \equiv 0$ or $3 \pmod{4}$, then all the elements of \hat{B}_n are of real type. If $n \equiv 1$ or $2 \pmod{4}$, then $T^{[r_1, \dots, r_n]}$ is of real or quaternionic type according as r_n is even or odd.*

Case IV: C_n ($n \geq 2$), the symplectic algebra in $2n$ dimensions.



The Weyl group contains multiplication by -1 ; so the elements of \hat{C}_n are all self-conjugate. For $j=1, \dots, n$ let

$$\beta_j = 2\alpha_j + \dots + 2\alpha_{n-1} + \alpha_n.$$

Then $\beta_1, \beta_2, \dots, \beta_n$ are pairwise strongly orthogonal roots, and we verify that

$$[0, \dots, 0, 2, 0, \dots, 0] = \beta_1 + \dots + \beta_p \quad ('2' \text{ in } p\text{th place, } p=1, \dots, n).$$

Therefore Lemma 4 gives:

PROPOSITION 4. *The element $T^{[r_1, \dots, r_n]}$ of \hat{C}_n is of real or quaternionic type according as $r_1 + r_3 + \dots + r_m$ is even or odd (where $m=n$ if n is odd and $m=n-1$ if n is even).*

Now come the exceptional algebras. We are indebted to Calvin Moore for a table of the roots of these algebras, on which the following calculations are based.

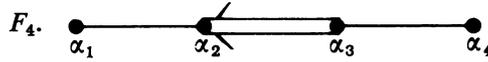
Case V:



There are no nontrivial automorphisms of the simple root system, so by Corollary 2 of Theorem 4 all the elements of \hat{G}_2 are self-conjugate. The roots α_1 and $\beta = 3\alpha_1 + 2\alpha_2$ are strongly orthogonal, and we verify that $[2, 0] = \alpha_1 + \beta$, $[0, 1] = \beta$. So Lemma 4 gives:

PROPOSITION 5. *All the elements of \hat{G}_2 are self-conjugate of real type.*

Case VI:



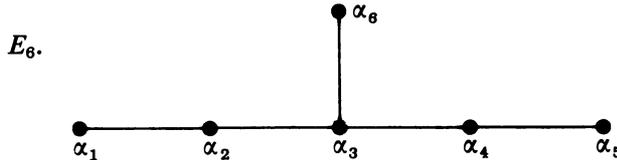
There being no nontrivial automorphisms of the simple root system, all the elements of \hat{F}_4 are self-conjugate. The four roots α_3 , $\beta=2\alpha_2+\alpha_3$, $\gamma=2\alpha_1+2\alpha_2+\alpha_3$, and $\delta=2\alpha_1+4\alpha_2+3\alpha_3+2\alpha_4$ are pairwise strongly orthogonal, and we verify that

$$\begin{aligned} [2, 0, 0, 0] &= \gamma + \delta, \\ [0, 2, 0, 0] &= \beta + \gamma + 2\delta, \\ [0, 0, 2, 0] &= \alpha_3 + \beta + \gamma + 3\delta, \\ [0, 0, 0, 1] &= \delta. \end{aligned}$$

So by Lemma 4 we obtain:

PROPOSITION 6. *All the elements of \hat{F}_4 are self-conjugate of real type.*

Case VII:



The Weyl group of E_6 does *not* contain multiplication by -1 . The four roots α_3 , $\beta=\alpha_2+\alpha_3+\alpha_4$, $\gamma=\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5$, and $\delta=\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+\alpha_5+2\alpha_6$ are pairwise strongly orthogonal. If S_α denotes reflection in the null-hyperplane of α , one verifies that the Weyl group element

$$w = S_{\alpha_3}S_\beta S_\gamma S_\delta$$

carries α_j into $-\alpha_{6-j}$ for $j=1, 2, 3, 4, 5$, and α_6 into $-\alpha_6$. It follows that the characteristic automorphism of the simple root system (corresponding to a compact real form of E_6) is that which sends α_j into α_{6-j} for $j=1, \dots, 5$, and α_6 into α_6 . Thus:

$$(T^{r_1, r_2, r_3, r_4, r_5, r_6})^- = T^{r_5, r_4, r_3, r_2, r_1, r_6}.$$

In particular, the element T^{r_1, \dots, r_6} of \hat{E}_6 is self-conjugate if and only if $r_1=r_5$ and $r_2=r_4$.

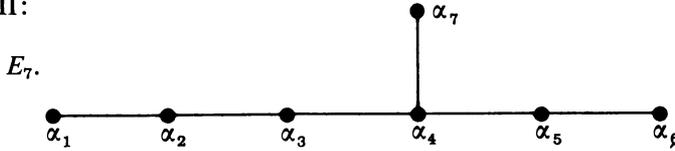
We verify that

$$\begin{aligned} [0, 0, 0, 0, 0, 1] &= \delta, \\ [1, 0, 0, 0, 1, 0] &= \gamma + \delta, \\ [0, 1, 0, 1, 0, 0] &= \beta + \gamma + 2\delta, \\ [0, 0, 2, 0, 0, 0] &= \alpha_3 + \beta + \gamma + 3\delta. \end{aligned}$$

Thus, by Lemma 4, we have:

PROPOSITION 7. All the self-conjugate elements of \hat{E}_6 are of real type.

Case VIII:



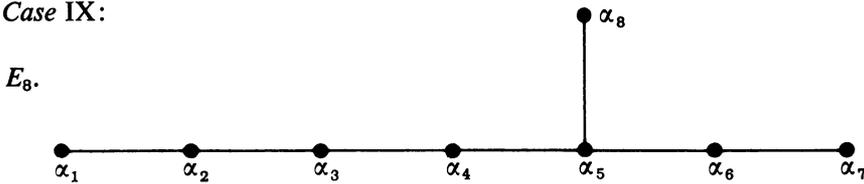
The simple root systems have no nontrivial automorphisms, all the elements of \hat{E}_7 are self-conjugate. The following seven positive roots are pairwise strongly orthogonal: $\alpha_1, \alpha_3, \alpha_5, \alpha_7, \beta = \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_7, \gamma = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_7,$ and $\delta = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7.$ We verify that

$$\begin{aligned} [2, 0, 0, 0, 0, 0, 0] &= \alpha_1 + \gamma + \delta, \\ [0, 1, 0, 0, 0, 0, 0] &= \gamma + \delta, \\ [0, 0, 2, 0, 0, 0, 0] &= \alpha_3 + \beta + 2\gamma + 3\delta, \\ [0, 0, 0, 1, 0, 0, 0] &= \beta + \gamma + 2\delta, \\ [0, 0, 0, 0, 2, 0, 0] &= \alpha_5 + \beta + \gamma + 3\delta, \\ [0, 0, 0, 0, 0, 1, 0] &= \delta, \\ [0, 0, 0, 0, 0, 0, 2] &= \alpha_7 + \beta + \gamma + 2\delta. \end{aligned}$$

Thus we have:

PROPOSITION 8. The element $T^{[r_1, r_2, \dots, r_7]}$ of \hat{E}_7 is of real or quaternionic type according as $r_1 + r_3 + r_7$ is even or odd.

Case IX:



The simple root system having no nontrivial automorphisms, all elements of \hat{E}_8 are self-conjugate. The following eight positive roots are pairwise strongly orthogonal: $\alpha_2, \alpha_4, \alpha_6, \alpha_8, \beta = \alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_8, \gamma = \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_8, \delta = \alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + 2\alpha_8,$ and $\epsilon = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8.$ We verify that

$$\begin{aligned} [1, 0, 0, 0, 0, 0, 0, 0] &= \epsilon, \\ [0, 2, 0, 0, 0, 0, 0, 0] &= \alpha_2 + \gamma + \delta + 3\epsilon, \\ [0, 0, 1, 0, 0, 0, 0, 0] &= \gamma + \delta + 2\epsilon, \\ [0, 0, 0, 2, 0, 0, 0, 0] &= \alpha_4 + \beta + 2\gamma + 3\delta + 5\epsilon, \\ [0, 0, 0, 0, 1, 0, 0, 0] &= \beta + \gamma + 2\delta + 3\epsilon, \\ [0, 0, 0, 0, 0, 2, 0, 0] &= \alpha_6 + \beta + \gamma + 3\delta + 4\epsilon, \\ [0, 0, 0, 0, 0, 0, 1, 0] &= \delta + \epsilon, \\ [0, 0, 0, 0, 0, 0, 0, 2] &= \alpha_8 + \beta + \gamma + 2\delta + 3\epsilon. \end{aligned}$$

Thus we have:

PROPOSITION 9. *All the elements of \hat{E}_8 are self-conjugate of real type.*

Combining Lemma 4 with the preceding special calculations, we obtain finally:

THEOREM 6. *Let \mathfrak{g}_0 be a compact semisimple real Lie algebra, \mathfrak{h}_0 a Cartan subalgebra of \mathfrak{g}_0 , and T an irreducible finite-dimensional complex representation of \mathfrak{g}_0 whose dominant weight on $\mathfrak{h} = \mathfrak{h}_0 + i\mathfrak{h}_0$ (with respect to some lexicographic ordering) is ϕ . Then the following two conditions are equivalent:*

(i) *T is self-conjugate;*

(ii) *we can write $2\phi = m_1\gamma_1 + \dots + m_p\gamma_p$, where $\gamma_1, \dots, \gamma_p$ are pairwise strongly orthogonal positive roots of \mathfrak{h} , and m_1, \dots, m_p are positive integers.*

If (i) and (ii) hold, then T is of real or quaternionic type according as $\sum_{j=1}^p m_j$ is even or odd.

Proof. (ii) \Rightarrow (i) by Lemma 4.

To prove the converse, suppose that T is self-conjugate. Let $\mathfrak{g}_0 = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_t$, where each \mathfrak{g}_j is a compact simple Lie algebra. Then T is of the form $T_1 \otimes \dots \otimes T_t$, where T_j is a self-conjugate irreducible representation of \mathfrak{g}_j . Now in the preceding Cases I–IX we have verified the implication (i) \Rightarrow (ii) for all the simple algebras, in particular for the \mathfrak{g}_j . So, if ϕ_j is the dominant weight of T_j (on a Cartan subalgebra \mathfrak{h}_j of \mathfrak{g}_j), we have

$$2\phi_j = \sum_i m_j^i \gamma_j^i,$$

where the m_j^i are positive integers and, for each j , the γ_j^i (i varying) are pairwise strongly orthogonal roots of \mathfrak{h}_j . Extending ϕ_j and γ_j^i trivially to all of $\mathfrak{h} = \sum_j \mathfrak{h}_j$, and noting that $\phi = \sum_j \phi_j$ is the dominant weight of T , we obtain

$$2\phi = \sum_{i,j} m_j^i \gamma_j^i,$$

where the γ_j^i are pairwise strongly orthogonal roots for all i and j . This proves that (i) \Rightarrow (ii).

The final statement of the theorem was part of Lemma 4.

REMARK. The sets of strongly orthogonal roots which appeared “from nowhere” in Cases I–IX were actually constructed by the following uniform procedure due to B. Kostant and C. Moore (unpublished): Let D be the Dynkin diagram of the simple algebra \mathfrak{g} . Take the largest root α of the given algebra, and let D' be the diagram consisting of those simple roots in D which are orthogonal to α . Take the connected components D'_1, \dots, D'_r of D' , and let $\alpha'_1, \dots, \alpha'_r$ be the largest roots of the algebras with diagrams D'_1, \dots, D'_r respectively. Repeating this procedure with each of the α'_j , we finally obtain a maximal collection of strongly orthogonal roots, in terms of which the 2ϕ of Theorem 6 can always be expressed.

BIBLIOGRAPHY

1. N. Bourbaki, *Groupes et algèbres de Lie*, Chapter I, Hermann, Paris, 1960.
2. Harish-Chandra, *On some applications of the universal enveloping algebra of a semi-simple Lie algebra*, Trans. Amer. Math. Soc. **70** (1951), 28–96.
3. ———, *Representations of a semi-simple Lie group on a Banach space. I*, Trans. Amer. Math. Soc. **75** (1953), 185–243.
4. S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.
5. G. W. Mackey, *Induced representations of locally compact groups. II; Frobenius reciprocity theorem*, Ann. of Math. **58** (1953), 193–221.
6. ———, *Unitary representations of group extensions. I*, Acta Math. **99** (1958), 265–311.
7. ———, *Mimeographed notes on group representations*, Univ. of Chicago, Chicago, Ill., 1956.
8. C. C. Moore, *Extensions and low dimensional cohomology theory of locally compact groups. I*, Trans. Amer. Math. Soc. **113** (1964), 40–63.
9. Seminaire Sophus Lie, *Théorie des algèbres de Lie, topologie des groupes de Lie*, Paris, 1955.

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