THE REALIZATION OF A SEMISIMPLICIAL BUNDLE MAP IS A k-BUNDLE MAP

BY
S. WEINGRAM

0. **Outline of main results.** Let $p: E \to B$ be an ss fiber bundle map [2] with group Γ and fiber F. We will prove that the geometric realization [5] $|p|: |E| \to |B|$ has a fiber-bundle like structure. The easiest way to describe it is to call it "a fiber bundle in the category of k-spaces," by which we mean the following. A Hausdorff space X is a *k-space* if it has the weak topology with respect to its compact subsets. A map f of a k-space X into any space Y is continuous if and only if its restriction to each compact subset is continuous. Each CW complex is a k-space. Let $\mathfrak{C}_k$ denote the category of k-spaces and continuous maps. It is easy to check that the category $\mathfrak{C}_k$ is closed under finite products. If X, Y are k-spaces, then the category product $X \times_k Y$ is the product space retopologized with the weak topology with respect to compact subsets.

Note that if X, Y are CW complexes, then $X \times_k Y$ is always a CW complex, the one whose cells are the products of those of X by those of Y, and whenever the product space $X \times Y$ is a CW complex, then $X \times_k Y = X \times Y$.

Using this definition of category product, we can define in the usual way a k-group (a group in the category $\mathfrak{C}_k$), and the operation in $\mathfrak{C}_k$ of the k-group Γ on the k-space F.

0.1. Definition. A map $p: X \to Y$ in $\mathfrak{C}_k$ is a coordinate *k-bundle map* (*=fiber bundle in* $\mathfrak{C}_k$) if there is a covering of Y by the interiors of closed neighborhoods $\{U_j\}$, coordinate functions $\phi_j: U_j \times F \to p^{-1}(U_j)$, transition functions $g_{ij}: U_i \cap U_j \to \Gamma$, etc., as in Steenrod [6], except that "group," "operation," "product" always are meant in the sense of $\mathfrak{C}_k$.

It is easy to see from [5] and the definition of k-product, etc., that for any ss complexes X, Y, that $|X \times Y| = |X| \times_k |Y|$; that the realization of an ss group is a k-group; and that if the ss group G operates on F, that $|G|$ operates on $|F|$ in the category $\mathfrak{C}_k$.

Our interest is in bundle structures which are closely related to the given ss structure. For example:

0.2. Theorem A. *Let B be an ss complex such that $|B|$ is covered by the interiors $\{U_\alpha\}$ of subcomplexes $\{|C_\alpha|\}$ which are contractible. Let $p: E \to B$ be an ss fiber bundle map with group Γ and fiber F. Then $|p|$ has a coordinate k-bundle structure in which the $\{U_\alpha\}$ are coordinate neighborhoods, in fact, in which $|p|$ is trivial not only over*

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U_α but also over $|C_\alpha|$ for each α. The transition functions can be chosen to be the restrictions to the U_α of realizations of ss maps $|\gamma_{\alpha\beta}| : |C_\alpha \cap C_\beta| \to |\Gamma|$.

This condition on B is sometimes satisfied (for example if B is an ordered simplicial complex) but not very widely.

We may then state our main result as follows. Its proof will be completed in 5.11.

0.3. Theorem B. *If B is a finite ss complex and $p: E \to B$ is an ss fiber bundle map with group Γ and fiber F, then $|p|: |E| \to |B|$ is a k-bundle. The k-group of this bundle is $|\Gamma|$, the fiber $|F|$. Moreover, we may choose coordinate neighborhoods to be subcomplexes (whose interiors cover $|B|$) of a subdivision of $|B|$, and the transition functions to be realizations of ss maps from the intersections of these subcomplexes to a subdivision of $|\Gamma|$.*

0.4. Corollary. *If F and Γ are countable (and B finite), then $|p|$ is a standard fiber bundle in the sense of Steenrod* [6].

Proof. In this case, all the products $|\Gamma| \times_k |\Gamma|$, $|\Gamma| \times_k |F|$, $V_i \times_k |F|$, etc., which appear in the definitions of "group," "operation," "local product structure" are products of countable CW complexes and therefore standard product spaces [5]. Hence all the structures reduce to the standard ones.

0.5. Corollary. *If $|B|$ is a finite CW complex, then the realization of any ss fiber bundle map $p: E \to B$ is a locally trivial fibration.*

Proof. If $|B|$ is finite, so is any subdivision, and therefore by Theorem 0.3, we have the k-bundle structure on the subdivision of $|B|$ where we may extend the local product structure, etc., over closed neighborhoods (subcomplexes) of $|B|$. Since these subcomplexes are finite, their k-product with any $|F|$ is always the product space. Hence, the k-local product structure is a standard local product structure. ∎

0.6. The realization $|p|: |E| \to |B|$ of an ss fiber bundle map is a Serre fibration, over any $|B|$.

Proof. By 0.5 it is a local product over each finite subcomplex of $|B|$. Now if K is any polyhedron and $F: K \times I \to |B|$ is any homotopy, then the image of F is compact and therefore contained in a finite subcomplex $|C|$ of $|B|$. Therefore any lifting f_0 of $F \mid K \times (0)$ to a map into $|E|$ can be lifted to a map f' of all $K \times I$ into $|E|$. ∎

0.7. Corollary. *If $p: E \to B$ is a Kan fibration in the ss category, then $|p|: |E| \to |B|$ admits a strong fiberwise deformation into a Serre fibration and therefore $|p|$ is a quasi-fibration. (For the definition of a quasi-fibration, see* [7].)

Proof. In [2] it is proved that for any Kan fibration there is a strong fiberwise deformation retraction into an ss fiber bundle map. This realizes as a strong fiberwise deformation retraction of $|p|$ into a Serre fibration (by 0.6) and therefore $|p|$ is a quasifibration. ∎

We remark that the first statement of Theorem 0.3 can be extended to the case that B is arbitrary. Then the Corollary 0.4 extends to the case that $|B|$ is countable, but Corollary 0.5, of course, needs not only $|B|$ but also $|F|$ countable in order that the local k-product structure be a true local product structure.

In §7 we will draw some conclusions from this theorem about the realization of ss function complexes. For example:

0.8. Theorem C. *If* $\mathscr{A}=(X: A_1, \ldots, A_n)$ *and* $\mathscr{B}=(Y: B_1, \ldots, B_n)$ *are* n*-ads of ss complexes such that the intersections* $\bigcap_i A_i$ *and* $\bigcap_i B_i$ *are nonvoid, then there is a natural singular homotopy equivalence* $P: |\mathscr{A}^{\mathscr{B}}_{\circ}| \to |\mathscr{A}|^{|\mathscr{B}|}_{\circ}$ *where the function space has the compact-open topology and the* $\circ$ *refers to the component of the constant map. If* Y *is a finite complex, then* P *is actually a homotopy equivalence.*

The proof of Theorem 0.3 will proceed as follows.

We will introduce a subdivision functor D which, unlike barycentric subdivision, is compatible with ss maps upon realization and which, together with its iterates, subdivides a product $X \times Y$ into the product $DX \times DY$ of the factors. From this second property, and a few others, it follows that the subdivision of an ss group is an ss group, and the subdivision of an ss fiber bundle map $p: E \to B$ with fiber F and group Γ is an ss fiber bundle map $Dp: DE \to DB$ with fiber DF and group $D\Gamma$.

After introducing D and proving some properties we need (3.1), we will prove the following result. (5.10.)

0.9. Proposition D. *If* B *is an ss complex whose realization is a finite CW complex, then there is an integer* q *so large that each point of* $|D^qB|$ *lies in the interior of a contractible subcomplex.*

Thus we can apply 0.2 to the fiber bundle map $D^qp: D^qE \to D^qB$. Because of the first property of D-subdivision given above, the realization of $|D^qp|$ will be (homeomorphic to) the given map $|p|$, proving 0.3.

Throughout, "ss fiber bundle map" and the related vocabulary is meant in the sense of [2]. "Realization" is meant in the sense of [5]. In particular, we use Milnor's description of the "standard geometric n-simplex" $|\Delta^n|$ as the subset of $\mathbf{R}^{n+2}$ consisting of all points $(t_0, \ldots, t_{n+1})$ where $0=t_0 \leqq t_1 \leqq \cdots \leqq t_n \leqq t_{n+1}=1$. By "coordinates" of a point, we always mean these coordinates. The topic of k-spaces is covered in Kelley [10] and in Spanier [11].

We will use ▮ to indicate the end of a proof, or, if it appears immediately after a statement, that no proof is forthcoming.

1. Proof of Theorem A.

1.1. Proposition. *Let* $p: E \to B$ *be an ss fiber bundle map. If the base complex* B *has the property that its realization* $|B|$ *is contractible as a CW complex, then* p *is a trivial bundle.*

Proof. It suffices to prove this for the associated principal bundle, so we assume that the group complex Γ of the bundle is also the fiber of p.

In [2] it is proved that the principal Γ-bundles over B are in 1-1 correspondence with the homotopy classes of maps of B into the Kan complex $\overline{W}(\Gamma)$, and that to the class of the constant map corresponds the trivial bundle. The proposition thus follows from the following statement.

1.2. Lemma. *Let B be an ss complex with a contractible realization $|B|$. Then any map of B into a Kan complex K is homotopic to the constant map.*

Proof. Let X be an ss complex; define $i: X \to S(|X|)$ (the singular complex of the space $|X|$) to be as in [5], sending the n-simplex x of X with characteristic map $f: \Delta^n \to X$ into the n-simplex $|f|: |\Delta^n| \to |X|$ of $S(|X|)$. It is proven in [5] that i is a transformation of functors $i: 1 \to S \circ R$, where R is realization and S the singular functor; and also that for each Kan complex K, $i: K \to S(|K|)$ is a homotopy equivalence.

Under our hypotheses then, for any map f we have a diagram:

$$\begin{array}{ccc} B & \xrightarrow{f} & K \\ \downarrow{\scriptstyle i_B} & & \downarrow{\scriptstyle i_K} \\ S(|B|) & \xrightarrow{S(|f|)} & S(|K|) \end{array}$$

Since $|B|$ is contractible, so is $S(|B|)$; if j_K is a homotopy inverse to i_K, then we have $f \simeq (j \circ S(|f|) \circ i_B$ so that f is homotopic to a map factoring through the contractible complex $S(|B|)$ and hence is homotopic to the constant map. ▮

1.3. Theorem A. *Let $p: E \to B$ be a principal Γ-bundle map with the property that $|B|$ is covered by the interiors of contractible subcomplexes. Then $|p|: |E| \to |B|$ is a k-bundle with fiber and group $|\Gamma|$.*

Moreover, we can take for coordinate neighborhoods a set of contractible subcomplexes whose interiors cover $|B|$, and transition functions which are the realization of ss maps from intersections of these subcomplexes to $|\Gamma|$.

Proof. Let $\{|K_\alpha|\}$ be a family of contractible subcomplexes whose interiors, as subsets of $|B|$, cover $|B|$, and let $K_{\alpha\beta}$ denote $K_\alpha \cap K_\beta$. Let $E_\alpha = p^{-1}(K_\alpha)$, and let $p_\alpha = p|E_\alpha$. Thus $p_\alpha: E_\alpha \to K_\alpha$ is the map induced by the inclusion $i: K_\alpha \to B$. By 1.2, each such bundle map must be trivial. Hence there are strong equivalences $\phi_\alpha: F \times K_\alpha \to E_\alpha$ such that $p_\alpha \phi_\alpha = \pi_2$, the projection map. These ss bijections realize as corresponding homeomorphisms $|\phi_\alpha|: |F| \times_k |K_\alpha| \to |E_\alpha|$, a set of coordinate functions for the local k-product structure. Since Γ is the group of the bundle, a composition $\phi_\beta^{-1}\phi_\alpha : F \times K_{\alpha\beta} \to F \times K_{\alpha\beta}$ is a map sending (f, k) into $(\gamma_{\alpha\beta}(k)f, k)$, where $\gamma_{\alpha\beta}$ is an ss map sending $K_\alpha \cap K_\beta$ into Γ. Upon realization $|\gamma_{\alpha\beta}|$ will be a transition function associated with the homeomorphism $|\phi_\beta|^{-1}|\phi_\alpha|$. For, $\phi_\beta^{-1}\phi_\alpha$ is the composition of the following maps:

$$F \times K_{\alpha\beta} \xrightarrow{1 \times d} F \times K_{\alpha\beta} \times K_{\alpha\beta} \xrightarrow{1 \times \gamma_{\alpha\beta} \times 1} F \times \Gamma \times K_{\alpha\beta} \xrightarrow{m \times 1} F \times K_{\alpha\beta},$$

where d is the diagonal map, and $m: F\times\Gamma \to F$ is the operation of Γ on F. Hence, the realization $|\phi_\beta^{-1}\phi_\alpha| = |\phi_\beta|^{-1}|\phi_\alpha|$ is the composition of the following maps:

$$|F|\times_k|K_{\alpha\beta}| \xrightarrow{1\times d} |F|\times_k|K_{\alpha\beta}|\times_k|K_{\alpha\beta}|$$

$$\xrightarrow{1\times|\gamma_{\alpha\beta}|\times 1} |F|\times_k|\Gamma|\times_k|K| \xrightarrow{|m|\times 1} |F|\times_k|K_{\alpha\beta}|,$$

so that $|\gamma_{\alpha\beta}| : |K_{\alpha\beta}| \to |\Gamma|$ determines a system of transition functions for the local k-product structure, with the (k-) operation of the (k-) group $|\Gamma|$ exactly as in the case of standard bundles, with the exception that "product" and hence all the structures which derive from it—group, operation, local product structure—are meant in the sense of the product in the category $\mathfrak{C}_k$. ▮

2. **Direct limit functors (Kan [3]).**

2.1. Let $\mathscr{C}$ be a category and $\mathscr{M}$ a proper subcategory (one whose maps form a set). For each object $X\in\mathscr{C}$, let $T(X)$ be the direct system in $\mathscr{M}$ whose objects are all pairs (M,f), where f is a map in $\mathscr{C}$ of M into X, and whose maps $m: (M,f) \to (M',f')$ are all maps in $\mathscr{M}$ such that $f=f'm$. Similarly if $g: X\to Y$, we define $T(g)$ to be the transformation $T(X)\to T(Y)$ which sends $T(M,f)\to T(M,g,f)$, etc. We say *$\mathscr{C}$ is modeled on $\mathscr{M}$* if for all objects X and maps g in $\mathscr{C}$, inj lim $T(X)=X$ and inj lim $T(g)=g$. (Cf. Kan [3].) If $\mathscr{C}$ is modeled on $\mathscr{M}$ and F is any functor from $\mathscr{M}$ to a direct category $\mathscr{D}$, there is a unique extension to a functor. $\tilde{F}: \mathscr{C}\to\mathscr{D}$ by defining $\tilde{F}=$ inj lim FT. If $\alpha: F\to G$ is a transformation on $\mathscr{M}$, then α extends to $\tilde{\alpha}=$ inj lim $\alpha FT : \tilde{F}\to\tilde{G}$. If $\alpha=$ identity, $\tilde{\alpha}=$ identity; if α has an inverse β (as a functor from $\mathscr{M}$ to $\mathscr{D}$), then $\tilde{\beta}$ extends to an inverse of $\tilde{\alpha}$. We say *$\tilde{F}$ is modeled on F*, and call $\tilde{F}$ a *direct limit functor*.

For example, the semisimplicial category $\mathscr{C}$ is modeled on the subcategory $\mathscr{M}$ whose objects consist of one abstract n-simplex Δ^n (cf. [2]) for each $n\geqq 0$ and the identity maps, the maps $d_i^*: \Delta^n\to\Delta^{n+1}$ sending $\delta^n\to d_i\delta^{n+1}$, and $s_i^*: \Delta^n\to\Delta^{n+1}$ sending $\delta^n\to s_i\delta^{n-1}$ together with all compositions. The geometric realization functor R is the extension $\tilde{F}$ of F which sends Δ^n into $|\Delta^n|$. The barycentric subdivision functor is modeled on $Sd: \Delta^n\to Sd\Delta^n$, the usual subdivision functor on ordered simplicial complexes.

If $\mathscr{C}$, $\mathscr{M}$ are the semisimplicial and model categories and $\mathscr{D}$ is either the topological category or $\mathscr{C}$, this extension has a special form (which Barratt in [1] calls a "*star functor*").

Let Φ be the monoid with generators $\{1, d_0, d_1, \dots, s_0, s_1, \dots\}$ satisfying the semisimplicial identities. We may represent the operation of an anti-isomorphic monoid $\Phi^*=\{1, d_0^*, d_1^*, \dots, s_0^*, \dots\}$ in $\mathscr{C}$ or $\mathscr{D}$ by writing it as right operation of Φ. Thus $d_i^*\Delta^n$, $s_i^*\Delta^n$ may be written $\Delta^n d_i$, $\Delta^n s_i$.

2.2. LEMMA. *Let F map $\mathscr{M}$ to $\mathscr{D}$. For each ssc X,* inj lim $F\circ T(X)$ *is the quotient of the disjoint union $\bigcup\{F(\Delta^n,f) \mid (\Delta^n,f)\in T(X)\}$ by the equivalence relation generated by: $P\in F(\Delta^n, f\circ\phi)$ is equivalent to $P\phi\in(F\Delta^n\phi, f)$, for all $\phi\in\Phi$. (If we denote*

the elements of (Δ^n, f) *by* $P \otimes f$, *then this is the same as saying* $P \otimes \phi f \sim P\phi \otimes f$ *for any* $\phi \in \Phi$.) *If F maps* $\mathscr{M}$ *into* $\mathscr{C}$, *the semisimplicial structure on* $\tilde{F}(X)$ *is given as follows*: d_i *and* s_i *send the class* $P \otimes x$ *into* $d_iP \otimes x$, *and* $s_iP \otimes x$. ▮

2.3. Lemma. *Assume further that F on* $\mathscr{M}$ *has these properties*:

(i) *each* $Q \in F(\Delta^n)$ *is* $P\delta = F(\delta^*)P$, *for some unique face operator* δ *and interior* P (*not in the image of any* d_i); *and*

(ii) *whenever any element* $P \in F(\Delta^n)$ *is interior, so is any* $Ps_i = F(s_i^*)P$. *Then there is a unique "irreducible" representative* $P \otimes x$ *in each class* $P \otimes x \in \tilde{F}(X)$, *where* P *is interior and* x *nondegenerate. The element is found as follows*: *in* $Q \otimes y$, *write* $Q = P_i\delta$ *as in* (i), *and let* $\delta y = \sigma x$ *where* x *is nondegenerate and* σ *a degeneracy operator. Then* $Q \otimes y \sim P_i\sigma \otimes x$ *has the desired form and is unique.*

Proof. As in [5, Lemma 3, p. 358]. ▮

Hence, as in [5], for any ssc X, $\tilde{F}(X)$ is partitioned into subsets of form $\{P \otimes x \mid P \in F(\Delta^n) \text{ interior}\}$ where x varies over all nondegenerate simplices of X.

3. The *D*-subdivision functor.

3.1. We introduce a subdivision functor D in the category of ss complexes by defining D on the model subcategory $\mathscr{M} = \{\Delta^n, s_i^*, d_i^*\}$ and extending as a direct limit functor. It will have the following properties:

(1) *D is a division functor*. That is, there is a transformation $\alpha(X)\colon |DX| \to |X|$, a homeomorphism for any ss complex X such that for any ss map $f\colon X \to Y$, we have:

$$\begin{array}{ccc} |DX| & \xrightarrow{|Df|} & |DY| \\ \Big\downarrow{\scriptstyle \alpha(X)} & & \Big\downarrow{\scriptstyle \alpha(Y)} \\ |X| & \xrightarrow{|f|} & |Y| \end{array}$$

(2) D is *multiplicative*: for any complexes X, Y, $\eta : D(X \times Y) \to DX \times DY$ induced by projections is an isomorphism. (D is an equivalence of functors of two variables.)

(3) $D\Delta^1$ is the ordered simplicial complex with three vertices $[d_0\delta^1] > [\delta^1] > [d_1\delta^1]$. The map $\alpha(\Delta^1) : |D\Delta^1| \to |\Delta^1|$ sends the vertices $[d_0\delta^1]$ into $t=1$, $[\delta^1]$ into $t=\frac{1}{2}$, and $[d_1\delta^1]$ into $t=0$ in $|\Delta^1| = [0, 1]$, and is a linear homeomorphism.

3.2. Lemma. *There is at most one direct limit functor D satisfying* (1), (2), *and* (3) *above.*

Proof. It suffices to show that D is uniquely determined on $\mathscr{M}$ by these three conditions.

We remark first that condition (1) implies that the subdivision Df of an ss map f is injective if and only if f is also injective. For, the realization $|f|$ is injective if and only if f itself is.

Secondly, for each n there is an embedding $\phi_n\colon \Delta^n \to (\Delta^1)^n$ with the properties that

(i) ϕ_n is an injection for all n;

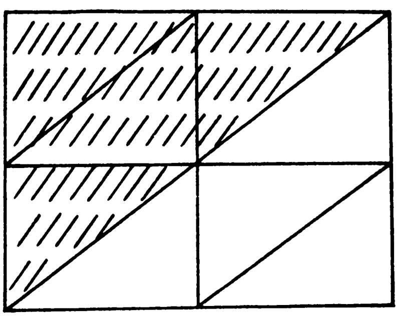
(ii) if S_i is the projection map $(\Delta^1)^n \to (\Delta^1)^{n-1}$ deleting the ith factor, then $\phi_{n-1}s_i^* = S_i\phi_n$; and

(iii) if T_i is the map $(\Delta^1)^n \to (\Delta^1)^{n+1}$ which repeats the $(i+1)$st factor, then $\phi_{n+1}d_i^* = T_i\phi_n$.

Namely, let ϕ_n map the generator δ^n of Δ^n into $\phi_0\delta^1 \times \cdots \times \phi_{n-1}\delta^1$, where $\phi_i = s_{n-1}\cdots s_{i+1}s_{i-1}\cdots s_0$.

By conditions (1), (2), and (3), $(D\Delta^1)^n$, DT_i, DS_i are determined. By condition (1), since ϕ_n is an injection, $|D\phi_n|$ is also an injection and therefore also $D\phi_n$. The image of $D\phi_n$ must be the subcomplex of $(\Delta^1)^n$ whose realization is the subdivision of the CW subcomplex $\alpha((\Delta^1)^n)(|\phi_n|(\Delta^n))$ of the subdivided n-cube $|D\Delta^1|^n = |(D\Delta^1)^n|$. Properties (ii) and (iii) of the embedding ϕ_n also determine d_i^* and s_i^* in terms of maps determined by the three given conditions, and therefore the operation of D on $\mathscr{M}$ is determined by those three conditions. Because D is to be extended as a direct limit functor, there can be at most one such functor. ∎

In order that D be multiplicative, we must have $|D\Delta^1|^n = D\{|\Delta^1|^n\}$, so that the simplicial complex $(\Delta^1)^n$ must be subdivided by D as follows. Drop the simplicial



subdivision on $|\Delta^1|^n$, and regard it just as a cube of edge 1; divide the edges in half, thus cutting the unit cube into subcubes of edge $\frac{1}{2}$.

Each subcube is now taken to be the product of ordered 1-simplicies (regarding each edge of the large cube to be $|D\Delta^1|$), and hence has an associated simplicial subdivision. The vertices of the subdivision are all $v_0 \times \cdots \times v_n$, each v_i a vertex of $|\Delta^1|$ or a median point. Reimpose the original simplicial subdivision of $|\Delta^1|^n$; the second subdivision is a simplicial subdivision of the original. An illustration is given above for the case $n=2$. $|D\Delta^2|$ is the shaded subset. ∎

We now show that there is a functor D satisfying conditions (1), (2), (3).

3.3. Definition. The ordered simplicial complex $D\Delta^n$ is defined for each n as follows. The *vertices* of $D\Delta^n$ are $(n+2)$-tuples $(\varepsilon_0, \ldots, \varepsilon_{n+1})$ where

$$\varepsilon_0 = 0 \leqq \varepsilon_1 \leqq \cdots \leqq \varepsilon_n \leqq \varepsilon_{n+1} = 1,$$

and each can take on the values 0, $\frac{1}{2}$, or 1. The *order* on these vertices is componentwise: $(\varepsilon_0 \cdots \varepsilon_{n+1}) \leqq (\varepsilon'_0 \cdots \varepsilon'_{n+1})$ if $\varepsilon_i \leqq \varepsilon'_i$ for each i. The *simplices* of $D\Delta^n$, say in dimension k, are all sets $(V^0, \ldots, V^k)$ of vertices which satisfy (1) $V^0 \leqq \cdots \leqq V^k$, and (2) for each i, the numbers $\varepsilon_i^{(0)}, \varepsilon_i^{(1)}, \ldots, \varepsilon_i^{(k)}$ can take on either one value or two consecutive values from $(0, \frac{1}{2}, 1)$. We represent the k-simplex σ^k of $D\Delta^n$ by a matrix whose $(i+1)$th row vector is the coordinates of $V^{(i)} = (\varepsilon_0^{(i)}, \varepsilon_1^{(i)}, \ldots, \varepsilon_{n+1}^{(i)})$, the ith vertex of σ^k.

$$(\sigma^k) = \begin{pmatrix} \varepsilon_0^{(0)} = 0 & \varepsilon_1^{(0)} & \cdots & \varepsilon_n^{(0)} & \varepsilon_{n+1}^{(0)} = 1 \\ \varepsilon_0^{(1)} = 0 & \varepsilon_1^{(1)} & \cdots & \varepsilon_n^{(1)} & \varepsilon_{n+1}^{(1)} = 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \varepsilon_0^{(k)} = 0 & \varepsilon_1^{(k)} & \cdots & \varepsilon_n^{(k)} & \varepsilon_{n+1}^{(k)} = 1 \end{pmatrix}.$$

We will regard this matrix corresponding to the k-simplex σ to be an $(n+2)$-tuple of $(k+1)$-dimensional (column) vectors $(\mathbf{0}, C_1, C_2, \ldots, C_n, \mathbf{1})$ satisfying $\mathbf{0} \leqq C_0 \leqq \cdots \leqq C_n \leqq \mathbf{1}$, (order by components), each C_i satisfying condition (2) above.

3.4. Definition. We define the map $Ds_i^*\colon D\Delta^n \to D\Delta^{n-1}$ to be the map which deletes the $(i+1)$th column from the matrix of σ^k, and Dd_i^* sending $D\Delta^n$ into $D\Delta^{n+1}$ to be the map which repeats the ith column vector.

If the row vectors of the matrix (σ^k) are the vertices of a k-simplex in $D\Delta^n$, then the row vectors of $Ds_i^*(\sigma^k)$ and $Dd_i^*(\sigma^k)$ are also vertices of a k-simplex in $D\Delta^{n-1}$, $D\Delta^{n+1}$ respectively. Hence Ds_i^* and Dd_i^* are simplicial maps.

Denote by $D\mathscr{M}$ the category whose objects are one $D\Delta^n$ for each $n = 0, 1, 2, \ldots$ and all maps which are identity maps, Dd_i^*, Ds_j^*, for all i, j, and all compositions. Then the assignment $\Delta^n \to D\Delta^n$, $s_i^* \to Ds_i^*$, $d_i^* \to Dd_i^*$ is a covariant functor $D\colon \mathscr{M} \to D\mathscr{M}$.

Definition. The direct limit functor induced by D is called the D *subdivision functor*.

The projection maps $X \times Y \to X, Y$ induce maps $D(X \times Y) \to DX, DY$. These determine a map (clearly natural) $\eta\colon D(X \times Y) \to DX \times DY$.

3.5. Proposition. *The map* $\eta\colon D(X \times Y) \to D(X) \times D(Y)$ *is a natural ss bijection.*

Proof. A typical nondegenerate element of $X \times Y$ is a simplex $Tx \times T'y$, where x, y are nondegenerate, T, T' are degeneracy operators without a common factor s_i. Hence a generic nondegenerate element in $D(X \times Y)$ is $\sigma^k \otimes (Tx \times T'y)$, with σ^k interior and T, T', x, y as above. Then η sends this element into

$$(\sigma^k \otimes Tx) \times (\sigma^k \otimes T'y)$$

in $DX \times DY$, which is the same as $(\sigma^k T \otimes x) \times (\sigma^k T' \otimes y)$, a product of nondegenerate k-simplices.

Conversely, suppose we have a product $V = (\sigma^k \otimes x) \times (\tau^k \otimes y)$, where these are nondegenerate k-simplices and nondegenerate representations. We will construct $\eta^{-1}(v)$.

LEMMA. *Let σ^k, τ^k be interior simplices of $D\Delta^n$, $D\Delta^m$ respectively. Then there is a unique q, ω^k, T, T' for which*

(1) *ω^k is an interior simplex in $D\Delta^q$ and*

(2) *the deletion maps DT^* and $D(T')^*$ send ω^k into σ^k, τ^k, respectively.*

Proof. Let σ^k, τ^k have the representations $(C_0=0, C_1, \ldots, C_n, C_{n+1}=1)$, $(C'_0=0, C'_1, \ldots, C'_m, C'_{m+1}=1)$ respectively in terms of column vectors. (Each column vector consists of $(k+1)$ numbers $(\varepsilon^{(0)}\cdots\varepsilon^{(k)})$ where $1\geqq\varepsilon^{(k)}\geqq\cdots\geqq\varepsilon^{(0)}\geqq 0$ and the $\{\varepsilon^{(i)}\}$ can take on only at most two consecutive values from $(0, \frac{1}{2}, 1)$). Construct a new matrix as follows. Take out the distinct vectors from among the C's and the (C')'s, call these $C''_0=0, C''_1, \ldots, C''_q, C''_{q+1}=1$. These can be arranged so that $C''_0=0\leqq C''_1\leqq\cdots\leqq C''_{q+1}=1$. (By first moving the C''_i with the largest number of zeros to first place, then the one with second largest, etc. After we have moved all the vectors containing 0's (and hence only 0's and $\frac{1}{2}$'s), move the vector with largest number of $\frac{1}{2}$'s into position behind the ones containing 0's, then the one with next largest number of $\frac{1}{2}$'s behind the one just placed, etc.) All the vectors in the first batch have only 0's and $\frac{1}{2}$'s and are in decreasing order reading down so that we have $0=C''_0\leqq C''_1\leqq\cdots\leqq C''_i\leqq$ for these; and the remainder have just $\frac{1}{2}$'s and 1's, also in decreasing order reading down, hence they too are ordered so that $C''_{i+1}\leqq\cdots\leqq C''_{q+1}=1$. Hence the whole set is arranged in order.)

The simplex $(C''_0\cdots C''_{q+1})$ is interior to $D\Delta^q$ since it has no repeated column vectors; the deletion operators T, T' required are those which strike out from $(C_0\cdots C_{q+1})=\omega^k$ all the columns except those of σ^k, τ^k respectively. The uniqueness of ω^k, T, T' is clear. ∎

With the lemma, we finish the proof. Let $V=(\sigma^k\otimes x)\times(\tau^k\otimes y)$ be as above. There is a unique ω^k interior to $D\Delta^q$, with T, T' as in the lemma. Let $\bar{\eta}(V)=\omega^k\otimes(Tx\times T'y)$; η sends $\omega^k\otimes(Tx\times T'y)$ into

$$(\omega^kT\otimes x)\times(\omega^kT'\otimes y) = (\sigma^k\otimes x)\times(t^k\otimes y),$$

so $\bar{\eta}\eta=1$, and clearly $\eta\bar{\eta}=1$. Hence η is a bijection. ∎

3.6. We now define a map α sending $|D\Delta^n|$ into $|\Delta^n|$ for each n. To each vertex of $|D\Delta^n|$ (or of $D\Delta^n$), say $v=(0, \varepsilon_1, \ldots, \varepsilon_n, 1)$, α makes the corresponding point of $|\Delta^n|$ with these coordinates. To the simplex $|e|$ of $|D\Delta^n|$ with vertices $\{v_0, v_1, \ldots, v_k\}$, α makes correspond linearly the convex hull of the points $\{\alpha(v_0), \ldots, \alpha(v_1), \ldots, \alpha(v_k)\}$ in $|\Delta^n|$. Since the standard simplex is itself convex, α defines a mapping of the simplicial complex $|D\Delta^n|$ into $|\Delta^n|$. If u has coordinates $(t_1\cdots t_n)$ in the simplex of $|D\Delta^n|$ corresponding to the matrix σ, $\alpha(u)$ is the following point of $|\Delta^n|$. Let the barycentric coordinates $(t_1, t_2-t_1, t_3-t_2, \ldots, 1-t_n)$ of u define the row vector $\boldsymbol{t}$. Then $\alpha(u)=\boldsymbol{t}\cdot\sigma$, product of the row vector $\boldsymbol{t}$ with the matrix of σ.

3.7. PROPOSITION. *α is a bijection.*

Proof. Construct α^{-1} as follows. Let $P=(u_1\cdots u_n)\in|\Delta^n|$. For each u_i, write $\varepsilon_i=2[u_i/2]$, and let $r_i=u_i-\varepsilon_i/2$. Rearrange the r_i's in size place, so that whenever two are equal the one with larger index succeeds the one with smaller. Let this sequence be $0\leqq r_{\pi(n)}\leqq r_{\pi(n-1)}\leqq\cdots\leqq r_{\pi(1)}<\frac{1}{2}$. Let $V^0=(\varepsilon_1/2,\varepsilon_2/2,\ldots,\varepsilon_n/2)$. Let

$$V^1=\left(\frac{\varepsilon_1}{2},\frac{\varepsilon_2}{2},\ldots,\frac{1}{2}(\varepsilon_{\pi(1)}+1),\ldots\right),$$

differing from V^0 only in $\pi(1)$st place. Let

$$V^2=\left(\frac{\varepsilon_1}{2},\frac{\varepsilon_2}{2},\ldots,\frac{1}{2}(\varepsilon_{\pi(2)}+1),\ldots,\frac{1}{2}(\varepsilon_{\pi\omega}+1),\ldots\right)$$

differing from V^1 only in $\pi(2)$nd place; etc. Let

$$\lambda_0=2(1-r_{\pi(1)}),\ \lambda_1=2(r_{\pi(1)}-r_{\pi(2)}),\ldots,\ \lambda_n=2(r_{\pi(n)}).$$

It is easy to see that $(V^0,\ldots,V^n)$ describe a simplex of $D\Delta^n$. If we let $Q=\alpha^{-1}(P)$ be the point of this simplex with barycentric coordinates $(\lambda_0\cdots\lambda_n)$, then α^{-1} so defined is an inverse to α. ▌

3.8. Lemma. *The map α is natural with respect to (order-preserving) simplicial maps.*

Proof. We need only prove this for the maps

$$s_i^*:|\Delta^n|\to|\Delta^{n-1}|\qquad\text{and}\qquad d_i^*:|\Delta^n|\to|\Delta^{n+1}|,$$

any other simplicial map being a composition of such maps. In the situation

$$\begin{tikzcd} {|D\Delta^n|} \arrow[r, "|Ds_i^*|"] \arrow[d, "\alpha_1"] & {|D\Delta^{n-1}|} \arrow[d, "\alpha_2"] \\ {|\Delta^n|} \arrow[r, "|s_i^*|"] & {|\Delta^{n-1}|} \end{tikzcd}$$

let P be a point of $|\Delta^n|$, $\sigma\otimes\delta^n$ an n-simplex of $D\Delta^n$, and $P\otimes(\sigma\otimes\delta^n)$ the corresponding point of $|D\Delta^n|$. Then

$$\begin{aligned}|Ds_i^*|(P\otimes(\sigma\otimes\delta^n)) &= P\otimes Ds_i^*(\sigma\otimes\delta)=P\otimes(\sigma\otimes s_i^*\delta)\\ &= P\otimes(\sigma\otimes s_i\delta^{n-1})=P\otimes(Ds_i^*(\sigma)\otimes\delta^{n-1}),\end{aligned}$$

and hence $\alpha(|Ds_i^*|\bar{P}\otimes(\sigma\otimes\delta))=\boldsymbol{P}'.\ ((Ds_i^*)\sigma)$ (where $P'=$barycentric coordinate vector of P).

Also $\alpha(P\otimes(\sigma\otimes\delta))=\boldsymbol{P}'\cdot(\sigma)$, and so $|s_i^*|\alpha(P\otimes(\sigma\otimes\delta))=|s_i^*|(P'\cdot\sigma)$.

But $P'\cdot((Ds_i^*)\sigma)$ is the product of P' by the matrix σ with the ith column deleted, while $|s_i^*|(P'\cdot\sigma)$ so the product $P'\cdot\sigma$ with ith coordinates deleted and they are obviously equal.

In a similar way one proves the naturality of α with respect to Dd_i^*. ▌

3.9. Corollary. *α is a natural equivalence of functors $\alpha\colon R\circ D\to R$, where R is the realization functor.*

Proof. From Lemma 3.8, on the model subcategory $\mathscr{M}$ of $\mathscr{C}$ the map α is an equivalence of functors. By 2.1, this implies that α extends to an equivalence on all $\mathscr{C}$. ▮

4. **Division of bundle maps.** Since D is a multiplicative functor which sends points into points, the subdivision $D\Gamma$ of a group complex Γ is a group complex. If Γ is abelian, so is $D\Gamma$. The same statements hold for any iterate D^m.

4.1. Proposition. *Let $p\colon E\to B$ be an ss fiber bundle map with fiber F and group Γ. Then $Dp\colon DE\to DB$ is an ss fiber bundle map with fiber DF and group $D\Gamma$.*

Proof. That Dp is surjective is obvious, since p is. Let $f\colon\Delta^n\to DB$ be any ss map, say the one sending δ^n into $\sigma^n\otimes b_q$, where σ^n is interior and b_q is nondegenerate in B. We must first demonstrate a strong equivalence between E^f, the induced map, and $\Delta^n\times DF$.

Observe that the map f is the composition of

$$\Delta^n\xrightarrow{\sigma} D\Delta^q\xrightarrow{Db} DB,$$

where σ is the inclusion of Δ^n of $D\Delta^q$, and b is the map sending δ^q onto b_q. Hence we have maps

$$\begin{tikzcd} E^f \arrow[r, "\bar\sigma"] \arrow[d, "p^f"] & D(F\times\Delta^q) \arrow[r, "D_\beta(b)"] \arrow[d, "Dp_*"] & DE \arrow[d, "Dp"] \\ \Delta^n \arrow[r, "\sigma"] & D\Delta^q \arrow[r, "Db"] & DB \end{tikzcd}$$

($p_*=$projection), since induced maps are transitive. Hence, p^f is the map induced by σ from the projection map $Dp_*\colon DF\times D\Delta^q\to D\Delta^q$; and $(\bar\sigma,\sigma)$ gives the strong equivalence with a projection over Δ^n from $DF\times\Delta^n$.

Now we must prove $D\Gamma$ to be the group of this bundle. Let Δ_i^{n-1} be an $(n-1)$ face of Δ^n. If it does not lie in a face of $D\Delta^q$ ($=$if it is not in the image of any Dd_i^*) then the bundle structure over this face of Δ^n is compatible with that of Δ^n; both are derived from the structure over Δ^q (which means that $\sigma(d_i\delta^n)$ lies in the image of some Dd_j^*). Then we proceed as follows.

Suppose $\sigma(d_i\delta^n)=(Dd_i^*)(\tau)$, where τ lies in $D\Delta^{q-1}$. If α is the atlas of the bundle map p, then $\xi_j(b)=\alpha(d_jb)^{-1}\circ\alpha_j(b)\colon\Delta^{q-1}\times F\to\Delta^{q-1}\times F$ is the map γ_l, for some $\gamma\colon\Delta^{q-1}\times F\to\Gamma$, hence a $(q-1)$ simplex of Γ. Let $\{\bar\alpha(\sigma^n\otimes b_q)\}$ be the atlas determined by maps $(\bar\sigma,\sigma)$ as in the first paragraph. Then the associated transition function $\bar\xi_i(\sigma^n\otimes b_q)=\alpha(d_i\sigma^n\otimes b_q)^{-1}\circ\alpha_i(\sigma^n\otimes b_q)$ mapping $\Delta^{n-1}\times DF$ into $\Delta^{n-1}\times DF$ is the restriction (via $(\bar\sigma,\sigma)$) of the map $D\gamma_l\colon D\Delta^{q-1}\times DF\to D\Delta^{q-1}\times DF$ to a subcomplex $\Delta^{q-1}\times DF$, hence is the restriction to $\Delta^{q-1}\times DF$ of the map $D\gamma\colon D\Delta^{q-1}\times DF\to D\Gamma$ to be the group of the bundle map Dp. ▮

5. **The existence of contractible subcomplexes.** From this point to the end of this section, we will be dealing with an arbitrary but fixed finite ssc K. Recall that $|K|$

can be considered as the quotient of the simplicial complex $\bigcup_p (|\Delta^p| \times x_p) = M(K)$ by the map $\phi\colon M(K) \to |K|$, where x_p runs over all simplices of K and $|\Delta^p| \otimes x_p$ is the p-simplex indexed by x_p. The map ϕ sends the point $P \times x_p$ into its equivalence class $P \otimes x_p$ in $|K|$.

We will prove that, given any point P of $|B|$, there is an integer N_0 such that for $N \geqq N_0$ there is a contractible subcomplex of $|D^N B|$ containing P. The idea is that the preimage $\phi^{-1}(P)$ in $M(K)$ consists of a union $\bigcup L_i$ of linear subspaces (in each simplex of $M(K)$), and that if we choose points Q sufficiently near P, that $\phi^{-1}(Q)$ consists of a union $\bigcup L_i$ of similar and parallel subspaces. If we choose a suitable small neighborhood of P (a subcomplex in a sufficiently high D-subdivision), then its ϕ-preimage F will be $\bigcup L_i \times Q_i$, the union of products where each Q_i is a convex subset of euclidean space. These can be deformed down to the preimage of a point v so that the projection on $|B|$ defines a contraction of $\phi(F)$ to v.

5.1. Lemma. *$\phi^{-1}(P)$ is the union $\bigcup L_i$ of restrictions of linear subspaces (the "linear summands" of $\phi^{-1}(P)$) of $\boldsymbol{R}^{n+2}$ to Δ^n. If P' is another interior point of $|x_n|$, the cell of $|B|$ whose interior contains P, the $\phi^{-1}(P') = \bigcup L_i'$, where L_i' is a subspace of the same dimension as and parallel to L_i.*

Proof. Let P have coordinates $(\xi) = (\xi_0, \ldots, \xi_{n+1})$ in $|x_n|$; that is, P is the image of the interior point (ξ) in $|\Delta^n| \times x_n$ in $M(K)$, for nondegenerate x_n. Suppose $Q \times y$ is any point of $M(K)$ which projects onto P, where Q is a point of some $|\Delta^q| \times y$. $Q \times y$ can be equivalent to the nondegenerate point $P \times x$ if and only if the following is true. For some degeneracy operator σ and face operator δ, we must have $Q = Q_1\delta$, $\delta y = \sigma x$ and $P = Q_1\sigma$. Thus to find all such points $Q \times y$ for any y having a degeneracy of x for a face, find all (finitely many) pairs (δ, σ) such that $\delta y = \sigma x$; given such a (δ, σ), choose all points Q_1 of $|\Delta^q|$ such that $Q_1\sigma = P$; and then for each Q_1, let $Q = Q_1\delta$.

If a point of $|\Delta^q| \times y$ has coordinates $(\eta_0, \eta_1, \ldots, \eta_{q+1})$, where $\eta_0 = 0$, $\eta_{q+1} = 1$ are the dummy coordinates, then via these $(q+2)$ coordinates, we regard $|\Delta^q| \times y$ to be the subset of points of euclidean $(q+2)$ space which satisfy $\eta_0 = 0 \leqq \eta_1 \leqq \cdots \leqq \eta_q \leqq \eta_{q+1} = 1$. It follows that for each pair (δ, σ) the corresponding points of $\Delta^q \times y$ which map onto P are the restriction of a linear subspace L of R^{q+2} to $|\Delta^q|$. The equations of this subspace, determined by the deletion operators s_i^* and the repetition operators d_i^* have the following form:

(i) Certain coordinates are fixed equal to those of P:

$$\eta_{a(0)} = \xi_0 = 0,\ \eta_{a(1)} = \xi_1,\ \ldots,\ \eta_{a(n+1)} = \xi_{n+1} = 1;$$

(*)

(ii) certain coordinates are repeated:

$$\eta_{j(1)} = \eta_{j(1)+1} = \eta_{j(1)+2} = \cdots;\ \cdots;\ \eta_{j(k)} = \eta_{j(k)+1} = \cdots;$$

where $a(p)$ means a_p, and $j(k)$ means j_k. A run of equal coordinates might include, and hence all be equal to, one of the coordinates of P.

As we vary the pairs (δ, σ), we run over a finite family L_i of linear subspaces of $R^{q+2} \cap |\Delta^q|$, and these make up the preimage $\phi^{-1}(P)$ in $|\Delta^q| \times y$. We observe that if P' is any other interior point of $|x|$, say with coordinates $(\xi') = (\xi'_0, \ldots, \xi'_{n+1})$ then the face and degeneracy operators will be the same as in the case of $\phi^{-1}(P)$ so $\phi^{-1}(P')$ will consist of a family $\{L'_i\}$ of similar subspaces, each parallel to the corresponding L_i. The only difference in the equations for the L'_i would be that the quantities (ξ'_i) replace the corresponding quantities (ξ_i). ▮

5.2. REMARKS. (a) We shall now pass to iterated subdivisions $D^\mu K$ $(\mu = 0, 1, 2, \ldots)$, we note that because D is a division functor, we have commutativity in the diagram:

$$\begin{tikzcd} D^\mu M(K) \arrow[r, "\alpha'"] \arrow[d, "\phi^\mu"] & M(K) \arrow[d, "\phi"] \\ |D^\mu K| \arrow[r, "\alpha"] & |K| \end{tikzcd}$$

where $D^\mu M(K)$ is $M(K)$ with each simplex $|\Delta^q| \times y$ subdivided to $|D^\mu \Delta^q| \times y$. In other words, we can regard the space $|D^\mu K|$ to be obtained by realizing subdivided simplices and identifying.

(b) We will use $\mathrm{St}^\mu v$ to denote the closed star of the vertex v of $|D^\mu K|$ in $|D^\mu K|$; this is the subcomplex of all closed cells having v for a vertex.

5.3. LEMMA. *For all integers i, μ, and M such that $0 \leqq i \leqq q+1$, $0 \leqq M \leqq 2^\mu$, the hyperplanes H and H' with the respective equations $\eta_i = M/2^\mu$ and $\eta_i - \eta_{1-1} = M/2^\mu$ are subcomplexes of $|D^\mu \Delta^q|$. (More precisely, they are the images of subcomplexes under the homeomorphism $|D^\mu \Delta^q| \to |\Delta^q|$.)*

Proof. Let σ be the degeneracy operator which deletes all but the ith coordinate from $(\eta_0, \ldots, \eta_{q+1})$. Then σ corresponds to a simplicial map $\sigma' : |\Delta^q| \to |\Delta^1|$; and hence to a simplicial map $|D^\mu \sigma| : |D^\mu \Delta^q| \to |D^\mu \Delta^1|$. Then H is the preimage of the vertex of $|D^\mu \Delta^1|$ with coordinate $M/2^\mu$, hence a subcomplex of $|D^\mu \Delta^q|$.

Similarly, the hyperplane H' is a subcomplex of $|D^\mu \Delta^q|$. For let σ be the deletion operator which eliminates all but the coordinates (η_{i+1}, η_i); then with σ is associated a simplicial map $|\sigma| : |\Delta^q| \to |\Delta^2|$, hence a simplicial map

$$|D^\mu \sigma| : |D^\mu \Delta^q| q \to |D^\mu \Delta^2|.$$

In the iterated subdivision of $|\Delta^2|$, it is easy to see that $\eta_2 - \eta_1 = M/2^\mu$, a line parallel to the hypotenuse of the triangle $0 \leqq \eta_1 \leqq \eta_2 \leqq 1$, is a subcomplex for each integer M between 0 and 2^μ. Hence H, the preimage of a subcomplex of $|D^\mu \Delta^2|$, is a subcomplex of $|D^\mu \Delta^q|$. ▮

5.4. COROLLARY. *For any integers M, M' between 0 and 2^μ, the regions of $|\Delta^q|$ and*

$$\left| \eta_i - \frac{M}{2^\mu} \right| \leqq \frac{M'}{2^\mu}, \qquad |\eta_{i+1} - \eta_i| \leqq \frac{M}{2^\mu}$$

are q dimensional subcomplexes of $|D^\mu \Delta^q|$. ▮

5.5. Corollary. (1) *Suppose that L_i is a subspace of $|\Delta^q|$ given by all points $(\eta_0, \ldots, \eta_{n+1})$ whose coordinates satisfy the equations* (*) *of Lemma* 5.1. *For any other integers N_i, $N_{j,i}$ between* 0 *and* 2^μ, *the subset F of $|\Delta^q|$ determined by a set of inequalities of the form*:

$$(**)\qquad \begin{aligned} &|\eta_{a(i)} - \xi_i| \leqq (1/2^\mu)N_i \quad (i = 0, 1, \ldots, n+1);\ \textit{and} \\ &\textit{one inequality}\colon |\eta_{j(r)+s} - \eta_{j(r)+s-1}| \leqq (1/2^\mu)N_{r,s} \\ &\textit{for each term } \eta_{j(r)+s-1} = \eta_{j(r)+s} \textit{ in } (*); \end{aligned}$$

is a convex set containing L_i and is a subcomplex of $|D^\mu\Delta^q|$.

(2) *For any such choice of integers N_i, $N_{i,j}$, F (as above) meets each face e^n of $|\Delta^q|$ in a similar subcomplex F' containing $L_i \cap e^n$, with constants $\{N_i', N_{i,j}'\}$ (in its inequalities) from among those in the set $\{N_i, N_{i,j}\}$.*

Proof. Immediate. ∎

5.6. Definition. If $\phi^{-1}(v)$ is the union $\bigcup L_i$ of linear subspaces of $|\Delta^q| \times y$ then, taking all the N_i's, $N_{i,j}$'s to be 1, for each L_i we have the corresponding convex subcomplex which we now label F_i, obtained as above. If the equations which give L_i are (*) of 5.1, then F_i will be the set of points satisfying (**). Note that each F_i is the restriction to $|\Delta^q|$ of a product $L_i \times Q_i$, where Q_i is a compact convex set. As $\mu \to \infty$, for fixed (ξ_i), note that the largest diameter of the Q_i tends to 0.

5.7. Fix P, an interior point of the n-cell $|x|$ of $|K|$ with coordinates $(\xi_0, \ldots, \xi_{n+1})$ in $|\Delta^n| \times x$ in $M(K)$. For each integer μ, let v^μ be a vertex interior to $|D^\mu x|$ in $|D^\mu K|$ for which P is interior to $\mathrm{St}^\mu(v^\mu)$. The preimage $\phi^{-1}(P)$ in $M(K)$ is the union of linear subspaces L_i in each $|\Delta^q| \times y$ of form (*) (of 5.1). The vertex v^μ (for each μ), also being an interior point of $|D^\mu x|$, has for preimage in the same simplex the union of linear subspaces $\{L_i^{(\mu)}\}$ where L_i is parallel to $L_i^{(\mu)}$ and for each i, we have a convex closed subcomplex $F_i^{(\mu)}$ determined as above. Each $F_i^{(\mu)}$ is the restriction to $|\Delta^q| \times y$ of the product $L_i^{(\mu)} \times Q_i^{(\mu)}$ where $\operatorname{diam}(Q_i^{(\mu)}) \to 0$ as $\mu \to \infty$. Denote by $F^{(\mu)}$ the union of all the $F_i^{(\mu)}$ for fixed μ. Each $F_i^{(\mu)}$ by construction contains not only $L_i^{(\mu)}$ but also L_i in its interior, so that as $\mu \to \infty$, the $F_i^{(\mu)}$ each tend to L_i (meaning that the intersection $\bigcap_\mu F_i^{(\mu)}$ is just L_i) and the $F_i^{(\mu)}$, for fixed i are a decreasing sequence of spaces. Hence also $\bigcap F^{(\mu)} = \phi^{-1}(P)$, and the $F^{(\mu)}$ are a decreasing family of unions of closed convex subspaces of $M(K)$.

We may consider $M(K)$ to be the union $\bigcup |\Delta^q| \times z$ only over nondegenerate simplices of K, for the quotient of this subcomplex is $|K|$, and this we now do; then $M(K)$ is a *finite* disjoint union of indexed simplices, hence is compact and so $F^{(\mu)}$ is compact for each μ.

Let C be the set of points which lie in simplices and faces of $M(K)$ which do not meet $\phi^{-1}(P)$.

5.8. Lemma. *There is a μ_1 so large that for $\mu \geqq \mu_1$, $F^\mu \cap C = \varnothing$, and for any set of linear summands $\{L_{i_{(1)}}, \ldots, L_{i_{(k)}}\}$ of $\phi^{-1}(P)$, the intersection $F^\mu_{i_{(1)}} \cap \cdots \cap F^\mu_{i_{(k)}} = \varnothing$ if and only if the corresponding intersection $L_{i_{(1)}} \cap \cdots \cap L_{i_{(k)}} = \varnothing$.*

Proof. There is a μ_0 for which for all larger μ, $F^{(\mu)}$ is disjoint from C. Assume μ at least this large. Let $\{L_i, \ldots, L_k\}$ be any collection of linear summands of $\phi^{-1}(P)$ in some $|\Delta^q| \times y$. For each μ, the sets $F_1^{(\mu)}, \ldots, F_k^{(\mu)}$ are compact; if the intersection $L_1 \cap \cdots \cap L_k$ is void, then for some sufficiently large μ, $F_1^{(\mu)} \cap \cdots \cap F_k^{(\mu)}$ is void. Hence for suitably large μ, for this set $\{L_1, \ldots, L_k\}$, $F_1^{(\mu)} \cap \cdots \cap F_k^{(\mu)}$ is void if and only if $L_1 \cap \cdots \cap L_k$ is void.

In any simplex of $M(K)$, the number of linear summands of $\phi^{-1}(P)$ is finite, and there are only finitely many simplices in all of $M(K)$. Hence there is a μ so large that in any simplex $|\Delta^q| \times y$ of $M(K)$, for any set of components $\{L_{i_1}, \ldots, L_{i_k}\}$ of $\phi^{-1}(P)$, the intersection $F_{i_1}^{(\mu)} \cap \cdots \cap F_{i_k}^{(\mu)}$ is void if and only if $L_{i_1} \cap \cdots \cap L_{i_k}$ is void. ∎

5.9. PROPOSITION. *If $\mu \geqq \mu_1$, the number from the preceding lemma, then $\phi(F^{(\mu)}(v^\mu))$ is a contractible subset of $|K|$, and therefore a contractible subcomplex of $|D^\mu K|$.*

Proof. In this proposition, we can assume $P = v^\mu$, so $\phi^{-1}(v^\mu) = \bigcup L_i$, etc.

We will inductively construct a strong deformation retraction H of the part of $F^{(\mu)}(v^\mu)$ lying in the n-skeleton of $M(K)$ to $\phi^{-1}(v^{(\mu)}) \cap M(K)^{(n)}$, in fact in each simplex deforming each $F_i^{(\mu)}$ to the corresponding $L_i^{(\mu)}$ through itself, and this will be done so that H is compatible with ϕ. This will give a strong deformation of $\phi(F^{(\mu)})$ to v^μ through itself.

The construction is obviously possible in the zero skeleton of $M(K)$, for if a vertex of $M(K)$ is in $F^{(\mu)}$, it lies in some L_i, so H is taken as the constant deformation. Assume inductively that H has been constructed on the part of $F^{(\mu)}$ lying in the $(q-1)$ skeleton of $M(K)$, so that it is compatible with ϕ and a strong deformation retraction of this part of $F^{(\mu)}$ to $\bigcup L_i$, and let e^q be $|\Delta^q| \times y$, a q-simplex of $M(K)$, or a q-dimensional face of some $|\Delta^r| \times z$, for z an r-simplex of K, with $r > q$. We distinguish two cases:

(i) e^q is "nondegenerate," which means that ϕ maps the interior of e^q homeomorphically into $|K|$, or

(ii) e^q is "degenerate," which means that ϕ collapses e^q to a cell of dimension less than q.

In case (i) it suffices to assume $e^q = |\Delta^q| \times y$ for some nondegenerate y of $M(K)$, and in (ii) that e^q is a face of some $|\Delta^r| \times z$, where $r > q$ and z is degenerate.

Case (i). Since ϕ is a homeomorphism on the interior of $|\Delta^q| \times y$, it makes no identifications, hence it suffices simply to extend the inductively given strong deformation over all of $F^{(\mu)}$ in $|\Delta^q| \times y$. First we find the maximal intersections $L_{i_1} \cap \cdots \cap L_{i_k}$ (those which are contained in no further $L_{i_1} \cap \cdots \cap L_{i_k} \cap L_r \neq \varnothing$). The corresponding $F_{i_1}, \ldots, F_{i_k}$ intersect in a convex neighborhood of the linear subspace $L_{i_1} \cap \cdots \cap L_{i_k}$ (all restricted to the convex set $|\Delta^q|$) and on a part of the boundary is given the deformation $\overline{H}$. We can thus surely extend this to a strong deformation over all $F_{i_1} \cap \cdots \cap F_{i_k}$; we can do this consecutively over all such

maximal intersections. Next, one extends H over next largest intersections, namely $L_{i_1} \cap \cdots \cap L_{i_{k-1}}$ which are contained in just one $L_{i_1} \cap \cdots \cap L_{i_{k-1}} \cap L_{i_k} \neq \varnothing$. Then H is given as before on part of the convex set $F_{i_1} \cap \cdots \cap F_{i_{k-1}}$ deforming it strongly to a linear subset $L_{i_1} \cap \cdots \cap L_{i_{k-1}}$. Again H extends over all $F_{i_1} \cap \cdots \cap F_{i_{k-1}}$ etc. Finally we arrive at the case: H is defined on part of the convex set F_i deforming it strongly to L_i, a linear subset, and H can again extend. This can be done for each $F_i^{(\mu)}$ and corresponding $L_i^{(\mu)}$. This extends H over all $F^{(\mu)}$ contained in $|\Delta^q| \times y$ and completes case (i).

Case (ii). In this case e^q is a face of $|\Delta^r| \times z$, for z nondegenerate, $r = \dim z > q$, and e^q corresponds to the degenerate q-dimensional face δz, δ the appropriate face operator. Thus $\delta z = \sigma z'$, where z' is nondegenerate and σ a degeneracy operator. Then ϕ mapping e^q into $|K|$ factors through the composition of first σ (acting on the right) sending e^q into $|\Delta^p| \times z$ ($p = \dim z'$) and then ϕ restricted to $|\Delta^p| \times z'$.

We observe the following fact about any degeneracy map σ: Given any T_4 space X, any closed subset A of X, any maps f, g as in the diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & |\Delta^q| \\ \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle \sigma} \\ X & \xrightarrow{f} & |\Delta^p| \end{array}$$

there is a lifting of f extending g.

Hence, since ϕ factors through $\phi \mid (|\Delta^p| \times z') \circ \sigma$, then in e^q the preimages $\phi^{-1}(P)$, $\phi^{-1}(v^\mu)$ are the preimages via σ of the corresponding preimages in $|\Delta^p|$. Hence if we denote the linear components in $|\Delta^p| \times z'$ by $\bar{L}_i$, $\bar{L}_i^{(\mu)}$, then L_i is the restriction to the subset $|\Delta^q|$ of $|\Delta^p| \times I^{q-p}$ of $\bar{L}_i \times I^{q-p}$, and $L_i^{(\mu)}$ that of $\bar{L}_i^{(\mu)} \times I^{(q-p)}$. Similarly, $F_i^{(\mu)} = \bar{F}_i^{(\mu)} \times I^{(q-p)} \cap |\Delta^q|$.

In $|\Delta^p| \times z'$, by inductive hypothesis, we have defined the strong deformation retraction of $\bar{F}^{(\mu)}$ to $\phi^{-1}(v^\mu)$ compatible with ϕ; we have only to "lift" this to a strong deformation in e^q of $F^{(\mu)}$ to $\phi^{-1}(v^\mu)$ compatible with σ for the deformation then to be compatible with ϕ. But this is a homotopy lifting problem (against σ), and it is always possible to lift such a homotopy compatibly with σ and prescribed to be the identity on $\phi^{-1}(\bar{L}_i^{(\mu)})$.

Hence H extends over e^q to a strong deformation covering the inductively defined H; this finishes case (ii) and the proof of the proposition. ▮

5.10. Proposition D. *For any finite ss complex K, for sufficiently large μ, $|D_\mu K|$ is covered by the interiors of contractible subcomplexes.*

Proof. Proposition 5.9 states that given any finite ssc K, and any point P, there is a μ_P so large that in $|D^\mu K|$ (for any $\mu \geqq \mu_P$) there is a contractible subcomplex with P as interior point. Because $|K|$ is compact, there is a μ_0 so large that for any $\mu \geqq \mu_0$, any point of $|D^\mu K|$ lies in the interior of a contractible subcomplex (for we cover $|K|$ with these sets and take a finite subcovering). ▮

5.11. As remarked in §0, Proposition D (5.10) together with Theorem A (1.3), Lemma 38, and Proposition 4.1 yield the proof of Theorem B (0.3), and Corollaries 0.4–0.7.

6. **Function spaces.**

6.1. We first recall some facts and definitions.

Definition. Let $\mathscr{A}, \mathscr{B}$ be $(n+1)$-tuples of ssc's $(X; A_1, \dots, A_n)$ and $(Y; B_1, \dots, B_n)$ respectively, where the A_i are subcomplexes of X and the B_i of Y. Then $\mathscr{A}^{\mathscr{B}}$ is the ssc which has for n-simplices ss maps f of $\Delta^n \times (Y; B_1, \dots, B_n)$ into $(X; A_1, \dots, A_n)$ and for face and degeneracy maps those induced by sending f into $f \circ (d_i^* \times 1)$ and $f \circ (s_i^* \times 1)$ respectively, with the notation as in [2].

If $\mathscr{A}^{\mathscr{B}}$ contains the constant map of Y into a point, we denote its component by $\mathscr{A}_0^{\mathscr{B}}$.

We define the ss evaluation map $P\colon X^Y \times Y \to X$ by $P(f_n, y_n) = f_n(\delta^n \times y_n)$, where δ^n is the fundamental n-simplex of Δ^n.

With notation as above, if X and each A_i are Kan complexes, then so is $\mathscr{A}^{\mathscr{B}}$. If $\mathscr{A}$ is an $(n+1)$-tuple of Kan complexes, then the restriction map $p\colon \mathscr{A}^{\mathscr{B}} \to \mathscr{A}^{\mathscr{B} \cap C}$ is a Kan fiber map, where C is any subcomplex of Y, and $C \cap \mathscr{B}$ is the $(n+1)$-tuple $(C; B_1 \cap C, \dots, B_n \cap C)$ ([9]). An immediate consequence of the definition of P is that it is natural with respect to maps to X or Y.

If X is a group complex, then X_0^Y is a normal subgroup complex of X^Y.

We will also use the following fact.

6.2. Lemma. *Let X and A be k-spaces. Then a map $f\colon X \to B^A$ is continuous (where the function space has the compact-open topology) if and only if the associated map $F\colon X \times_k A \to B$ is continuous.*

Proof. In view of the fact that maps of k-spaces are continuous if and only if they are continuous when restricted to compact subsets, this follows from exercise [12, 10a, p. 24]. ∎

6.3. Proposition. *Let $p\colon E \to B$ be an ss fiber bundle map with fiber F, and let E, B be Kan complexes and Y be any ssc. Then $p'\colon E_0^Y \to B_0^Y$ is an ss bundle with fiber F^Y.*

Proof. (1) p' is surjective.

(2) Let f be any map of Δ^n into B_0^Y, and let $p^f\colon E^f \to \Delta^n$ be the induced map. We must show that p^f is strongly equivalent to the projection onto Δ^n from $F^Y \times \Delta^n$. Let $(e, \Phi\delta^n)$ be a typical q-simplex of E^f. Then e is a simplex of E^Y for which $p'(e) = f(\Phi\delta^n)$ be a typical q-simplex of E^f. Then e is a simplex of E^Y for which $p(e) = f(\Phi\delta^n)$. Thus we have (the solid lines in) the diagram:

$$\begin{tikzcd}
\Delta^q \times Y \arrow[r, "e"] \arrow[d, "\Phi \times 1"] & E \arrow[d, "p"] & \Delta^n \times Y \times F \arrow[l, "\beta(f)"'] \arrow[d, "p_*"] \\
\Delta^n \times Y \arrow[r, "f"] & B & \Delta^n \times Y \arrow[l, "f"']
\end{tikzcd}$$

Since f is trivial, the map induced by f over $\Delta^n \times Y$ is equivalent to the product map, and we fix any such equivalence. Thus we have the entire diagram commutative. Note that the image of the injection $\beta(f)$ includes the image of e. Hence the map $\beta(f)^{-1} \circ e \colon \Delta^q \times Y \to \Delta^n \times Y \times F$ is well defined, and composing with the projection onto F gives a correspondence $e \to \pi \circ \beta(f)^{-1} \circ e \in F^Y$. If we look at any face $d_i e$ of e, then $\beta(f)^{-1} \circ d_i e$ is the map $\beta(f)^{-1} \circ e \circ d_i$, so that this correspondence commutes with face operators, and similarly with degeneracy operators. Hence this is an ss map which we call w.

We define $\tilde{\alpha}(f) \colon E^f \to \Delta^n \times F^Y$ by $\tilde{\alpha}(f)(\Phi\delta^n, e) = (\Phi\delta^n, w(e))$. The map satisfies $p^f = p_* \circ \tilde{\alpha}(f)$ by inspection. It is injective, for if $\tilde{\alpha}(\Phi\delta^n, e) = \tilde{\alpha}(\Phi\delta^n, e')$ then $w(e) = w(e')$, and since $\beta(f)$ is injective this means $e = e$. It is surjective, for if $(\Phi\delta^n, h)$ is a q-simplex of $\Delta^n \times F^Y$, then let e be the map of $\Delta^q \times Y$ into E given by $e = \beta(f) \circ [(\Phi \times 1) \wedge h]$, the inner map obtained by composing the diagonal map with $(\Phi \times 1) \times h$. Then $w(e)$ is the projection onto F of $\beta(f)^{-1}(e)$, and the latter is $(\Phi \times 1) \times h$ so that $w(e)$ is just h. Hence $\tilde{\alpha}$ is a strong equivalence and p is an ss bundle map. ∎

Let X, Y be ss complexes; $A_1, \dots, A_n$; $B_1, \dots, B_n$ n-tuples of subcomplexes. Let $\mathscr{A} = (X; A_1, \dots, A_n)$, $\mathscr{B} = (Y; B_1, \dots, B_n)$, and let $|\mathscr{B}| = (|Y| \colon |B_1|, \dots, |B_n|)$.

The map $P \colon \mathscr{A}^{\mathscr{B}} \times \mathscr{B} \to \mathscr{A}$ induces upon realization the map

$$|P| \colon |\mathscr{A}^{\mathscr{B}} \times \mathscr{B}| \to |\mathscr{A}|.$$

Since $|U \times V|$ is the product $|U| \times_k |V|$ in the category $\mathfrak{C}_k$ of k-spaces, $|P|$ is a map of $|\mathscr{A}^{\mathscr{B}}| \times_k |\mathscr{B}|$ into $|\mathscr{A}|$. Thus $|P|$ induces a (unique) map $\bar{P} \colon |\mathscr{A}_0^{\mathscr{B}}| \to |\mathscr{A}|_0^{|\mathscr{B}|}$, where the function space on the right is given the compact-open topology.

In the following sections, we will prove that $\bar{P}$ is a singular homotopy equivalence. (If X is a $k(\pi, n)$ complex, then Thom has essentially proven in [8] that $\bar{P}$ is a singular equivalence.)

We will denote by $|X|_0^{|Y|}$ the trivial component of $|X|^{|Y|}$.

6.4. Lemma. *Suppose $p \colon E \to B$ is an ss bundle map with fiber F, and Y any ssc. Then if two of the three maps*

(1) $\bar{P} \colon |E_0^Y| \to |E|_0^{|Y|}$,

(2) $\bar{P} \colon |F^Y| \to |F|^{|Y|}$,

(3) $\bar{P} \colon |B_0^Y| \to |B|_0^{|Y|}$

are singular equivalences, then so is the third.

Proof. We know $E_0^Y \to B_0^Y$ is a fiber bundle map with fiber F^Y, so that $|E_0^Y| \to |B_0^Y|$ is a k-bundle map. Since $|E| \to |B|$ is a Serre fiber map, so is $|E|_0^{|Y|} \to |B|_0^{|Y|}$, with fiber $|F|^{|Y|}$. Then $\bar{P}$ is a fiber preserving map, and maps the exact sequences of the fiber spaces. By the five lemma, the conclusion follows. ∎

6.5. Lemma. *If X is a $K(\pi, 0)$ complex, then $P \colon |X^Y| \to |X|^{|Y|}$ is a singular equivalence (and a continuous bijection).*

Proof. We may replace X by a subcomplex having 0-simplices 1-1 with the elements of π and no other nondegenerate simplices, and Y by a subcomplex consisting of one "point" (=vertex) from each component. Then $|X^Y|$ is the discrete set π^N, N the set of components of Y, and $|X|^{|Y|}$ is this set, perhaps with a nondiscrete topology. Then P is clearly continuous and bijective in this case. ∎

6.6. Proposition. *If X is an abelian $K(\pi, n)$ complex, then $P: |X^Y| \to |X|^{|Y|}$ is an equivalence of singular homotopy types (a weak homotopy equivalence).*

Proof. If $n=0$, then Lemma 6.5 above is the proof. Assume inductively that the proposition is true in dimensions less than n, and consider the case that X is a $K(\pi, n)$ complex.

Let (E, p, X) be a classifying bundle for X (as in [2]). Then $(|E|, |p|, |X|)$ is a fiber map with fiber the realization of a $K(\pi, n-1)$ complex. By Lemma 6.4, $\bar{P}: |X_0^Y| \to |X|_0^{|Y|}$ is a singular equivalence.

Finally, the trivial component is carried by group operation in either space into any other component; since P is a group homomorphism, it itself must be a homotopy equivalence of singular complexes. ∎

6.7. Corollary. *If X is a $K(\pi, 1)$ complex, with π nonabelian, then $\bar{P}: |X_0^Y| \to |X|_0^{|Y|}$ is a singular equivalence.* ∎

6.8. Proposition. *If X is a Kan complex, Y an ss complex, then $\bar{P}: |X_0^Y| \to |X|_0^{|Y|}$ induces a singular homotopy equivalence.*

Proof. If X is a $K(\pi, n)$ complex, then 6.6 above demonstrates the singular homotopy equivalence. Let the natural Postnikov system of X be given by the sequence of fibrations

$$(X_{(0)}, X_{(1)}, \ldots, X_{(n)}, \ldots)$$

where $X_{(k)}=(X^{(k-1)}, p, X^{(k)})$ is a fiber map, and the fiber of p is an Eilenberg-MacLane complex of type $K(\pi_{n+1}(X); n+1)$.

Without altering the homotopy type of X, it may be assumed that each fibration is an ss fiber bundle map. One then applies Lemma 6.4 repeatedly, using Proposition 6.6 and Corollary 6.7. ∎

6.9. Corollary. *If X is a group complex, then $\bar{P}: |X^Y| \to |X|^{|Y|}$ is a singular equivalence.*

Proof. Existence of the group operation implies that all of the components of X^Y are equivalent to the trivial component, and P is then clearly an equivalence on all $|X^Y|$. ∎

6.10. Corollary. *If Y is a finite complex, then $\bar{P}: |X_0^Y| \to |X|_0^{|Y|}$ is a homotopy equivalence.*

Proof. In this case, Milnor [4] has proven that $|X|_0^{|Y|}$ has the homotopy type of a CW complex. ∎

Proposition 6.8 can be extended to n-ads.

6.11. THEOREM. *Let $\mathscr{A}=(X; A_1, \ldots, A_n)$, $\mathscr{B}=(Y; B_1, \ldots, B_n)$ be $(n+1)$-tuples of ss complexes such that X and each A_i are Kan complexes, and such that the intersections $\bigcap_i A_i$ and $\bigcap_i B_i$ are not void. Let $\mathscr{A}_0^{\mathscr{B}}$ be the component in $\mathscr{A}^{\mathscr{B}}$ of the constant map, and similarly for $|\mathscr{A}|_0^{|\mathscr{B}|}$. Then $\bar{P}\colon |\mathscr{A}_0^{\mathscr{B}}| \to |\mathscr{A}|_0^{|\mathscr{B}|}$ is a singular homotopy equivalence. If Y is a finite ss complex, then $\bar{P}$ is a homotopy equivalence.*

Proof. By induction on n. If $n=1$, this reduces to the previous result. Suppose the theorem true for values less than n. Apply the theorem to the case of the ss fibration $\mathscr{A}'^{\mathscr{B}'} \to \mathscr{A}'^{B_n \cap \mathscr{B}'}$, where $\mathscr{A}'=(X; A_1, \ldots, A_{n-1})$, and similarly for $\mathscr{B}'$, and $B_n \cap \mathscr{B}'=(B_n; B_n \cap B_1, \ldots, B_{n-1} \cap B_n)$, and p is the restriction map. Theorem 0.6 together with the induction hypothesis implies that

$$\bar{P}\colon |(X; A_1, \ldots, A_{n-1}, x)_0^{\mathscr{B}}| \to |(X; A_1, \ldots, A_{n-1}, x)|_0^{|\mathscr{B}|}$$

is a singular equivalence. Then consider the fibration $P\colon \mathscr{A}_0^{\mathscr{B}} \to (A_n \cap \mathscr{A}')_0^{(B_n \cap \mathscr{B}')}$ induced by restriction. Again, this has the same fiber as the previous fibration. Applying 6.4 and the induction hypothesis again, we get: $\bar{P}\colon |\mathscr{A}_0^{\mathscr{B}}| \to |\mathscr{A}|_0^{|\mathscr{B}|}$ is a singular equivalence.

If Y is finite, then Milnor has proven in [4] that the function space $|\mathscr{A}|_0^{|\mathscr{B}|}$ has the homotopy type of a CW complex, and therefore $\bar{P}$ is a homotopy equivalence. ∎

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PURDUE UNIVERSITY,
LAFAYETTE, INDIANA