SOME NEW HILBERT ALGEBRAS

BY

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1. Introduction. The object of the present paper is the introduction of a new class of commutative Hilbert algebras. These algebras are distinct from those produced by a number of variations of the original axioms of an $H^*$-algebra given by W. Ambrose [1], [2]. The structure theorems obtained in the previous modifications give decompositions of the algebra into orthogonal subspaces, each of which is a minimal left ideal. In the commutative case, these result in an orthogonal basis for the space consisting of minimal, usually selfadjoint, idempotents. By adopting a different set of axioms, the present author determines a more general set of commutative $H$-algebras in which the direct sum decomposition is not necessarily an orthogonal one. The set $\{e_i\}$ of minimal idempotents which appear need neither be selfadjoint nor orthogonal, but rather make up the elements of a Fischer-Riesz system, a concept introduced by N. Bary [3]. A brief summary of the theory of Bary's is given in §4.

This paper obtains structure theorems for three different formulations of the axioms. §2 discusses the concept of a regular algebra in which the fundamental assumption is that every maximal modular ideal $R$ have a complementary ideal. §3 introduces the concepts of an adjoint algebra and of a dual adjoint algebra. An adjoint algebra is an algebra with two binary operations which permit the appearance of nonselfadjoint idempotents in the commutative case. A dual adjoint algebra is an algebra with two inner products which again permit the occurrence of nonselfadjoint idempotents. §4 gives a brief summary of the concepts of Bessel system, Hilbert system, and Fischer-Riesz system introduced by N. Bary. It also contains a basic example of a more general type of $H$-algebra than the standard $H^*$-algebra.

The symbols $\{e_i\}$, $\{g_i\}$, ($i=1, 2, \ldots$), are used to denote the elements of a biorthogonal system in a Hilbert space $H$. The symbols $(e_i)$ and $[e_i]$ denote the linear hull and the closed linear hull respectively of the set of vectors $\{e_i\}$, ($i=1, 2, \ldots$). All of the algebras which appear in this paper are semisimple, commutative $H$-algebras, although this condition is not always expressly stated. The continuity in multiplication is given by the existence of a constant $K$ such that

$$\|xy\| \leq K\|x\|\|y\|.$$ 

All of the Hilbert spaces which appear as underlying spaces of the algebras are assumed to be separable. This convention is used primarily for notational convenience.

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2. Complemented algebras. All of the Banach algebras considered in §2 are semisimple, commutative $H$-algebras. An algebra $A$ is regular if every maximal modular ideal $R$ of $A$ is complemented, i.e., there exists an ideal $I$ of $A$ such that $A$ is the direct sum $R+I$.

**Lemma 2.1.** Every maximal modular ideal $R$ of a regular algebra $A$ has associated with it a unique minimal idempotent $e$ and a unique multiplicative element $g$. An element $x \in A$ is in $R$ if and only if either $ex$ is the zero vector or $(x, g)$ is the zero complex number. For every pair $x, y \in A$,

\[(xy, g) = (x, g)(y, g), \quad \text{and} \quad ex = (x, g)e.\]

**Proof.** The difference algebra $A/R$ is isomorphic to the complex numbers so that the natural mapping $n$ of $A$ onto $A/R$ can be regarded as a linear functional on $A$. Since $R$ is closed, $n$ is a bounded linear functional. Thus there exists a unique element $g \in A$ so that $n(x)$ and $(x, g)$ are equal for every $x \in A$. Since $n$ is a homomorphism of $H$,

\[(xy, g) = n(xy) = n(x)n(y) = (x, g)(y, g)\]

for every $x, y \in A$. This unique $g$ is called the associated multiplicative element of $R$. Note that if $e$ is any idempotent, then

\[(e, g) = (e^2, g) = (e, g)^2\]

so that $(e, g)$ must be either one or zero.

There exists at least one ideal $I$ complementary to $R$. Since $I$ is isomorphic as a vector space to $A/R$, it must be a one-dimensional minimal ideal of $A$. The idempotent generator $e$ of $I$ is a minimal idempotent of $A$. If $I$ and $I'$ are distinct complementary ideals of $R$ with idempotent generators $e$ and $e'$ respectively, the product $ee'$ must be zero. Consequently,

\[0 = (ee', g) = (e, g)(e', g).\]

It follows that either $e$ or $e'$ must be contained in $R$, a contradiction. Therefore the complementary minimal ideal $I$ and the minimal idempotent generator $e$ are uniquely determined by the maximal modular ideal $R$.

Note that $ex$ must be a multiple $ne$ of $e$ for any $x \in A$. Since $(e, g)$ is one, we see that

\[(2.04) \quad ex = (x, g)e.\]

A slight variation of the above argument gives the

**Lemma 2.2.** Every minimal idempotent $e$ of a regular algebra $A$ is the idempotent associated with some maximal regular ideal $R$ of $A$.

We emphasize the useful fact that $ex$ is zero for a minimal idempotent $e$ if and only if $(x, g)$ is zero for the associated multiplicative element $g$, i.e., if and only if $x$ is an element belonging to the corresponding maximal regular ideal $R$. 
Lemma 2.3. Let \( \{R_\pi\}, \{e_\pi\}, \{g_\pi\}, (\pi \in P) \), denote the set of all maximal modular ideals, associated minimal idempotents, and associated multiplicative elements respectively of a regular algebra \( A \). Then

\[
(e_\pi, g_{\pi'}) = \begin{cases} 
1, & \pi = \pi', \\
0, & \pi \neq \pi'. 
\end{cases}
\]

(2.05)

This lemma is merely a restatement of the preceding two. Nevertheless, it points up the important fact that the two systems \( \{g_\pi\} \) and \( \{e_\pi\} \) form a biorthogonal system for a regular algebra \( A \). It follows immediately from semisimplicity that the system \( \{g_\pi\} \) is complete.

Lemma 2.4. A regular algebra \( A \) contains at most a countably infinite number of distinct maximal regular ideals whenever \( A \) is a separable Hilbert space.

Proof. The set \( M \) of multiplicative elements \( \{g_\pi\} \) is a part of a biorthogonal system. By a result of S. Levin [7], no element \( g \) of \( M \) is contained in the closed linear hull \( [g'] \), \( (g' \in M, g' \neq g) \). Let the elements of \( M \) be labelled with the members of a well-ordered set \( P \), i.e., let

\[
M = \{g_\pi : \pi \in P\}.
\]

By induction, there exists an orthonormal set \( \{u_\pi : \pi \in P\} \) such that for every \( \pi' \in P \)

\[
[[g_\pi] : \pi < \pi'] = \{[u_\pi] : \pi < \pi'\}.
\]

The set \( \{u_\pi : \pi \in P\} \) must be countable which establishes the lemma.

Lemma 2.5. Every nonzero ideal \( I \) of a regular algebra \( A \) contains a minimal idempotent.

Proof. If \( I \) contains no minimal idempotent \( e_i \); then \( e_i x \) must be zero for every minimal idempotent \( e_i \) and every \( x \) in \( I \). It follows that \( I \) is the zero ideal, a contradiction.

Lemma 2.6. Every finite-dimensional ideal \( I \) of a regular algebra \( A \) is complemented, i.e., there exists a closed ideal \( J \) such that \( A \) is the direct sum \( I + J \).

Proof. Observe that every minimal idempotent \( e \) of \( A \) either belongs to \( I \) or else \( ex \) is zero for every \( x \) of \( I \). Since the set \( E \) of minimal orthogonal idempotents of \( A \) is finitely linearly independent; only a finite set, say \( \{e_i\}, (i=1,\ldots,n) \), of minimal idempotents is contained in \( I \). Denote by \( E_n \) the idempotent \( e_1 + \cdots + e_n \) of \( I \). Every \( x \) in \( A \) can be written

\[
x = E_n x + (x - E_n x).
\]

(2.06)

If \( y \) denotes \( x - E_n x \), then \( e_i y \) is zero for every \( e_i \) contained in \( I \). It follows that when \( x \) belongs to \( I \), \( e_i y \) vanishes for every \( e_i \) in \( A \), implying that \( y \) is zero. Hence every \( x \) of \( I \) can be written

\[
x = E_n x = (x, g_1)e_1 + \cdots + (x, g_n)e_n.
\]

(2.07)
Thus $I$ is the linear hull $(e_i), (e_i \in I)$. The set of all $y$ such that $e_y$ is zero for $e_i$ in $I$ is a closed ideal $J$ containing all elements of the form $x - E_n x$ where $x$ is any element of $A$. It follows that $A$ is the direct sum $I + J$.

In the case of $H^*$-algebras, orthogonality enables one to extend this lemma to any proper closed ideal of $A$. We recall that the socle of any commutative algebra $A$, Banach or not, is the algebraic sum of the minimal ideals of $A$.

**Lemma 2.7.** The socle of a regular algebra $A$ is dense if and only if every proper closed ideal $I$ of $A$ is contained in a maximal regular ideal $R$ of $A$.

**Proof.** Suppose the socle $(e_i), (i = 1, 2, \ldots)$, is dense in $A$. If $I$ is a proper closed ideal, then there exists a minimal idempotent $e_i$ not contained in $I$. Consequently, $e_i I$ is the zero ideal and $I$ is contained in $R_i$. Let every proper ideal $I$ of $A$ be contained in a maximal regular ideal $R$ of $A$. Then since $[e_i], (i = 1, 2, \ldots)$, is a closed ideal not contained in any maximal regular ideal $R_i$, it must coincide with $A$.

The structural analysis of $H^*$-algebras proceeds via the route of orthogonal, complete reducibility. At each stage, one is presented with a direct sum decomposition $I + J$ where $x \in I, y \in J$ imply that $(x, y)$ is zero. Any Hilbert algebra $A$ is said to be well separated if, whenever $A$ is the direct sum $I + J$ of closed ideals $I$ and $J$, there exists a constant $k$, with $0 < k < 1$, such that $x \in I, y \in J$ imply

\[(2.08) \quad |(x, y)| \leq k \|x\| \|y\|.

**Theorem 2.8.** Let $A$ be a well-separated, regular algebra in which every proper closed ideal is contained in a maximal regular ideal. Then there exists a basis of minimal idempotents, $\{e_i\}$, of $A$ and two sequences of positive numbers, $\{d_i\}$ and $\{D_i\}$, $(i = 1, 2, \ldots)$, such that

\[d_1 |x_1|^2 + \cdots + d_n |x_n|^2 + \cdots \leq \|x_1 e_1 + \cdots + x_n e_n + \cdots \|^2 \leq D_1 |x_1|^2 + \cdots + D_n |x_n|^2 + \cdots
\]

for every element $x_1 e_1 + \cdots + x_n e_n + \cdots$ of $A$. Furthermore, there exists a continuous homomorphism of an $H^*$-algebra $H_1$ into a dense subset of $A$ and a continuous homomorphism of $A$ into an $H^*$-algebra $H_2$.

**Proof.** Denote by $E$ the set of all minimal idempotents of $A$. Let $\{u_i\}$ be an orthonormal set of $A$ such that

\[(2.09) \quad \{(e_i), i = 1, \ldots, m\} = \{(u_i), i = 1, \ldots, m\}
\]

for every $m$. Since the socle is dense, it follows that the set $\{u_i\}$ is an orthonormal basis of $A$.

For each positive integer $n$, let $I_n$ and $E_n$ denote the ideal $(e_i), (i = 1, \ldots, n)$, and the idempotent $e_1 + \cdots + e_n$, respectively. According to Lemma 2.6, $A$ is a direct sum $I_n + J_n$ of closed ideals. Consequently, any $z \in A$ can be written $x + y$ with $x \in I_n, y \in J_n$. It follows that

\[\|z\|^2 \geq (1 - k)(\|x\|^2 + \|y\|^2).
\]
Also
\[ \|z\|^2 \leq (1+k)(\|x\|^2 + \|y\|^2). \]

Finally,
\[ a(\|x\|^2 + \|y\|^2) \leq \|z\|^2 \leq b(\|x\|^2 + \|y\|^2) \]
where \( 0 < 1-k = a < 1+k = b \). If \( c^2(1-k) \) is one, then
\[ \|E_nz\| \leq c\|z\|. \]

Thus the projections of the set \( \{E_i\} \), \( i=1, 2, \ldots \), are uniformly bounded by the number \( c \).

Now, given any \( x \) in \( A \) and any positive number \( \varepsilon \), there exists an integer \( N \) such that if
\[ x_n = (x, u_1)u_1 + \cdots + (x, u_n)u_n, \]
then
\[ \|x-x_n\| < \varepsilon/(1+c) \]
for \( n \) greater than \( N \). Note that equation (2.09) implies that \( x_n \in I_i \), whenever \( r \) exceeds \( n \). Now for \( N < n < r \), we have
\[ \|x-E_ix\| \leq \|x-x_n\| + \|x_n-E_ix\| \]
\[ \leq \|x-x_n\| + c\|x-x_n\| \]
\[ < \varepsilon. \]

Thus the sequence \( \{E_ix\} \), \( j=1, 2, \ldots \), where
\[ E_ix = (x, g_1)e_1 + \cdots + (x, g_j)e_j, \]
converges to \( x \) for every \( x \in A \). Consequently, the set \( E \) of minimal idempotents is a basis of \( A \).

Repeated applications of inequality (2.10) lead to the following inequality,
\[ |x_1|^2 a\|e_1\|^2 + \cdots + |x_n|^2 a^n\|e_n\|^2 \leq \|x_1e_1 + \cdots + x_ne_n\|^2 \]
\[ \leq |x_1|^2 b\|e_1\|^2 + \cdots + |x_n|^2 b^n\|e_n\|^2. \]

Define two new sequences, \( \{d_i\} \) and \( \{D_i\} \), by means of
\[ d_i = a^i\|e_i\|^2, \quad D_i = b^i\|e_i\|^2, \quad (i = 1, 2, \ldots). \]

Equation (2.14) asserts that given a sequence \( \{x_n\} \) such that
\[ D_1|x_1|^2 + \cdots + D_n|x_n|^2 + \cdots \]
converges, then there exists an \( x \) in \( A \) given by
\[ x = x_1e_1 + \cdots + x_ne_n + \cdots. \]
Furthermore, it states that if some \( x \) in \( A \) is given by equation (2.17), then the series
\[
(2.18) \quad d_1|x_1|^2 + \cdots + d_n|x_n|^2 + \cdots
\]
converges.

These results are quite similar to the standard ones for semisimple, commutative \( H^* \)-algebras. The \( H^* \)-algebras are distinguished by the fact that the two sequences \( \{d_i\} \) and \( \{D_i\} \) coincide, due to the orthogonality conditions on the minimal idempotents. We are now able to define a continuous homomorphism of an \( H^* \)-algebra \( H_1 \) into a dense subset of \( A \) and a continuous homomorphism of \( A \) into an \( H^* \)-algebra \( H_2 \).

Let \( H_1 \) denote an \( H^* \)-algebra with a sequence of minimal selfadjoint idempotents \( \{s_i\} \) satisfying the conditions that
\[
\|s_i\|^2 = D_i.
\]

If
\[
(2.19) \quad x = x_1s_1 + \cdots + x_ns_n + \cdots
\]
denotes any element of \( H_1 \), then
\[
\|x\|^2 = |x_1|^2 D_1 + \cdots + |x_n|^2 D_n + \cdots.
\]
The multiplication in \( H_1 \) satisfies a condition of the form
\[
(2.20) \quad \|xy\| \leq K\|x\|\|y\|
\]
for every \( x, y \in H_1 \). The mapping \( T \) which makes the element \( x \) of equation (2.19) correspond to
\[
T(x) = x_1e_1 + \cdots + x_ne_n + \cdots
\]
is a continuous homomorphism of \( H_1 \) into \( A \). Since the \( \text{Im} \ T \) includes all elements of the form, \( x_1e_1 + \cdots + x_ne_n \), the \( \text{Im} \ T \) is dense in \( A \). The construction of \( H_2 \) should be clear.

An equation such as equation (2.20) implies that \( 1/K' \leq \|e\| \) for an idempotent \( e \) of an algebra \( A \). However, there is no result in the theory of \( H^* \)-algebras bounding the norms of the minimal selfadjoint idempotents of the algebra. Nevertheless, an \( H^* \)-algebra is somewhat pathological when there is no uniform bound for these. Call a sequence \( \{x_n\} \) admissible with respect to the basis \( \{v_i\} \) if \( x_1v_1 + \cdots + x_nv_n + \cdots \) converges. For \( H^* \)-algebras with bases of uniformly bounded minimal idempotents \( \{v_i\} \), the sequence \( \{x_n\} \) and \( \{x_{n(p)}\} \) are either both admissible or both inadmissible, for any permutation \( p \) of the integers. An algebra \( A \) with a basis of minimal idempotents \( \{e_i\} \) is homogeneous if it enjoys this property. I. M. Gel'fand [4] has proved that a homogeneous basis is a Fischer-Riesz basis. Thus we have established most of the following theorem. The remainder of the proof is given in §4.

**Theorem 2.9.** A well-separated, regular algebra \( A \) in which every proper closed
ideal is contained in a maximal modular ideal is an isomorphic image of a homogeneous $H^*$-algebra if and only if $A$ is homogeneous.

We close this section with a characterization of regular algebras in terms of their structure space. We recall that for a commutative Banach algebra the structure space is the set of all maximal modular ideals $M$. Recall that the hull $h(I)$ of an ideal $I$ is the set of all maximal modular ideals containing $I$. The kernel $k(S)$ of a set $S$ of maximal modular ideals is their intersection, a closed ideal. The set $M$ of maximal modular ideals is topologized by defining the closure $\bar{S}$ of a set $S$ of $M$ to be the hull of the kernel of $S$.

**Theorem 2.10.** An algebra $A$ is regular if and only if the structure space $M$ of $A$ is discrete in the hull-kernel topology.

**Proof.** Let $M$ denote the structure space of a regular algebra $A$. Denote by $S_i$ the subset of $M$ consisting of all maximal regular ideals other than $R_i$. Let $J_i$ be $k(S_i)$, the kernel of $S_i$. Since $(e_i, g_i)$ is zero for $j$ different from $i$, it follows that $J_i$ contains the minimal ideal $Ae_i$. Thus

$$J_i = Ae_i + J_i \cap R_i$$

where the sum is direct. However, if $x$ is an element of $J_i \cap R_i$, then $x$ belongs to the radical and must be zero. Thus $J_i$ coincides with the minimal ideal $Ae_i$. On the other hand, since $(e_i, g_i)$ is zero only for $j$ not equal to $i$, it follows that $h(J_i)$ must be $S_i$.

Consequently,

$$S_i = h(J_i) = h(k(S_i)),$$

and $S_i$ is closed in the hull-kernel topology. Therefore $R_i$, the complement of $S_i$, must be open. Since $R_i$ is maximal, it follows that

$$R_i = h(R_i) = h(k(R_i))$$

so that $R_i$ is also closed. The structure space $M$ is discrete.

Suppose, on the other hand, that the structure space of a semisimple, commutative $H$-algebra $A$ is discrete. Using the notation $S_i$ as above, we see that $S_i$ is closed so that

$$S_i = h(k(S_i)).$$

Consequently, there exists a nonzero vector $x_i$ in $k(S_i)$ which is not in $R_i$. It follows that $(x_i, g_i)$ is not zero. For each $i$, define $e_i$ to be $x_i/(x_i, g_i)$. Then

$$(e_i, g_j) = 1, \quad i = j,$$

$$= 0, \quad i \neq j.$$  

(2.21)

Thus the sequences $\{e_i\}$, $\{g_i\}$, $(i=1, 2, \ldots)$, form a biorthogonal system. We note that $(e_i^2 - e_i, g_i)$ is zero for every $j$ which implies that

$$e_i^2 - e = 0, \quad \text{or} \quad e_i^2 = e_i.$$
Consequently, the sequence \( \{e_i\} \) consists entirely of idempotents. Furthermore, note that for every \( x \)

\[
(e_i x, g_j) = (e_i, g_j)(x, g_j).
\]

This equation implies that \( e_i x \) is an element of \( R_i \) whenever \( i \) differs from \( j \). To the contrary, \( e_i x \) belongs to \( R_i \) only when it is the zero vector. In addition, for any \( x \) in \( H \),

\[
(x - e_i x, g_i) = 0.
\]

Hence \( A \) is the direct sum \( R_i + A e_i \), i.e., every maximal modular ideal \( R_i \) is complemented and \( A \) is regular.

3. Adjoint algebras. Let \( H \) be a Hilbert space on which there exist two binary operations \( f \) and \( f^* \). The standard product of \( x, y \in H \) is the image \( f(x, y) \) and is denoted by \( xy \). The adjoint product is the image \( f^*(x, y) \) denoted by \( x \cdot y \) to distinguish it from the standard product. Assume further that constants \( K \) and \( K^* \) exist such that

\[
(3.01) \quad \|xy\| \leq K\|x\|\|y\|, \quad \|x \cdot y\| \leq K^*\|x\|\|y\|.
\]

In addition, there exists a bijection \( \pi \) on \( H \) with \( \pi(x) \) denoted by \( x^* \) such that

\[
\|x^*\| \leq b\|x\|, \quad \|x\| \leq b^*\|x^*\|
\]

for all \( x \). The set \( H \) is said to form an adjoint algebra \( A \) if it is a Hilbert algebra under each of the binary operations and the following equation is satisfied for all \( x, y, z \in H \)

\[
(3.02) \quad (xy, z) = (y, x^* \cdot z).
\]

We limit our considerations to the case where \( H \) is a semisimple, commutative \( H \)-algebra under each of the binary operations. We do not repeat these assumptions in the following paragraphs. We use the adjectives, standard and adjoint, in the expected way. A standard ideal \( I \) of \( A \) is a vector subspace which is closed under standard multiplication by any element in \( A \). An adjoint maximal modular ideal is a maximal modular ideal under the adjoint product. Given that each of the products is semisimple, the following rules are valid

\[
(3.03) \quad (e_1 x + e_2 y)^* = \overline{e_1} x^* + \overline{e_2} y^*,
\]

\[
(xy)^* = y^* \cdot x^* = x^* \cdot y^*
\]

where \( x, y \) belong to \( H \); \( e_1, e_2 \) are complex numbers.

The following lemma is clear.

**Lemma 3.1.** The orthogonal complement of a standard (adjoint) ideal \( I \) of the adjoint algebra \( A \) is an adjoint (standard) ideal \( J \) of \( A \).

**Lemma 3.2.** An adjoint algebra \( A \) is regular under each of its two products.

**Proof.** Let \( R \) be a standard maximal modular ideal of \( A \). Then \( A \) is the orthogonal direct sum \( R + J \) where \( J \) is a one-dimensional adjoint ideal of \( A \). It follows
that \( J \) is minimal and contains an idempotent generator \( e^* \) such that \( J \) and \( A \cdot e^* \) coincide. Let \( e \) denote \( \pi^{-1}(e^*) \) so that \( e \) is a standard idempotent by equation (3.03). For \( y \in R, x \in A \),

\[
(ye, x) = (y, x \cdot e^*) = 0
\]

so that \( ye \) is zero for every element \( y \in R \). On the other hand, if \( ye \) is zero then \( (y, x \cdot e^*) = (ye, x) = 0 \) for every \( x \) which implies that \( y \in R \), i.e., \( R \) is the annihilator of \( e \). Finally, \( x = (x - xe) + xe \) for every \( x \) in \( A \), that is

(3.04) \[
A = R + Ae
\]

is a direct sum decomposition of \( A \). Consequently, the standard maximal modular ideal \( R \) is complemented and \( A \) is regular with respect to the standard product. A similar argument is valid for the adjoint product.

Let \( \{e_i\} (i = 1, 2, \ldots) \) denote the complete set of standard minimal idempotents of the adjoint algebra \( A \). Then the set \( \{e^*_i\} (i = 1, 2, \ldots) \) where \( e^*_i = \pi(e_i) \) is a complete set of minimal adjoint idempotents of \( A \). Let \( \{g_i\}, (i = 1, 2, \ldots) \), be the set of associated functionals for the set \( \{e_i\} \). Note that

\[
\delta_j^i = (e^*_i, g_j) = (e^*_i, e_jg_i) = (g^*_i, e_i)
\]

which implies that the set \( \{g^*_i\} \) where \( g^*_i = \pi(g_i) \) is the set of associated functions for the set \( \{e_i\} \). In this section, we use the symbols \( \{e_i\}, \{g^*_i\}, \{e^*_i\}, \{g_i\}, (i = 1, 2, \ldots) \), to denote the biorthogonal sequences of minimal idempotents and associated linear functions in the cases of the standard and adjoint operations, respectively, of the adjoint algebra \( A \).

**Lemma 3.3.** The standard (adjoint) socle is dense in the adjoint algebra \( A \).

**Proof.** The closed linear hull \( [e_i], (e_i \in A) \), is a standard ideal \( I \) of \( A \). If the standard socle is not dense, then the orthogonal complement \( J \) of \( I \) is a proper adjoint ideal of \( A \) and

(3.05) \[
A = I + J.
\]

Let \( y \in A, x \in J, \) and \( e^*_i \) be any minimal adjoint idempotent. Then

(3.06) \[
(y, e^*_i \cdot x) = (e_i y, x) = 0
\]

so that \( e^*_i \cdot x \) is zero for every adjoint minimal idempotent which implies that \( x \) is in the adjoint radical. Thus \( J \) is a nonzero ideal contained in the adjoint radical, a contradiction. We do not argue the adjoint case since it is essentially the same.

**Lemma 3.4.** Let \( A \) be an adjoint algebra such that \( \pi \) is an involution. Then

(3.07) \[
(x, y) = (y^*, x^*)
\]

for all \( x, y \in A \).
Proof. First, consider the case where one of the elements is a minimal idempotent \( e_i \). Then for any \( x \in A \)

\[
(e_i, x) = (e_i^*, x) = (e_i, e_i^* \cdot x) = (e_i, x^* \cdot e_i^*) = (x^*, e_i^*).
\]

Now let

\[
z_n = z_{n1}e_1 + \cdots + z_{nn}e_n
\]

and \( y \) be any element of \( A \). Then

\[
(z_n, y) = (z_{n1}e_1 + \cdots + z_{nn}e_n, y)
= z_{n1}(e_1, y) + \cdots + z_{nn}(e_n, y)
= z_{n1}(y^*, e_1^*) + \cdots + z_{nn}(y^*, e_n^*)
= (y^*, z_{n1}e_1^* + \cdots + z_{nn}e_n^*)
= (y^*, z_n^*).
\]

We see, in particular, for elements \( z_n \), that

\[
(z_n, z_n) = (z_n^*, z_n^*) = \|z_n^*\|^2.
\]

Since the socle is dense, the stated result follows via continuity of the mapping \( \pi \) and of the inner product.

Lemma 3.5. Let \( e_i \) be a minimal idempotent of the adjoint algebra \( A \). Then

\[
e_i e_i^* = e_i e_i \quad \text{and} \quad e_i = \pi_i g_i
\]

where \( e_i \) and \( \pi_i \) are complex numbers different from zero.

Proof. First, observe that

\[
(e_i, e_j^*) = (e_i e_i^*, e_j^*) = (e_i, e_i^* \cdot e_j^*)
\]

which is zero whenever \( i \) differs from \( j \). However, if \( (e_i, e_j^*) \) is also zero, then \( e_i \) itself must be zero, an impossibility. It follows that \( e_i \) must be a nonzero multiple of \( g_i \), that is

\[
e_i = \pi_i g_i.
\]

Furthermore, if \( e_i e_i^* \) is zero, then

\[
0 = (e_i e_i^*, e_i^*) = (e_i^*, e_i^* \cdot e_i^*) = (e_i^*, e_i^*)
\]

so that \( e_i^* \) is zero, contradicting its definition as an adjoint minimal idempotent. Thus

\[
e_i e_i^* = e_i e_i
\]

where \( e_i \) is not zero.

We wish to observe that in the case of an \( H^* \)-algebra, the minimal idempotent \( e_i \) is selfadjoint and equation (3.10) holds with \( e_i \) equal to one. An adjoint algebra \( A \) is called normal if each \( e_i \) satisfying equation (3.10) is positive.
Theorem 3.6. Let $A$ be a normal adjoint algebra in which the minimal idempotents are bounded in norm by $B$. Then $A$ has a basis $\{e_i\}$ ($i=1, 2, \ldots$) of minimal idempotents which is a Fischer-Riesz system.

Proof. It follows from equations (3.09) and (3.10) that

$$e_i = (e_ie_i^*, g_i^*) = (e_i^*, g_i^*) = \pi_i(g_i^*, g_i^*)$$

which proves that $\pi_i$ is real and positive.

The following inequalities are valid in $A$:

$$1/B \leq ||g_i|| \leq K,$$
$$1/B \leq ||g_i^*|| \leq K^*,$$
$$1/K \leq ||e_i|| \leq B,$$
$$1/K^2 \leq ||e_i^*|| \leq B,$$
$$1/K^2 \leq \pi_i \leq B^2.$$

(3.11)

Now let $x$ be any element in $A$ and denote $(x, g_i^*)$ by $x_i$. Since the standard socle is dense, there exist sequences

$$z_n = x_{n1}e_1 + \cdots + x_{nn}e_n,$$
$$z_n^* = \bar{x}_{n1}e_1^* + \cdots + \bar{x}_{nn}e_n^* = \pi_1\bar{x}_{n1}g_1^* + \cdots + \pi_n\bar{x}_{nn}g_n^*,$$

(3.12)

such that $x$ is the limit of $z_n$. It follows from the continuity of the norm and inner product that

$$\lim ||z_n|| = ||x||, \quad \text{and} \quad \lim x_{nf} = x_f.$$

(3.13)

From relations (3.11) and (3.12), we see that

$$(1/K^2)(|x_{n1}|^2 + \cdots + |x_{nn}|^2) \leq \pi_1|x_{n1}|^2 + \cdots + \pi_n|x_{nn}|^2$$

$$= (z_n, z_n^*) \leq ||z_n|| ||z_n^*|| \leq b||z_n||^2.$$

Given $\varepsilon > 0$, there exists an integer $N$ such that

$$||z_n||^2 \leq (||x||^2 + \varepsilon)^2$$

whenever $n > N$. Thus, for this case,

$$(1/K^2)(|x_{n1}|^2 + \cdots + |x_{nn}|^2) \leq b(||x|| + \varepsilon)^2,$$

which implies that, for $v = K^2b$,

$$(|x_1|^2 + \cdots + |x_n|^2) \leq v(||x|| + \varepsilon)^2$$

for every $\varepsilon$. Consequently, we have

$$(|x_1|^2 + \cdots + |x_n|^2) \leq v||x||^2,$$

(3.14)

$$(|x_1|^2 + \cdots + |x_n|^2 + \cdots) \leq v||x||^2.$$
Equation (3.14) implies that the set \( \{e_i\} \) of minimal idempotents is a Bessel system. It follows from the results of Bary, that the system \( \{g_i^*\} \), biorthogonal to it, is a Hilbert system. Therefore, given any sequence \( \{c_i\} \) in \( l_2 \), the sequence \( \{z_n\} \) given by
\[
(3.15) \quad z_n = c_1 g_1^* + \cdots + c_n g_n^*
\]
converges to an element \( x \) of \( A \). Furthermore, there exists a constant \( m \) such that
\[
\|x\| \leq m(|c_1|^2 + \cdots + |c_n|^2 + \cdots).
\]
Note that the sequence \( \{c_i\} \) belongs to \( l_2 \) if and only if the sequence \( \{c_i^*\} \) belongs to \( l_2 \). Consequently, if \( \{c_i\} \) is in \( l_2 \), then the sequence \( \{z_n\} \) where
\[
(3.16) \quad z_n = c_1 e_1^* + \cdots + c_n e_n^*
\]
converges to an element \( x \) in \( A \). In addition,
\[
\|x\| \leq m(|\pi_1 c_1|^2 + \cdots + |\pi_n c_n|^2 + \cdots)
\]
\[
\leq mB^* (|c_1|^2 + \cdots + |c_n|^2 + \cdots)
\]
Thus, we see that the system \( \{e_i^*\} \) is a Hilbert system. However, \( \{e_i^*\} \) is a Bessel system by the same sort of argument that proved that \( \{e_i\} \) is a Bessel system. It follows that \( \{e_i^*\} \) and, consequently, \( \{e_i\} \) are Fischer-Riesz systems, as was to be shown.

We turn to one further variation in the axioms in which the results follow almost immediately.

Let \( A \) be a commutative, semisimple \( H \)-algebra with an involution \( \pi \) where the image \( \pi(x) \) is denoted by \( x^* \). Suppose that \( f \) and \( f^* \) are two inner products on \( A \) with \( f(x, y) \) and \( f^*(x, y) \) denoted by \( (x, y) \) and \( [x, y] \) respectively. In addition, assume there exist positive constants \( B \) and \( B^* \) such that
\[
(3.17) \quad |(x, y)|^2 \leq B^2 [x, x] [y, y], \quad \text{and} \quad ||[x, y]||^2 \leq (B^*)^2 (x, x)(y, y).
\]
The algebra \( A \) is called a dual adjoint algebra if the involution and the inner products satisfy the relations
\[
(3.18) \quad (xy, z) = [x, zy^*] = [y, x^*z].
\]

**Theorem 3.7.** The dual adjoint algebra \( A \) has a basis \( \{e_i\} \), \( i = 1, 2, \ldots \), of minimal idempotents. Furthermore, the basis \( \{e_i\} \) is a Fischer-Riesz system if and only if the minimal idempotents are uniformly bounded.

**Proof.** We define a new inner product on \( A \) under which it becomes an \( H^* \)-algebra. For \( x, y \in A \), let
\[
(3.19) \quad \{x, y\} = (x, y) + [x, y].
\]
Denote the norm of $x$ in this new inner product by $||x||$. It is easy to see that
\begin{equation}
||(x, y)|| \leq (1 + B^*) ||x|| ||y||, \quad \text{and}
\end{equation}
\begin{equation}
||x||^2 \leq ||x|| ||y|| \leq ||x||^2 ||y||
\end{equation}
where $||x||^2$ denotes $(x, x)$. Furthermore, if $||xy|| \leq K ||x|| ||y||$, then $||xy|| \leq K' ||x|| ||y||$ where $K'$ denotes $K(1 + B^*)^1/2$. Finally,
\begin{equation}
\{xy, z\} = \{y, x^*z\} = \{x, zy^*\}
\end{equation}
so that $A$ is a semisimple, commutative $H^*$-algebra in this new norm. We denote the “two” normed algebras by $A(\ , \ )$ and $A\{\ , \ \}$ respectively.

The fundamental theorem on $H^*$-algebras asserts that $A\{\ , \ \}$ has an orthonormal basis $\{e_i\}$, $(i=1, 2, \ldots)$, of minimal selfadjoint idempotents. The identity mapping of $A\{\ , \ \}$ onto $A(\ , \ )$ is a continuous automorphism. It follows that the set $\{e_i\}$ is a basis of minimal idempotents in the algebra $A(\ , \ )$, the original dual adjoint algebra. It is easy to see that $\{e_i\}$ is a Fischer-Riesz basis in each algebra if and only if the minimal idempotents are uniformly bounded.

4. Examples. The following examples were suggested by the results of N. Bary [3] in which the most natural generalization of orthonormal base is developed. Her more important results for biorthogonal systems $\{e_i\}, \{g_j\}, (i=1, 2, \ldots)$, in the separable Hilbert space $H$ are in terms of three basic concepts. The complete system $\{e_i\}$ is said to be a **Bessel system** if given any $x \in H$, the coefficients $\{(x, g_i)\}$ belong to $l_2$. The complete system $\{g_j\}$ is said to be a **Hilbert system** if given any sequence $\{c_j\}$ in $l_2$ there exists a unique $x \in H$ such that $\{c_j\}$ coincides with the sequence $\{(x, e_i)\}$. A complete system $\{e_i\}$ is said to be a **Fischer-Riesz system** if it is both a Bessel and a Hilbert system. The fundamental results of Bary’s are: A system $\{e_i\}$ is a Fischer-Riesz system if and only if it is the image of every orthonormal basis $\{u_i\}$ of $H$ under some continuous automorphism, not necessarily isometric, of $H$. A system $\{e_i\}$ is a Bessel system if and only if every orthonormal basis $\{u_i\}$ is the image of $\{e_i\}$ under some continuous endomorphism of $H$. A system $\{e_i\}$ is a Hilbert system if and only if it is the image of every unitary basis $\{u_i\}$ under some continuous endomorphism of $H$.

The results have the following additional interpretations. Let $\{e_i\}, \{g_j\}$ form a biorthogonal system. Let $x$ be in $H$ and consider the following two inequalities:
\begin{equation}
m(|(x, g_1)|^2 + \cdots + |(x, g_n)|^2 + \cdots) \leq ||x||^2.
\end{equation}
\begin{equation}
||x||^2 \leq M(|(x, g_1)|^2 + \cdots + |(x, g_n)|^2 + \cdots).
\end{equation}
A Fischer-Riesz system $\{e_i\}$ is a basis such that constants $m$ and $M$ exist for which both inequalities (4.01) and (4.02) are valid whenever $x$ belongs to $H$. Although a complete Hilbert system $\{e_i\}$ need not be a basis, there exists a constant $M$ such that inequality (4.02) is valid for every $x$ in $H$. Furthermore, a complete Bessel system $\{e_i\}$ need not be a basis, but there exists a constant $m$ such that inequality (4.01) is valid for every $x$ in $H$.

We need the following Lemmas before considering our examples.
Lemma 4.1. Let the Hilbert space $H$ be the direct sum $I + J$ of the nonorthogonal, closed subspaces $I$ and $J$. Let $P$ be the projection of $H$ on $I$ along $J$. Then if $u \in I$, $v \in J$

\begin{equation}
(u, v) \leq k\|u\|\|v\|, \quad 0 < k < 1.
\end{equation}

Proof. Let $u$ and $v$ be elements of $I$ and $J$, respectively, each being of unit norm. Since $u$ and $v$ are linearly independent, $\langle u, v \rangle < \|u\|\|v\| = 1$.

If $r$ is the vector $[u - (u, v)v]/\|1 - (u, v)^2\|$, then it is easy to see that

\begin{equation}
\|P\|^2 \geq \|r\|^2 = 1/[1 - |(u, v)|^2] \geq 1.
\end{equation}

From which it follows that

\begin{equation}
|(u, v)|^2 \leq \|P\|^2 - 1)/\|P\|^2 < k^2 < 1.
\end{equation}

For general $u$ and $v$, we have the desired result:

\begin{equation}
|(u, v)| \leq k\|u\|\|v\|.
\end{equation}

Whenever $T$ is a continuous automorphism of a Hilbert space $H$ onto itself, $T$ has a continuous inverse $T^{-1}$ such that $1 \leq \|T\|\|T^{-1}\|$.

Lemma 4.2. Let $T$ be a continuous automorphism of a Hilbert space $H$ onto itself. Then, if $\|T\|\|T^{-1}\|$ is one, $T$ is essentially unitary, i.e., there exists a positive number $r$ and a unitary transformation $U$ such that

\begin{equation}
T = rU.
\end{equation}

Proof. Suppose there exist vectors $x_1, x_2$ in $H$ such that

\begin{equation}
\|x_1\| = \|x_2\| = 1, \quad \|Tx_1\| < \|Tx_2\|.
\end{equation}

Then

\begin{equation}
1/(\|T^{-1}\|) \leq \|Tx_1\| < \|Tx_2\| \leq \|T\|
\end{equation}

so that

\begin{equation}
1 < \|T\|\|T^{-1}\|,
\end{equation}

a contradiction. Thus

\begin{equation}
\|Tx_1\| = \|Tx_2\| = r^2 > 0
\end{equation}

whenever $x_1$ and $x_2$ are of norm one. It follows that

\begin{equation}
((T/r)x, (T/r)x) = (x, x)
\end{equation}

for every $x$, i.e., $(T/r)$ is an isometry and, consequently, must be unitary.

Let $\{e_i\}$, $\{g_i\}$ be a biorthogonal system where both $\{e_i\}$ and $\{g_i\}$ are Fischer-Riesz systems. Let $\{u_i\}$ be an orthonormal basis of $H$ and $T$ be the continuous automorphism of $H$ such that

\begin{equation}
e_i = Tu_i, \quad \text{and} \quad 1/(\|T^{-1}\|) \leq \|e_i\| \leq \|T\|.
\end{equation}

The continuous automorphism $T' = (T^*)^{-1}$ has the property that

\begin{equation}
g_i = T'u_i, \quad \text{and} \quad 1/\|T^*\| \leq \|g_i\| \leq \|T'\|.
\end{equation}
Remark 4.3. Now let \( R \cup S \) be any partition of the integers \( \mathbb{Z} \) into disjoint subsets and
\[
I' = [u_i], \quad (i \in R); \quad J' = [u_i], \quad (i \in S);
I = [e_i], \quad (i \in R); \quad J = [e_i], \quad (i \in S).
\]
Then \( H \) is the orthogonal direct sum \( I' + J' \) of the subspaces \( I' \) and \( J' \). It follows from the fact that \( T \) is a continuous automorphism that \( H \) is also the direct sum \( I + J \).

Lemma 4.4. The subspaces \( I \) and \( J \) are well separated, i.e., there exists a constant \( k \) with \( 0 < k < 1 \) such that if \( u \in I, v \in J \) then \( |(u, v)| \leq k \|u\| \|v\| \) for any choice of the partitioning sets \( R \) and \( S \).

Proof. Let \( P \) and \( P' \) denote the orthogonal projections on the subspaces \( I' \) and \( J' \) respectively. The projections on \( I \) and \( J \) are given by
\[
E = T(P)T^{-1} \quad \text{and} \quad E' = T(P')T^{-1}
\]
from which it follows that each of them is bounded by the constant \( \|T\| \|T^{-1}\| \). It follows from Lemma 4.1 that if \( k \) is any constant such that
\[
((\|T\| \|T^{-1}\|)^2 - 1)/(\|T\| \|T^{-1}\|)^2 < 1
\]
then \( u \in I, v \in J \) implies \( |(u, v)| \leq k \|u\| \|v\| \).

Example 4.5. Let \( \{e_i\} \) be a Fischer-Riesz basis for the separable Hilbert space \( H \). If
\[
x = x_1 e_1 + \cdots + x_n e_n + \cdots, \quad y = y_1 e_1 + \cdots + y_n e_n + \cdots
\]
are any two elements in \( H \), let the product \( xy \) be defined by
\begin{equation}
(4.06) \quad xy = x_1 y_1 e_1 + \cdots + x_n y_n e_n + \cdots.
\end{equation}
Then it follows from inequalities (4.01) and (4.02) that
\[
\|xy\| \leq M(\|x_1 y_1\|^2 + \cdots + \|x_n y_n\|^2 + \cdots)
\leq M(\|x_1\|^2 + \cdots + \|x_n\|^2 + \cdots)(\|y_1\|^2 + \cdots + \|y_n\|^2 + \cdots)
= (M|m^2|)[m(\|x_1\|^2 + \cdots + \|x_n\|^2 + \cdots)][m(\|y_1\|^2 + \cdots + \|y_n\|^2 + \cdots)]
\leq (M|m^2|)^2 \|x\|^2 \|y\|^2.
\]
Consequently, we see that \( H \) becomes a Hilbert algebra \( A \) with this definition of product. Denote by \( R_i \) the set of all linear combinations
\[
x = x_1 e_1 + \cdots + x_n e_n + \cdots
\]
where the coefficient \( x_j \) is zero. Clearly, \( R_i \) is a maximal modular ideal. The algebra \( A \) is semisimple since the intersection of all the maximal modular ideals in the set \( \{R_i\}, (i = 1, 2, \ldots) \), is the zero ideal. The basis \( \{e_i\} \) is clearly a maximal set of minimal idempotents. Suppose that \( R \) is any maximal modular ideal of \( A \). Since \( R \) is closed, there exists an \( e_i \) which is not contained in \( R \). From this, it follows that \( R \) must be contained in \( R_i \) and, consequently, must coincide with \( R_i \). The minimal ideal \( Ae_i \) is complementary to \( R \). Thus \( A \) is a regular algebra. The sequence \( \{g_i\} \)
biorthogonal to \( \{e_i\} \) is the set of multiplicative elements associated with the set \( \{R_j\} \) of maximal modular ideals. Given that \( A \) is the direct sum \( I+J \) of two proper closed ideals, then \( I \) coincides with \( \{e_i\} \) (\( e_i \in I \)), and \( J \) coincides with \( \{e_i\} \) (\( e_i \in J \)). It follows from Lemma 4.4 that \( A \) is well separated. Inequalities (4.01) and (4.02) imply that \( A \) is homogeneous. Thus we see that \( A \) is an example of a homogeneous, semisimple, commutative well-separated regular algebra in which every proper closed ideal is contained in a maximal regular ideal.

Furthermore, the mapping \( T^{-1} \) of equation (4.04) is an isomorphism of \( A \) onto an \( H^* \)-algebra \( H \) generated by taking the orthonormal basis \( \{u_i\} \) as the selfadjoint minimal idempotents of \( H \). Consequently, every \( H \)-algebra \( A \) with a set \( \{e_i\} \) of minimal idempotents which form a Fischer-Riesz basis for the space \( A \) is isomorphic to an \( H^* \)-algebra. This result shows in particular that a homogeneous, well-separated, regular algebra \( A \) in which every proper closed ideal is contained in a maximal regular ideal is isomorphic to an \( H^* \)-algebra. When the isomorphism is also isometric, it is easy to see as well that the original algebra is actually an \( H^* \)-algebra and the isomorphism is a *-isomorphism.

We note that the definition of the multiplication can be framed somewhat differently. If \( x, y \in A \), then the product \( xy \) can be defined to be that unique element \( z \) in \( A \) such that

\[
(z, g_i) = (x, g_i)(y, g_i).
\]

A second product \( x \cdot y \) can be defined to be that unique element \( z' \) in \( A \) such that

\[
(z', e_i) = (x, e_i)(y, e_i).
\]

This last definition is equivalent to defining \( x \cdot y \) for

\[
x = x_1g_1 + \cdots + x_ng_n + \cdots,
\]
\[
y = y_1g_1 + \cdots + y_ng_n + \cdots
\]

to be given by

\[
x \cdot y = x_1y_1g_1 + \cdots + x_ny_ng_n + \cdots.
\]

A sequence \( \{x_n\} \) is admissible for the Fischer-Riesz basis \( \{e_i\} \) if and only if it is admissible for the system \( \{g_i\} \). Consequently, the mapping \( \pi \) defined by

\[
\pi(x_1e_1 + \cdots + x_ne_n + \cdots) = \bar{x}_1g_1 + \cdots + \bar{x}_ng_n + \cdots
\]

is a bijection on \( A \). Note that \( \pi \) is usually not an involution; however, if we denote \( \pi(x) \) by \( x^* \), then

\[
(x^* \cdot y) = (x, z^*) = (y, x^* \cdot z)
\]

so that \( A \) is an example of an adjoint algebra.

We briefly consider an algebra \( A \) which is a continuous automorphic image of an \( H^* \)-algebra \( H \) whose minimal idempotents \( \{g_i\} \) are uniformly bounded by \( B \). If \( T \) is the given automorphism, then \( T \) maps the set of all minimal selfadjoint
idempotents \{s_i\} of \(H\) onto the set of all minimal idempotents \(\{e_i\}\) of \(A\). It is easy to see that \(\{e_i\}\) is a Fischer-Riesz system so that all of the results on algebras with a Fischer-Riesz basis of minimal idempotents apply to \(A\). It should be clear now that if \(A\) is an algebra with a Fischer-Riesz basis of minimal idempotents, then \(A\) is a continuous isomorphic image of any \(H^*\)-algebra \(H\) whose minimal idempotents are uniformly bounded. This statement finishes the proof of Theorem 2.9 and completes our discussion of this particular kind of algebra.

We end with a counterexample:

**Example 4.6.** Let \(A\) be an \(H^*\)-algebra whose maximal set of selfadjoint minimal idempotents \(\{s_i\}\) has the property that \(\|s_k\|\) is equal to \(k\). It is easy to see that \(A\) is a well-separated, regular algebra in which every proper closed ideal is contained in a maximal regular ideal. Nevertheless, it is clear that there exists no continuous automorphism carrying a unitary basis \(\{u_i\}\) onto the set of minimal idempotents \(\{s_i\}\). Consequently, the set \(\{s_i\}\) is not a Fischer-Riesz basis and \(A\) is not homogeneous.

**References**