ON HOMEOMORPHISMS OF CERTAIN INFINITE DIMENSIONAL SPACES

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1. **Introduction.** All spaces concerned are taken to be separable metric. In this paper we prove various properties of homeomorphisms on l_2 and certain infinite product spaces, in particular, the Hilbert cube I^∞ and s (the countable infinite product of lines).

It has been shown in [5] and [6] by V. Klee that each homeomorphism on I^∞ (or on l_2) is isotopic to the identity mapping by means of into-homeomorphisms. He raised the question whether into-homeomorphisms can be replaced by self-homeomorphisms. Results in this paper give each of his questions a positive answer. We prove that any homeomorphism on a space such as I^∞, s, or l_2 is isotopic to the identity mapping. (Note that our definition of isotopy requires self-homeomorphisms. See 3.1.) In fact stronger theorems are obtained for homeomorphisms on spaces I^∞, s, and l_2. Namely, any homeomorphism on each of these spaces is stable. (For definition, see §4. In §4 we prove stability for homeomorphism on s and l_2. R. D. Anderson recently asserted the result for I^∞ [3].) It is easy to see (by a method of Alexander) that a homeomorphism on I^∞ (or s) is isotopic to the identity mapping if it is stable.

2. **Notation.** (1) If X is a space, by *a homeomorphism on* X (=self-homeomorphism) is meant a homeomorphism of X onto itself.

(2) If X is a space, by X^n is meant the finite product space $\prod_{i=1}^n X_i$, where $X_i = X$ and by X^∞ is meant the infinite product space $\prod_{i=1}^\infty X_i$ where $X_i = X$.

(3) J, J°, I, and I° will denote intervals $[-1, 1]$, $(-1, 1)$, $[0, 1]$, and $(0, 1)$ respectively.

(4) A mapping is a continuous function.

(5) "$\sim$" will mean "is *homeomorphic* to"; "$\sim^i$" will mean "is isotopic to."

(6) By "Hilbert cube" we mean the space J^∞ or I^∞ with metric $\rho(x, y) = \sum_{i \geq 1} |x_i - y_i|/2^i$. Hilbert space, l_2, is the space of all square summable sequences of real numbers with $d((x_i), (y_i)) = (\sum_{i=1}^\infty (x_i - y_i)^2)^{1/2}$. The space $(J^\circ)^\infty$ or $(I^\circ)^\infty$ is also denoted by s.

(7) e will always denote the identity mapping on the corresponding space.

(8) π_n and τ_n will denote the projecting functions of X^∞ onto X_n and X^n

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respectively; that is, if $x=(x_1, x_2, \ldots) \in X^\infty$, then $\pi_n(x)=x_n$ and $\tau_n(x) =(x_1, x_2, \ldots, x_n)$.

(9) $\varnothing$ = the empty set.

(10) Bd = Boundary, Int = Interior.

3. Isotopy theorems.

3.1. DEFINITIONS. (1) N = the set of all positive integers.

(2) For any $\alpha \subset N$, π_α will denote the projecting function of X^∞ onto $\prod_{i\in\alpha} X_i$; that is, if $x=(x_1, x_2, \ldots) \in X^\infty$, then $\pi_\alpha(x)=(x_i)_{i\in\alpha}$.

(3) For any $\alpha \subset N$, if h is a homeomorphism on $\prod_{i\in\alpha} X_i$, $\bar{h}$ will denote its natural extension on X^∞. More precisely, if $x \in X^\infty$, then $\bar{h}(x)$ is the point in X^∞ such that $\pi_\alpha(\bar{h}(x))=h(\pi_\alpha(x))$ and $\pi_i(\bar{h}(x))=\pi_i(x)$ for all $i \notin \alpha$.

(4) If h_1, h_0 are homeomorphisms on a space X, then h_1 is isotopic to h_0 if there is a mapping H of $X\times I$ onto X such that $H|_{X\times 1}=h_1$, $H|_{X\times 0}=h_0$ and for each $t\in I$, $H|_{X\times t}$ is a homeomorphism on X. In this case we say that $\{h_t=H|_{X\times t}\}_{t\in I}$ is an isotopy between h_1 and h_0.

(5) For any $\alpha \subset N$, a homeomorphism h on X^∞ is said to be fixed on the α coordinates if for each $x \in X^\infty$ and each $i \in \alpha$, $\pi_i(h(x))=\pi_i(x)$.

(6) If h_1, h_0 are homeomorphisms on X^∞ and $\alpha \subset N$, an isotopy $\{h_t\}_{t\in I}$ between h_1 and h_0 is said to be fixed on the α coordinates if each h_t is fixed on the α coordinates.

3.2. PROPERTY Φ. A space X satisfies property Φ if the homeomorphism g on X^∞ defined by $f(x_1, x_2, x_3, x_4, \ldots)=(x_2, x_1, x_3, x_4, \ldots)$ is isotopic to the identity mapping.

Let ϕ_n be the homeomorphism on $X_n \times X_{n+1}$ such that $\phi_n(x, y)=(y, x)$ and let $\bar{\phi}_n$ be the natural extension of ϕ_n on X^∞. X is said to have property Φ' if each $\bar{\phi}_n$ is isotopic to the identity mapping under an isotopy with the property that for $n>1$, the isotopy is fixed on the first $n-1$ coordinates.

LEMMA 3.1. *X satisfies property Φ if and only if X satisfies property Φ'.*

Proof. Obvious.

We shall prove several lemmas which will lead to the following theorem:

THEOREM 3.1. *A necessary and sufficient condition that each homeomorphism h on X^∞ is isotopic to the identity mapping is that X satisfies property Φ.*

Let X be a space satisfying property Φ (and hence Φ' by Lemma 3.1) and let h be any homeomorphism on X^∞. For each n, there is an isotopy $\{\phi_{n,t}\}_{t\in[(n-1)/n,\, n/(n+1)]}$ between $\bar{\phi}_n$ and e with the property that for any $n>1$ and any $t \in [(n-1)/n, n/(n+1)]$, $\phi_{n,t}$ leaves the first $n-1$ coordinates fixed.

For any $a \in X$ and any $n \in N$, define mappings $a^{(n)}$ and $\tilde{\pi}_n$ of X^∞ into itself as follows:

$$a^{(n)}(x_1, x_2, \ldots) = (x_1, \ldots, x_{n-1}, a, x_n, \ldots);$$
$$\tilde{\pi}_n(x_1, x_2, \ldots) = (x_1, \ldots, x_{n-1}, x_{n+1}, \ldots).$$

LEMMA 3.2. *If $P, P_i \in X^\infty$ such that $P_i \to P$, and for each i, $a_i \in X$, then $\tilde{\pi}_i(P_i) \to P$ and $a_i^{(i)}(P_i) \to P$.*

Proof. The lemma follows since for any fixed n, and for any $i>n$, $\pi_n(\tilde{\pi}_i(P_i)) = \pi_n(P_i) = \pi_n(a_i^{(i)}(P_i))$.

For $x \in X$, denote the function $x \to (\pi_n(x))^{(n)} h \tilde{\pi}_n(x)$ by $\tilde{h}_n$. The following two lemmas are evident.

LEMMA 3.3. *Each $\tilde{h}_n$ is a homeomorphism on X^∞ leaving the nth coordinate fixed.*

LEMMA 3.4. $\tilde{h}_{n+1} = \phi_n \tilde{h}_n \phi_n$.

We observe that from Lemma 3.4, it follows that for any n, $\tilde{h}_{n+1}$ is isotopic to $\tilde{h}_n$ by means of the isotopy $\{h_{n,t} = \phi_{n,t} \tilde{h}_n \phi_{n,t}\}_{t \in [(n-1)/n, n/(n+1)]}$.

LEMMA 3.5. *If $P, P_i \in X^\infty$ such that $P_i \to P$ and $\{f_i\}_{i \geqq 1}$ is a sequence of functions satisfying* (1) *each $f_i = \phi_{n,t}$ for some $t \in [(n-1)/n, n/(n+1)]$ and* (2) *for a fixed n, there are at most finitely many f_i such that $f_i = \phi_{n,t}$. Then $f_i(P_i) \to P$.*

Proof. The lemma follows since for any fixed n, there exists a large enough K_n such that $\pi_n(f_i(P_i)) = \pi_n(P_i)$ for all $i > K_n$.

LEMMA 3.6. *If $P_i, P \in X^\infty$ such that $P_i \to P$, then $\tilde{h}_i(P_i) \to h(P)$.*

Proof. By Lemma 3.2, $\tilde{\pi}_i(P_i) \to P$. Hence $h(\tilde{\pi}_i(P_i)) \to h(P)$. Applying Lemma 3.2 again, we get $(\pi_i(P_i))^{(i)} h \tilde{\pi}_i(P_i) \to h(P)$. But this means $\tilde{h}_i(P_i) \to h(P)$.

LEMMA 3.7. $\tilde{h}_1 \sim {}^i h$.

Proof. Define a function H of $X^\infty \times I$ onto X^∞ as follows: $H|_{X^\infty \times 1} = h$, $H|_{X^\infty \times t} = h_{n,t}$ where $t \in [(n-1)/n, n/(n+1)]$. (We recall that $\{h_{n,t} = \phi_{n,t} \tilde{h}_n \phi_{n,t}\}_{t \in [(n-1)/n, n/(n+1)]}$ is an isotopy between $\tilde{h}_{n+1}$ and $\tilde{h}_n$.) It suffices to show H is continuous on $X^\infty \times 1$. Let $\{(P_i, t_i)\}_{i \geqq 1}$ be a sequence of points in $X^\infty \times I$ such that $(P_i, t_i) \to (P, 1)$. We may assume $t_i < 1$ for all i. $H(P_i, t_i) = h_{n,t_i}(P_i) = \phi_{n,t_i} \tilde{h}_n \phi_{n,t_i}(P_i)$. Note that the sequence $\{\phi_{n,t_i}\}_{i \geqq 1}$ satisfies the conditions in Lemma 3.5, hence $\phi_{n,t_i}(P_i) \to P$. By Lemma 3.6, $\tilde{h}_n \phi_{n,t_i}(P) \to h(P)$. Apply Lemma 3.5 again, $\phi_{n,t_i} \tilde{h}_n \phi_{n,t_i}(P_i) \to h(P)$ and the lemma is proved.

Proof of Theorem 3.1. The necessity is obvious. We now show the sufficiency. By Lemma 3.3, $\tilde{h}_1$ is the natural extension of a homeomorphism $\tilde{g}_1$ on $\prod_{i>1} X_i$. We can repeat the same argument on $\prod_{i>1} X_i$ and show that $\tilde{g}_1$ can be isotopic to a homeomorphism $\tilde{g}_2$ with the property that $\tilde{g}_2$ is the natural extension of a homeomorphism $\tilde{f}_2$ on $\prod_{i>2} X_i$. This means $\tilde{h}_1$ can be isotopic to $\tilde{g}_2$ by means of an isotopy leaving the 1st coordinate fixed. Note that $\tilde{g}_2$ leaves the first two coordinates fixed. Iterating this process on $\prod_{i>2} X_i$, on $\prod_{i>3} X_i$, and so on, we see easily that h is isotopic to the identity mapping.

3.3. We proceed now to show that both J and J° satisfy property Φ. Lemmas in the following are stated merely for J; similar lemmas for J° can be stated.

Let (τ, θ) be the polar coordinate system on the plane. Define homeomorphisms f on J^2, β, γ on the unit disk D as follows:

$$\begin{aligned} f(r, \theta) &= (|r\cos\theta|, \theta) \text{ if } -\pi/4 \leqq \theta \leqq \pi/4 \text{ or } 3\pi/4 \leqq \theta \leqq 5\pi/4; \\ &= (|r\sin\theta|, \theta) \text{ if } \pi/4 \leqq \theta \leqq 3\pi/4 \text{ or } 5\pi/4 \leqq \theta \leqq 7\pi/4; \\ \beta(r, \theta) &= (r, \theta+\pi) \text{ and } \gamma(r, \theta) = (r, \theta+\pi/4). \end{aligned}$$

Clearly both β, γ are isotopic to e. Denote isotopies between β and e by $\{\beta_t\}_{t\in I}$, between γ and e by $\{\gamma_t\}_{t\in I}$.

LEMMA 3.8. *$F=f^{-1}\gamma f$ is a homeomorphism on J^2 such that* (1) *if $F(x, y)=(x', y')$, then $F(y, x)=(-x', y')$ and* (2) *$F \sim^i e$.*

Proof. We omit the straightforward proof of this lemma.

LEMMA 3.9. *If ω is the homeomorphism on J^2 such that $\omega(x, y)=(-x, -y)$, then $\omega \sim^i e$.*

Proof. $\omega=f^{-1}\beta f$ and $\{f^{-1}\beta_t f\}_{t\in I}$ is the necessary isotopy.

LEMMA 3.10. *If σ is the homeomorphism on J_1 such that $\sigma(x)=-x$, then $\bar{\sigma} \sim^i e$ on J^∞.*

Proof. For each n, define ω_n on $J_n\times J_{n+1}$ by $\omega_n(x, y)=(-x, -y)$ and let $\{\Psi_{n,t}\}_{t\in[(n-1)/n,n/(n+1)]}$ be an isotopy between ω_n and e on $J_n\times J_{n+1}$. Let

$$\bar{h}_n = \bar{\omega}_n\cdots\bar{\omega}_2\bar{\omega}_1.$$

Then $\bar{h}_1$ is isotopic to e on J^∞ by $\{h_{1,t}=\Psi_{1,t}\}_{t\in[0,1/2]}$ and for $n>1$, $\bar{h}_n$ is isotopic to $\bar{h}_{n-1}$ by $\{h_{n,t}=\bar{\Psi}_{n,t}\bar{h}_{n-1}\}_{t\in[(n-1)/n,n/(n+1)]}$. Now define a mapping H of $J^\infty\times I$ onto J^∞ by $H|_{J^\infty\times t}=h_{n,t}$ if $t\in[(n-1)/n, n/(n+1)]$ and $H|_{J^\infty\times 1}=\bar{\sigma}$.

THEOREM 3.2. *Any homeomorphism on the Hilbert cube is isotopic to the identity mapping.*

Proof. By virtue of Theorem 3.1, it suffices to show that J satisfies property Φ. Let g be the homeomorphism on J^∞ defined by

$$g(x_1, x_2, x_3, x_4, \ldots) = (x_2, x_1, x_3, x_4, \ldots),$$

and let F, σ be defined as before. Clearly $g=\bar{F}^{-1}\bar{\sigma}\bar{F}$. Then by Lemmas 3.8, 3.10, $g \sim^i e$.

Similarly we can show that J° satisfies property Φ, hence

THEOREM 3.3. *Any homeomorphism on s is isotopic to the identity mapping.*

THEOREM 3.4. *Any homeomorphism on l_2 is isotopic to the identity mapping.*

Proof. This is an immediate consequence of the fact $l_2\sim s$ [2] and of Theorem 3.3.

4. Stable homeomorphisms. A homeomorphism h on a space X is *stable* (in the sense of Brown-Gluck) if h can be written as a composition of finitely many homeomorphisms on X each of which is the identity on some open set in X. s will denote the space $(I^{\circ})^{\infty}$. K_1 will denote the set $\{x \in I^{\infty} : \pi_1(x) = 1\}$ and H will denote the space $[0, 2] \times \prod_{i>1} I_i$, where each $I_i = I$. Our main result is: Any homeomorphism on s or l_2 is stable. It is easy to see (as will be shown in Corollary 4.3) that (by means of Alexander's method which was originally used for n-cells) a homeomorphism on s is isotopic to the identity mapping if it is stable. For further discussion of stable homeomorphisms on manifolds, refer to Brown-Gluck [4].

THEOREM 4.1. $s \cup K_1 \sim s$.

THEOREM 4.2. *If K is a compact subset in s and h is a homeomorphism of K into s, then h can be extended to a stable homeomorphism $\tilde{h}$ on s.*

For the proof of Theorem 4.1, refer to [1]. A theorem like Theorem 4.2 was proved by Klee [7] in a somewhat different context (without stressing stability). Later on Theorem 4.2 was also proved by R. D. Anderson using Klee's method [1]. Note that in Anderson's paper, stability of the homeomorphism $\tilde{h}$ was not explicitly proved, but it was explicitly observed that stability can be easily achieved for the homeomorphisms considered there.

COROLLARY 4.1. *If $s' \sim s$, then any homeomorphism h' from a compact subset K' of s' into s' can be extended to a stable homeomorphism $\tilde{h}'$ on s'.*

Proof. Let f be a homeomorphism of s' onto s, and let $h = fh'f^{-1}$. h is a homeomorphism of $f(K')$ into s, hence can be extended to a stable homeomorphism $\tilde{h}$ on s. Write $\tilde{h} = f_n \cdots f_2 f_1$, where each f_i is a homeomorphism on s which is the identity on some open set in s. Then define

$$\tilde{h}' = f^{-1}\tilde{h}f = f^{-1}f_n \cdots f_2 f_1 f = (f^{-1}f_n f) \cdots (f^{-1}f_2 f)(f^{-1}f_1 f).$$

COROLLARY 4.2. *If h is a homeomorphism on $s \cup K_1$ and K is a compact subset in $s \cup K_1$, then there exists a stable homeomorphism f on $s \cup K_1$ such that fh is the identity on K.*

Proof. $h|_K$ is a homeomorphism of K into $s \cup K_1$, hence by Theorem 4.1 and Corollary 4.1, $h|_K$ can be extended to a stable homeomorphism g on $s \cup K_1$. Then let $f = g^{-1}$.

LEMMA 4.1. *If X, Y are spaces such that $X \sim Y$, then every homeomorphism on X is stable if and only if every homeomorphism on Y is stable.*

Proof. Obvious, by means of the method used to prove Corollary 4.1.

LEMMA 4.2. *If for each i, $i = 1, 2, \ldots, n$, h_i is a homeomorphism on X which is isotopic to the identity mapping, then $h = h_n \cdots h_2 h_1$ is a homeomorphism on X such that h is isotopic to the identity mapping.*

Proof. Obvious.

THEOREM 4.3. *Any homeomorphism h on s is stable.*

Proof. By virtue of Theorem 4.1 and Lemma 4.1, it suffices to show that any homeomorphism on $s \cup K_1$ is stable. By Corollary 4.2, there is a stable homeomorphism f on $s \cup K_1$ such that fh is the identity on K_1. Hence there exists an open set V in $s \cup K_1$ and a real number r such that $\sup \{\pi_1(V \cup fh(V))\} < r < 1$. Let φ be the extension of fh onto $H' = s \cup [1, 2] \times \prod_{i>1} I_i$ by taking φ as the identity outside of $s \cup K_1$. Let α be a homeomorphism on $[0, 2]$ such that α is the identity on $[0, r]$ and $\alpha(1) = 3/2$. Define a homeomorphism g on H by $g(x_1, x_2, \ldots) = (\alpha(x_1), x_2, \ldots)$. Then $\theta = g^{-1}\varphi g|_{s \cup K_1}$ is a homeomorphism on $s \cup K_1$. Clearly θ is the identity on some neighborhood of K_1 and $\theta^{-1}(fh)$ is the identity on V. But $fh = \theta[\theta^{-1}(fh)]$, hence $h = f^{-1}\theta[\theta^{-1}(fh)]$, a finite composition of stable homeomorphisms. Therefore h is stable.

THEOREM 4.4. *Any homeomorphism on l_2 is stable.*

This is an immediate consequence of the fact that $l_2 \sim s$ [2] and of Lemma 4.1.

COROLLARY 4.3. *Any homeomorphism h on s is isotopic to the identity mapping.*

Proof. h is stable by Theorem 4.3. Hence, by Lemma 4.2, it suffices to prove the theorem for the case that h leaves some open set V fixed. We now use Alexander's method applied to s. For some large number n, there is an open set W in $(I^\circ)^{n+1}$ such that

(1) $W = \prod_{i=1}^{n} (a_i, b_i) \times (a_{n+1}, 1)$ where for each $i \leq n$, $0 < a_i < b_i < 1$ and $0 < a_{n+1} < 1$ and

(2) $W \times \prod_{i>n+1} I_i^\circ \subset V$.

Let $\overline{W}$ be the closure of W in I^{n+1}, $\mathrm{Int}(\overline{W})$ the interior of $\overline{W}$ in I^{n+1} and let $0 = (0, 0, \ldots) \in I^{n+1}$. There exists a positive number K such that $[0, 1/K]^{n+1} \cap \overline{W} = \varnothing$. For each $x = t/K \in [0, 1/K]$, let $Q_t = [0, x]^{n+1}$. Let $\mathrm{Bd}(\overline{W})$, $\mathrm{Bd}(Q_t)$ denote the boundaries of $\overline{W}$ and Q_t in I^{n+1} respectively. Evidently there is a mapping H of $I^{n+1} \times I$ onto I^{n+1} such that:

(1) $g_1 = H|_{I^{n+1} \times 1}$ is the identity mapping on I^{n+1}.

(2) For each $0 < t \leq 1$, $g_t = H|_{I^{n+1} \times t}$ is a homeomorphism on I^{n+1} such that $g_t(I^{n+1} \backslash \mathrm{Int}(\overline{W})) = Q_t$ for $0 < t \leq \frac{1}{2}$.

(3) $g_t(0) = 0$ for all $t \in I$ and $H|_{I^{n+1} \times 0}(I^{n+1} \backslash \mathrm{Int}(\overline{W})) = 0$.

Now the desired mapping F from $s \times I$ onto s is defined as follows: $F|_{s \times t} = \bar{g}_t h \bar{g}_t^{-1}$ for $0 < t \leq 1$ and $F|_{s \times 0} = e$ on s.

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