

ON HOMEOMORPHISMS OF CERTAIN INFINITE DIMENSIONAL SPACES

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1. Introduction. All spaces concerned are taken to be separable metric. In this paper we prove various properties of homeomorphisms on l_2 and certain infinite product spaces, in particular, the Hilbert cube I^∞ and s (the countable infinite product of lines).

It has been shown in [5] and [6] by V. Klee that each homeomorphism on I^∞ (or on l_2) is isotopic to the identity mapping by means of into-homeomorphisms. He raised the question whether into-homeomorphisms can be replaced by self-homeomorphisms. Results in this paper give each of his questions a positive answer. We prove that any homeomorphism on a space such as I^∞ , s , or l_2 is isotopic to the identity mapping. (Note that our definition of isotopy requires self-homeomorphisms. See 3.1.) In fact stronger theorems are obtained for homeomorphisms on spaces I^∞ , s , and l_2 . Namely, any homeomorphism on each of these spaces is stable. (For definition, see §4. In §4 we prove stability for homeomorphism on s and l_2 . R. D. Anderson recently asserted the result for I^∞ [3].) It is easy to see (by a method of Alexander) that a homeomorphism on I^∞ (or s) is isotopic to the identity mapping if it is stable.

2. Notation. (1) If X is a space, by a *homeomorphism on X* (=self-homeomorphism) is meant a homeomorphism of X onto itself.

(2) If X is a space, by X^n is meant the finite product space $\prod_{i=1}^n X_i$, where $X_i = X$ and by X^∞ is meant the infinite product space $\prod_{i=1}^\infty X_i$ where $X_i = X$.

(3) J , J° , I , and I° will denote intervals $[-1, 1]$, $(-1, 1)$, $[0, 1]$, and $(0, 1)$ respectively.

(4) A mapping is a continuous function.

(5) " \sim " will mean "is *homeomorphic* to"; " \sim^t " will mean "is isotopic to."

(6) By "Hilbert cube" we mean the space J^∞ or I^∞ with metric $\rho(x, y) = \sum_{i \geq 1} |x_i - y_i|/2^i$. Hilbert space, l_2 , is the space of all square summable sequences of real numbers with $d((x_i), (y_i)) = (\sum_{i=1}^\infty (x_i - y_i)^2)^{1/2}$. The space $(J^\circ)^\infty$ or $(I^\circ)^\infty$ is also denoted by s .

(7) e will always denote the identity mapping on the corresponding space.

(8) π_n and τ_n will denote the projecting functions of X^∞ onto X_n and X^n

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respectively; that is, if $x=(x_1, x_2, \dots) \in X^\infty$, then $\pi_n(x)=x_n$ and $\tau_n(x)=(x_1, x_2, \dots, x_n)$.

(9) \emptyset = the empty set.

(10) Bd = Boundary, Int = Interior.

3. Isotopy theorems.

3.1. DEFINITIONS. (1) N = the set of all positive integers.

(2) For any $\alpha \subset N$, π_α will denote the projecting function of X^∞ onto $\prod_{i \in \alpha} X_i$; that is, if $x=(x_1, x_2, \dots) \in X^\infty$, then $\pi_\alpha(x)=(x_i)_{i \in \alpha}$.

(3) For any $\alpha \subset N$, if h is a homeomorphism on $\prod_{i \in \alpha} X_i$, \bar{h} will denote its natural extension on X^∞ . More precisely, if $x \in X^\infty$, then $\bar{h}(x)$ is the point in X^∞ such that $\pi_\alpha(\bar{h}(x))=h(\pi_\alpha(x))$ and $\pi_i(\bar{h}(x))=\pi_i(x)$ for all $i \notin \alpha$.

(4) If h_1, h_0 are homeomorphisms on a space X , then h_1 is isotopic to h_0 if there is a mapping H of $X \times I$ onto X such that $H|_{X \times 1}=h_1$, $H|_{X \times 0}=h_0$ and for each $t \in I$, $H|_{X \times t}$ is a homeomorphism on X . In this case we say that $\{h_t=H|_{X \times t}\}_{t \in I}$ is an isotopy between h_1 and h_0 .

(5) For any $\alpha \subset N$, a homeomorphism h on X^∞ is said to be fixed on the α coordinates if for each $x \in X^\infty$ and each $i \in \alpha$, $\pi_i(h(x))=\pi_i(x)$.

(6) If h_1, h_0 are homeomorphisms on X^∞ and $\alpha \subset N$, an isotopy $\{h_t\}_{t \in I}$ between h_1 and h_0 is said to be fixed on the α coordinates if each h_t is fixed on the α coordinates.

3.2. PROPERTY Φ . A space X satisfies property Φ if the homeomorphism g on X^∞ defined by $f(x_1, x_2, x_3, x_4, \dots)=(x_2, x_1, x_3, x_4, \dots)$ is isotopic to the identity mapping.

Let ϕ_n be the homeomorphism on $X_n \times X_{n+1}$ such that $\phi_n(x, y)=(y, x)$ and let $\bar{\phi}_n$ be the natural extension of ϕ_n on X^∞ . X is said to have property Φ' if each $\bar{\phi}_n$ is isotopic to the identity mapping under an isotopy with the property that for $n > 1$, the isotopy is fixed on the first $n-1$ coordinates.

LEMMA 3.1. X satisfies property Φ if and only if X satisfies property Φ' .

Proof. Obvious.

We shall prove several lemmas which will lead to the following theorem:

THEOREM 3.1. A necessary and sufficient condition that each homeomorphism h on X^∞ is isotopic to the identity mapping is that X satisfies property Φ .

Let X be a space satisfying property Φ (and hence Φ' by Lemma 3.1) and let h be any homeomorphism on X^∞ . For each n , there is an isotopy $\{\phi_{n,t}\}_{t \in [(n-1)/n, n/(n+1)]}$ between $\bar{\phi}_n$ and e with the property that for any $n > 1$ and any $t \in [(n-1)/n, n/(n+1)]$, $\phi_{n,t}$ leaves the first $n-1$ coordinates fixed.

For any $a \in X$ and any $n \in N$, define mappings $a^{(n)}$ and $\tilde{\pi}_n$ of X^∞ into itself as follows:

$$a^{(n)}(x_1, x_2, \dots) = (x_1, \dots, x_{n-1}, a, x_n, \dots);$$

$$\tilde{\pi}_n(x_1, x_2, \dots) = (x_1, \dots, x_{n-1}, x_{n+1}, \dots).$$

LEMMA 3.2. *If $P, P_i \in X^\infty$ such that $P_i \rightarrow P$, and for each $i, a_i \in X$, then $\tilde{\pi}_i(P_i) \rightarrow P$ and $a_i^{(i)}(P_i) \rightarrow P$.*

Proof. The lemma follows since for any fixed n , and for any $i > n$, $\pi_n(\tilde{\pi}_i(P_i)) = \pi_n(P_i) = \pi_n(a_i^{(i)}(P_i))$.

For $x \in X$, denote the function $x \rightarrow (\pi_n(x))^{(n)}h\tilde{\pi}_n(x)$ by \tilde{h}_n . The following two lemmas are evident.

LEMMA 3.3. *Each \tilde{h}_n is a homeomorphism on X^∞ leaving the n th coordinate fixed.*

LEMMA 3.4. $\tilde{h}_{n+1} = \phi_n \tilde{h}_n \phi_n$.

We observe that from Lemma 3.4, it follows that for any n , \tilde{h}_{n+1} is isotopic to \tilde{h}_n by means of the isotopy $\{h_{n,t} = \phi_{n,t} \tilde{h}_n \phi_{n,t}\}_{t \in [(n-1)/n, n/(n+1)]}$.

LEMMA 3.5. *If $P, P_i \in X^\infty$ such that $P_i \rightarrow P$ and $\{f_i\}_{i \geq 1}$ is a sequence of functions satisfying (1) each $f_i = \phi_{n,t}$ for some $t \in [(n-1)/n, n/(n+1)]$ and (2) for a fixed n , there are at most finitely many f_i such that $f_i = \phi_{n,t}$. Then $f_i(P_i) \rightarrow P$.*

Proof. The lemma follows since for any fixed n , there exists a large enough K_n such that $\pi_n(f_i(P_i)) = \pi_n(P_i)$ for all $i > K_n$.

LEMMA 3.6. *If $P_i, P \in X^\infty$ such that $P_i \rightarrow P$, then $\tilde{h}_i(P_i) \rightarrow h(P)$.*

Proof. By Lemma 3.2, $\tilde{\pi}_i(P_i) \rightarrow P$. Hence $h(\tilde{\pi}_i(P_i)) \rightarrow h(P)$. Applying Lemma 3.2 again, we get $(\pi_i(P_i))^{(i)}h\tilde{\pi}_i(P_i) \rightarrow h(P)$. But this means $\tilde{h}_i(P_i) \rightarrow h(P)$.

LEMMA 3.7. $\tilde{h}_1 \sim {}^4h$.

Proof. Define a function H of $X^\infty \times I$ onto X^∞ as follows: $H|_{X^\infty \times 1} = h$, $H|_{X^\infty \times t} = h_{n,t}$ where $t \in [(n-1)/n, n/(n+1)]$. (We recall that $\{h_{n,t} = \phi_{n,t} \tilde{h}_n \phi_{n,t}\}_{t \in [(n-1)/n, n/(n+1)]}$ is an isotopy between \tilde{h}_{n+1} and \tilde{h}_n .) It suffices to show H is continuous on $X^\infty \times 1$. Let $\{(P_i, t_i)\}_{i \geq 1}$ be a sequence of points in $X^\infty \times I$ such that $(P_i, t_i) \rightarrow (P, 1)$. We may assume $t_i < 1$ for all i . $H(P_i, t_i) = h_{n,t_i}(P_i) = \phi_{n,t_i} \tilde{h}_n \phi_{n,t_i}(P_i)$. Note that the sequence $\{\phi_{n,t_i}\}_{i \geq 1}$ satisfies the conditions in Lemma 3.5, hence $\phi_{n,t_i}(P_i) \rightarrow P$. By Lemma 3.6, $\tilde{h}_n \phi_{n,t_i}(P) \rightarrow h(P)$. Apply Lemma 3.5 again, $\phi_{n,t_i} \tilde{h}_n \phi_{n,t_i}(P_i) \rightarrow h(P)$ and the lemma is proved.

Proof of Theorem 3.1. The necessity is obvious. We now show the sufficiency. By Lemma 3.3, \tilde{h}_1 is the natural extension of a homeomorphism \tilde{g}_1 on $\prod_{i>1} X_i$. We can repeat the same argument on $\prod_{i>1} X_i$ and show that \tilde{g}_1 can be isotopic to a homeomorphism \tilde{g}_2 with the property that \tilde{g}_2 is the natural extension of a homeomorphism \tilde{f}_2 on $\prod_{i>2} X_i$. This means \tilde{h}_1 can be isotopic to \tilde{g}_2 by means of an isotopy leaving the 1st coordinate fixed. Note that \tilde{g}_2 leaves the first two coordinates fixed. Iterating this process on $\prod_{i>2} X_i$, on $\prod_{i>3} X_i$, and so on, we see easily that h is isotopic to the identity mapping.

3.3. We proceed now to show that both J and J° satisfy property Φ . Lemmas in the following are stated merely for J ; similar lemmas for J° can be stated.

Let (r, θ) be the polar coordinate system on the plane. Define homeomorphisms f on J^2 , β, γ on the unit disk D as follows:

$$\begin{aligned} f(r, \theta) &= (|r \cos \theta|, \theta) \text{ if } -\pi/4 \leq \theta \leq \pi/4 \text{ or } 3\pi/4 \leq \theta \leq 5\pi/4; \\ &= (|r \sin \theta|, \theta) \text{ if } \pi/4 \leq \theta \leq 3\pi/4 \text{ or } 5\pi/4 \leq \theta \leq 7\pi/4; \\ \beta(r, \theta) &= (r, \theta + \pi) \text{ and } \gamma(r, \theta) = (r, \theta + \pi/4). \end{aligned}$$

Clearly both β, γ are isotopic to e . Denote isotopies between β and e by $\{\beta_t\}_{t \in I}$, between γ and e by $\{\gamma_t\}_{t \in I}$.

LEMMA 3.8. $F=f^{-1}\gamma f$ is a homeomorphism on J^2 such that (1) if $F(x, y)=(x', y')$, then $F(y, x)=(-x', y')$ and (2) $F \sim^t e$.

Proof. We omit the straightforward proof of this lemma.

LEMMA 3.9. If ω is the homeomorphism on J^2 such that $\omega(x, y)=(-x, -y)$, then $\omega \sim^t e$.

Proof. $\omega=f^{-1}\beta f$ and $\{f^{-1}\beta_t f\}_{t \in I}$ is the necessary isotopy.

LEMMA 3.10. If σ is the homeomorphism on J_1 such that $\sigma(x)=-x$, then $\bar{\sigma} \sim^t e$ on J^∞ .

Proof. For each n , define ω_n on $J_n \times J_{n+1}$ by $\omega_n(x, y)=(-x, -y)$ and let $\{\Psi_{n,t}\}_{t \in [(n-1)/n, n/(n+1)]}$ be an isotopy between ω_n and e on $J_n \times J_{n+1}$. Let

$$\bar{h}_n = \bar{\omega}_n \cdot \dots \cdot \bar{\omega}_2 \bar{\omega}_1.$$

Then \bar{h}_1 is isotopic to e on J^∞ by $\{h_{1,t}=\Psi_{1,t}\}_{t \in [0, 1/2]}$ and for $n > 1$, \bar{h}_n is isotopic to \bar{h}_{n-1} by $\{h_{n,t}=\Psi_{n,t}\bar{h}_{n-1}\}_{t \in [(n-1)/n, n/(n+1)]}$. Now define a mapping H of $J^\infty \times I$ onto J^∞ by $H|_{J^\infty \times t} = h_{n,t}$ if $t \in [(n-1)/n, n/(n+1)]$ and $H|_{J^\infty \times 1} = \bar{\sigma}$.

THEOREM 3.2. Any homeomorphism on the Hilbert cube is isotopic to the identity mapping.

Proof. By virtue of Theorem 3.1, it suffices to show that J satisfies property Φ . Let g be the homeomorphism on J^∞ defined by

$$g(x_1, x_2, x_3, x_4, \dots) = (x_2, x_1, x_3, x_4, \dots),$$

and let F, σ be defined as before. Clearly $g = \bar{F}^{-1}\bar{\sigma}\bar{F}$. Then by Lemmas 3.8, 3.10, $g \sim^t e$.

Similarly we can show that J° satisfies property Φ , hence

THEOREM 3.3. Any homeomorphism on s is isotopic to the identity mapping.

THEOREM 3.4. Any homeomorphism on l_2 is isotopic to the identity mapping.

Proof. This is an immediate consequence of the fact $l_2 \sim s$ [2] and of Theorem 3.3.

4. Stable homeomorphisms. A homeomorphism h on a space X is *stable* (in the sense of Brown-Gluck) if h can be written as a composition of finitely many homeomorphisms on X each of which is the identity on some open set in X . s will denote the space $(I^\circ)^\infty$. K_1 will denote the set $\{x \in I^\infty : \pi_1(x) = 1\}$ and H will denote the space $[0, 2] \times \prod_{i>1} I_i$, where each $I_i = I$. Our main result is: Any homeomorphism on s or I_2 is stable. It is easy to see (as will be shown in Corollary 4.3) that (by means of Alexander's method which was originally used for n -cells) a homeomorphism on s is isotopic to the identity mapping if it is stable. For further discussion of stable homeomorphisms on manifolds, refer to Brown-Gluck [4].

THEOREM 4.1. $s \cup K_1 \sim s$.

THEOREM 4.2. *If K is a compact subset in s and h is a homeomorphism of K into s , then h can be extended to a stable homeomorphism \tilde{h} on s .*

For the proof of Theorem 4.1, refer to [1]. A theorem like Theorem 4.2 was proved by Klee [7] in a somewhat different context (without stressing stability). Later on Theorem 4.2 was also proved by R. D. Anderson using Klee's method [1]. Note that in Anderson's paper, stability of the homeomorphism \tilde{h} was not explicitly proved, but it was explicitly observed that stability can be easily achieved for the homeomorphisms considered there.

COROLLARY 4.1. *If $s' \sim s$, then any homeomorphism h' from a compact subset K' of s' into s' can be extended to a stable homeomorphism \tilde{h}' on s' .*

Proof. Let f be a homeomorphism of s' onto s , and let $h = fh'f^{-1}$. h is a homeomorphism of $f(K')$ into s , hence can be extended to a stable homeomorphism \tilde{h} on s . Write $\tilde{h} = f_n \cdots f_2 f_1$, where each f_i is a homeomorphism on s which is the identity on some open set in s . Then define

$$\tilde{h}' = f^{-1}\tilde{h}f = f^{-1}f_n \cdots f_2 f_1 f = (f^{-1}f_n f) \cdots (f^{-1}f_2 f)(f^{-1}f_1 f).$$

COROLLARY 4.2. *If h is a homeomorphism on $s \cup K_1$ and K is a compact subset in $s \cup K_1$, then there exists a stable homeomorphism f on $s \cup K_1$ such that fh is the identity on K .*

Proof. $h|_K$ is a homeomorphism of K into $s \cup K_1$, hence by Theorem 4.1 and Corollary 4.1, $h|_K$ can be extended to a stable homeomorphism g on $s \cup K_1$. Then let $f = g^{-1}$.

LEMMA 4.1. *If X, Y are spaces such that $X \sim Y$, then every homeomorphism on X is stable if and only if every homeomorphism on Y is stable.*

Proof. Obvious, by means of the method used to prove Corollary 4.1.

LEMMA 4.2. *If for each $i, i = 1, 2, \dots, n, h_i$ is a homeomorphism on X which is isotopic to the identity mapping, then $h = h_n \cdots h_2 h_1$ is a homeomorphism on X such that h is isotopic to the identity mapping.*

Proof. Obvious.

THEOREM 4.3. *Any homeomorphism h on s is stable.*

Proof. By virtue of Theorem 4.1 and Lemma 4.1, it suffices to show that any homeomorphism on $s \cup K_1$ is stable. By Corollary 4.2, there is a stable homeomorphism f on $s \cup K_1$ such that fh is the identity on K_1 . Hence there exists an open set V in $s \cup K_1$ and a real number r such that $\sup \{\pi_1(V \cup fh(V))\} < r < 1$. Let φ be the extension of fh onto $H' = s \cup [1, 2] \times \prod_{i>1} I_i$ by taking φ as the identity outside of $s \cup K_1$. Let α be a homeomorphism on $[0, 2]$ such that α is the identity on $[0, r]$ and $\alpha(1) = 3/2$. Define a homeomorphism g on H by $g(x_1, x_2, \dots) = (\alpha(x_1), x_2, \dots)$. Then $\theta = g^{-1}\varphi g|_{s \cup K_1}$ is a homeomorphism on $s \cup K_1$. Clearly θ is the identity on some neighborhood of K_1 and $\theta^{-1}(fh)$ is the identity on V . But $fh = \theta[\theta^{-1}(fh)]$, hence $h = f^{-1}\theta[\theta^{-1}(fh)]$, a finite composition of stable homeomorphisms. Therefore h is stable.

THEOREM 4.4. *Any homeomorphism on l_2 is stable.*

This is an immediate consequence of the fact that $l_2 \sim s [2]$ and of Lemma 4.1.

COROLLARY 4.3. *Any homeomorphism h on s is isotopic to the identity mapping.*

Proof. h is stable by Theorem 4.3. Hence, by Lemma 4.2, it suffices to prove the theorem for the case that h leaves some open set V fixed. We now use Alexander's method applied to s . For some large number n , there is an open set W in $(I^\circ)^{n+1}$ such that

(1) $W = \prod_{i=1}^n (a_i, b_i) \times (a_{n+1}, 1)$ where for each $i \leq n$, $0 < a_i < b_i < 1$ and $0 < a_{n+1} < 1$ and

(2) $W \times \prod_{i>n+1} I_i^\circ \subset V$.

Let \bar{W} be the closure of W in I^{n+1} , $\text{Int}(\bar{W})$ the interior of \bar{W} in I^{n+1} and let $0 = (0, 0, \dots) \in I^{n+1}$. There exists a positive number K such that $[0, 1/K]^{n+1} \cap \bar{W} = \emptyset$. For each $x = t/K \in [0, 1/K]$, let $Q_t = [0, x]^{n+1}$. Let $\text{Bd}(\bar{W})$, $\text{Bd}(Q_t)$ denote the boundaries of \bar{W} and Q_t in I^{n+1} respectively. Evidently there is a mapping H of $I^{n+1} \times I$ onto I^{n+1} such that:

(1) $g_1 = H|_{I^{n+1} \times 1}$ is the identity mapping on I^{n+1} .

(2) For each $0 < t \leq 1$, $g_t = H|_{I^{n+1} \times t}$ is a homeomorphism on I^{n+1} such that $g_t(I^{n+1} \setminus \text{Int}(\bar{W})) = Q_t$ for $0 < t \leq \frac{1}{2}$.

(3) $g_t(0) = 0$ for all $t \in I$ and $H|_{I^{n+1} \times 0}(I^{n+1} \setminus \text{Int}(\bar{W})) = 0$.

Now the desired mapping F from $s \times I$ onto s is defined as follows: $F|_{s \times t} = \bar{g}_t h \bar{g}_t^{-1}$ for $0 < t \leq 1$ and $F|_{s \times 0} = e$ on s .

BIBLIOGRAPHY

1. R. D. Anderson, *Topological properties of the Hilbert cube and the infinite product of open intervals*, Trans. Amer. Math. Soc. **126** (1967), 200-216.
2. ———, *Hilbert space is homeomorphic to the countable infinite product of lines*, Bull. Amer. Math. Soc. **72** (1966), 515-519.
3. ———, *On extending homeomorphisms on the Hilbert cube*, Abstract 634-56, Notices Amer. Math. Soc. **13** (1966), 375.

4. M. Brown and H. Gluck, *Stable structures on manifolds. I*, Ann. of Math. (2) **79** (1964), 1–17.
5. V. Klee, *Convex bodies and periodic homeomorphisms in Hilbert space*, Trans. Amer. Math. Soc. **74** (1953), 36.
6. ———, *Homogeneity of infinite-dimensional parallelotopes*, Ann. of Math. (2) **66** (1957), 454–460.
7. ———, *Some topological properties of convex sets*, Trans. Amer. Math. Soc. **78** (1955), 30–45.
8. ———, *A note on topological properties of normed linear spaces*, Proc. Amer. Math. Soc. **7** (1956), 673–674.

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