SEMISIMPLE MAXIMAL QUOTIENT RINGS

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Notation and Introduction. R denotes an arbitrary associative ring. A right R-module A over R will be denoted A_R. B_R is a large submodule of A_R (A_R is an essential extension of B_R), if B_R is a submodule of A_R having nonzero intersection with every nonzero submodule of A_R. A right ideal I of R is a large right ideal, if I_R is a large submodule of R_R.

Given A_R, $Z(A_R)$ is the singular submodule of A_R [9], which consists of all those elements of A whose annihilators in R are large right ideals.

Following Johnson [9], Q is a right quotient ring of R if Q is a ring with identity containing R as a subring (the identity of Q is the identity of R if R has one) and R_R is a large submodule of Q_R.

The quotient rings considered by Goldie in [6], [7] will be called classical quotient rings. Q is a classical right quotient of R if every regular element (nonzero divisor) of R is a unit in Q and every element of Q is of the form ab^{-1}, $a, b \in R$, b regular in R. In general, a ring R need not possess a classical right quotient ring.

Goldie [7], has given necessary and sufficient conditions that a ring possess a classical right quotient ring which is semisimple. Here semisimple means semisimple with minimum condition [8].

This paper is concerned with the question of characterizing those rings which have a semisimple maximal right quotient ring [4], [9], [10], [11] and in this case generalizing some simple well-known results about commutative integral domains, their quotient rings and modules over these domains. Johnson [9] has shown that R has a regular maximal right quotient ring Q if and only if $Z(R_R)=0$, where Q is a regular ring [13] if every finitely generated right (left) ideal of Q is generated by an idempotent. In this case Q_R is injective [3] as a right R-module, hence the injective hull of R [2].

A ring R has a semisimple maximal right quotient ring Q if and only if $Z(R_R)=0$ and dim R_R is finite, where a right R-module M is of finite dimension if every direct sum of submodules of M has only finitely many nonzero summands. This is the main result of §1. In addition another characterization is given for rings which possess a semisimple classical right quotient ring, namely, R has a semisimple classical right quotient ring if and only if $Z(R_R)=0$ and if I is a large right ideal of R, then there is an element $a \in I$ such that aR is a large right ideal of R.

If R has a semisimple classical right quotient ring Q, then it is known [3], that

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Q is the maximal right quotient ring of R. The converse is not valid, since there are rings with or without identity that have a semisimple maximal right quotient ring Q and Q is not the classical right quotient ring of R. Let Q be the ring of $n \times n$ matrices over a division ring Δ and R the set of upper triangular (strictly upper triangular) matrices of Q. It is easily verified that Q is a right quotient ring of R but Q is not a classical right quotient ring of R. Since Q is semisimple Q is the maximal right quotient ring of R.

It is also shown in §1 that if R has a semisimple maximal right quotient ring Q, then $Z(A/Z(A))=0$ for every right R-module A. This generalizes the analogous result if R is a commutative integral domain, since then $Z(A_R)$ is the torsion subgroup of A.

In §2 rings with identity are considered.

In this case the following generalizations of results known [1] when R is a commutative integral domain hold, thus extending some of the results of Gentile [5] also.

1. R has a semisimple maximal right quotient ring Q if and only if $A \to A \otimes_R Q$ has $Z(A_R)$ for its kernel for every unitary right R-module A.

If R has a semisimple maximal right quotient ring Q, then

2. ${}_RQ$ is flat [1] as a left R-module.
3. Every unitary left Q-module is flat as a left R-module.
4. If $Z(A_R)=0$, then $0 \to A \to A \otimes_R Q$ is the injective hull of A, a unitary R-module.
5. $\mathrm{Tor}_1^R(A, Q/R) \cong Z(A_R)$ for every unitary right R-module A.

Another result with weaker hypothesis is valid.

Any direct sum of injective right R-modules, each with zero singular submodule is injective if dim R_R is finite.

The following generalizes a result of Matlis [12]. If H_R is an epimorphic image of an injective right R-module E_R and $Z(H/Z(H))=0$, then $Z(H))$ is a direct summand of H with complementary summand injective. The proof given here is simpler in that it does not appeal to any quotient ring of R as was done in Matlis [12, Theorem 1.1] when R is a commutative integral domain. Also Proposition 2.1, Proposition 2.2, and Proposition 2.4 of [12] can be generalized to a noncommutative ring R which has a semisimple maximal right quotient ring utilizing identical proofs.

1. **Arbitrary rings.**

DEFINITION 1.1. If M is a right R-module, then the set of all large submodules of M is denoted by $L(M_R)$.

It is useful to recall the following results, which are essentially in [9].

PROPOSITION 1.2.
1. *If $A, B \in L(M_R)$, then $A+B \in L(M_R)$ and $A \cap B \in L(M_R)$.*
2. *If $A \in L(M_R)$ and B a submodule of M containing A, then $B \in L(M_R)$.*

3. *If A is a submodule of M, then there exists a submodule B of M maximal with respect to the property that $A \cap B=0$ and consequently $A+B \in L(M_R)$.*

4. *If $f \in \mathrm{Hom}_R(M, N)$ and $A \in L(N_R)$, then $f^{-1}(A)=\{x \in M \mid f(x) \in A\} \in L(M_R)$.*

COROLLARY. *If $A_1, \ldots, A_n \in L(A_R)$ and $x_1, \ldots, x_n \in A$, then $I=\{r \in R \mid x_i r \in A_i$ for all $i\} \in L(R_R)$.*

Proof. $\{r \in R \mid x_i r \in A_i\}=I_i$ for each i is the counter-image of A_i by the right R homomorphism from R into A given by left multiplication by x_i, hence I is the intersection of finitely many large right ideals of R, so $I \in L(R_R)$.

Goldie [7] calls a right R-module M of finite dimension, dim M_R finite, if every direct sum of nonzero submodules of M has only a finite number of direct summands which are nonzero.

If M is a right R-module and $x_1, \ldots, x_n \in M$, then $[x_1, \ldots, x_n]$ will denote the submodule of M generated by $\{x_1, \ldots, x_n\}$. For $x \in M$, $xR=\{xr \mid r \in R\}$. If R is a ring with identity and M a unitary R-module, then clearly $[x]=xR$. A module M_R will be called regular if for $0 \neq x \in M$, $xR \neq 0$. Clearly if R is a ring with identity every unitary R-module is regular. Also if R is arbitrary then M_R is regular if $Z(M_R)=0$.

THEOREM 1.3. *Let M_R be a right R-module, and consider the following conditions.*

(a) *dim M_R is finite.*

(b) *If $K \in L(M_R)$, there are $x_1, \ldots, x_n \in K$ such that $[x_1, \ldots, x_n] \in L(M_R)$.*

(c) *If K is a submodule of M, then there are $x_1, \ldots, x_n \in K$ such that $[x_1, \ldots, x_n] \in L(K_R)$.*

(b*) *If $K \in L(M_R)$, there are $x_1, \ldots, x_n \in K$ such that $\sum x_i R \in L(M_R)$.*

(c*) *If K is a submodule of M, then there are $x_1, \ldots, x_n \in K$ such that $\sum x_i R \in L(K_R)$.*

If M_R is any R-module then (a), (b), *and* (c) *are equivalent. If M_R is a regular R-module then* (a), (b*), *and* (c*) *are equivalent, hence all the statements are equivalent.*

Proof. Only the equivalence of (a), (b), and (c) will be shown as the equivalence of (a), (b*), and (c*) when M_R is regular has an analogous proof.

(a) implies (b). Let $K \in L(M_R)$ and suppose (b) is denied. Let $0 \neq x_1 \in K$, then $[x_1] \notin L(K_R)$, hence there is $0 \neq x_2 \in K$ such that $[x_1] \cap [x_2]=0$. Suppose $x_1, \ldots, x_n \in K$ such that $0 \neq [x_i]$ and the sum $\sum [x_i]$ is direct. Since $\sum [x_i]=[x_1, \ldots, x_n]$, $\sum [x_i] \notin L(K_R)$, hence there is $0 \neq x_{n+1} \in K$ such that $[x_1, \ldots, x_n] \cap [x_{n+1}]=0$. Thus, there is an infinite sum $[x_1] \oplus \cdots \oplus [x_n] \oplus \cdots \subseteq K \subseteq M$ contradicting (a), so for some n $[x_1, \ldots, x_n] \in L(K_R)$, hence $[x_1, \ldots, x_n] \in L(M_R)$ since $K \in L(M_R)$.

(b) implies (c). Let K be a submodule of M_R. By Zorn's lemma there is a submodule L of M which is maximal with respect to the property that $K \cap L=0$ and consequently $K+L \in L(M_R)$. By (b) there exist finitely many $a_1, \ldots, a_n \in K+L$ such that $[a_1, \ldots, a_n] \in L(M_R)$. Now $a_i=x_i+y_i$, $k_i \in K$, $y_i \in L$. The counter image of $[x_i, y_i]_{i=1}^n \in L(M_R)$ by the inclusion map $K \rightarrow K+L$ is $[x_1, \ldots, x_n]$, hence is large in K, so (c) follows.

(c) implies (a). Given K a direct sum of nonzero submodules of M, then there are finitely many $x_1, \ldots, x_n \in K$ such that $[x_1, \ldots, x_n] \in L(K_R)$. Now $[x_1, \ldots, x_n]$ is contained in the direct sum of finitely many of the submodules of K and $[x_1, \ldots, x_n]$ thus has zero intersection with the others so the others are zero and (a) follows.

Similar proofs show the equivalence of (a), (b*), and (c*).

PROPOSITION 1.4. *For a unitary module A_R, the following statements are equivalent.*

(a) *A is semisimple (sum of simple submodules).*

(b) *A is a direct sum of simple submodules.*

(c) *Every submodule of A_R is a direct summand of A.*

(d) $L(A_R)=\{A_R\}$.

Proof. The equivalence of (a), (b), and (c) is well known, e.g., [8]. Clearly (c) implies (d). Conversely, if B is a submodule of A, then there is a submodule C of A such that $B \cap C=0$ and $B+C \in L(A_R)$, by (d) $A=B \oplus C$ so (c) follows.

LEMMA 1.5. *If Q is a right quotient ring of R with $Z(R_R)=0$ and A, B R-submodules of Q_R such that $A \cap B=0$, then $AQ \cap BQ=0$.*

Proof. If $x \in AQ \cap BQ$, then $x=\sum a_i q_i=\sum b_i p_i$, $a_1, \ldots, a_n \in A$, $b_1, \ldots, b_n \in B$, $p_1, \ldots, p_n, q_1, \ldots, q_n \in Q$. $I=\{r \in R \mid q_i r \in R, p_i r \in R$ for all $i\} \in L(R_R)$ and $xI=0$ so $x \in Z(Q_R)=0$, since $Z(R_R)=0$.

COROLLARY. *If Q is a right quotient ring of R, $Z(R_R)=0$ and $B \in L(Q_Q)$, then $B \cap L(R_R)$.*

Proof. If $B \cap J=0$ for a right ideal J of R, then $BQ \cap JQ=0$ by the lemma, hence $J=0$ since $BQ=B$.

THEOREM 1.6. *Let $Z(R_R)=0$, and Q the maximal right quotient ring of R, then the following statements are equivalent.*

(a) *$IQ=Q$ for every $I \in L(R_R)$.*

(b) *For $I \in L(R_R)$ there are $a_1, \ldots, a_n \in I$ such that $\sum a_i R \in L(R_R)$.*

(c) dim R_R *is finite.*

(d) *If I is a right ideal of R, then there are $a_1, \ldots, a_n \in I$ such that $\sum a_i R \in L(I_R)$.*

(e) *Q is a semisimple ring.*

Proof. The equivalence of (b), (c), and (d) follows from Theorem 1.3 since $Z(R_R)=0$.

(a) implies (b). If $I \in L(R_R)$, then $IQ=Q$, hence there are $a_1, \ldots, a_n \in I, q_1, \ldots, q_n \in Q$ such that $\sum a_i q_i=1$. $J=\{r \in R \mid q_i r \in R$ for all $i\} \in L(R_R)$ and clearly $J \subseteq \sum a_i R$ so (b) follows.

(b) implies (e). If $B \in L(Q_Q)$, then $B \cap R \in L(R_R)$ by the corollary to Lemma 1.5. So $B \cap R$ has elements $a_1, \ldots, a_n$ such that $I=\sum a_i R \in L(R_R)$. IQ is a finitely generated right ideal of Q. Since $Z(R_R)=0$, Q is a regular ring, hence $IQ=eQ$, $e=e^2 \in Q$. However, $(1-e)I=0$ so $1-e \in Z(Q_R)=0$ so $IQ=Q$, but $IQ \subseteq B$ so $B=Q$, that is $L(Q_Q)=\{Q_Q\}$ so Q is a semisimple ring.

(e) implies (a). If $I \in L(R_R)$, then $IQ=eQ$ for some $e=e^2 \in Q$ since Q is semisimple. Since Q is the maximal right quotient ring of R and Q is semisimple, then Q is a regular ring so $Z(Q_R)=0$ by [9]. Therefore, since $(1-e)I=0$, $(1-e) \in Z(Q_R) =0$, so $1=e$ and $IQ=Q$.

Now the case of a semisimple classical right quotient ring of R will be considered.

THEOREM 1.7. *For a ring R, the following statements are equivalent.*

(a) *R has a semisimple classical right quotient ring.*

(b) $Z(R_R)=0$ *and for $I \in L(R_R)$ there is $a \in I$ such that $aR \in L(R_R)$.*

(c) *R is a semiprime ring,* dim R_R *is finite and R satisfies the ascending chain condition on right annihilators.*

Proof. The equivalence of (a) and (c) was shown by Goldie [7].

(a) implies (b)(2). Let Q be a semisimple classical right quotient ring of R, then Q is the maximal right quotient ring of R. By Theorem 1.6 $Z(R_R)=0$ and for $I \in L(R_R)$, there are $a_1, \ldots, a_n \in I$, $q_1, \ldots, q_n \in Q$ such that $\sum a_i q_i = 1$. Since $q_1 \in Q$, $q_1 = c_1 d_1^{-1}$, c_1, $d_1 \in R$, d_1 regular in R, hence $d_1 = \sum_{i>1} a_i q_i d_1 + a_1 c_1$. Since $q_2 d_1 \in Q$, $q_2 d_1 = c_2 d_2^{-1}$, c_2, $d_2 \in R$, d_2 regular in R, so $d_1 d_2 = a_1 c_1 d_2 + a_2 c_2 + \sum_{i>2} a_i q_i d_1 d_2$. Continuing in this fashion it follows that there exist regular elements $d_1, \ldots, d_n \in R$ such that $d=d_1 \cdots d_n \in \sum a_i R \subseteq I$. If $dR \cap J=0$ for a right ideal J of R, then $dRQ \cap JQ=0$, but since $R \in L(R_R)$, $RQ=Q$ so $dRQ=Q$ since d is regular so $J=0$, hence $dR \in L(R_R)$.

(b) implies (a). Let Q be the maximal right quotient ring of R and $q \in Q$, then $I=\{r \in R \mid qr \in R\} \in L(R_R)$. By (b) there is $a \in I$ such that $aR \in L(R_R)$. By Theorem 1.6 $Q=aRQ=aQ$ so a has right inverse. Since Q is semisimple and a has a right inverse a has a left inverse so a is a regular element of R and $q=ba^{-1}$. If $a \in R$ and a is regular, then the right annihilator of a in R is zero, hence in Q also. Since Q is semisimple a is regular in Q and (a) follows.

It is not valid in general that for a right R-module A_R, $Z(A/Z(A_R))=0$. Let R be a local ring with Jacobson radical $N \neq 0$ such that $N^2=0$. For instance $I/(p^2)$, I the ring of integers and p a prime. Since N is the unique maximal right ideal of R, $N \in L(R_R)$, hence it follows that $Z(R_R)=N$. Since $(R/N)N=0$, $Z(R/N)=R/N \neq 0$.

THEOREM 1.8. *If $Z(R_R)=0$ and* dim R_R *is finite, then $Z(A/Z(A))=0$ for every right R-module A_R.*

Proof. If $x+Z(A) \in Z(A/Z(A))$, then $I=\{r \in R \mid xr \in Z(A)\} \in L(R_R)$ by definition. By Theorem 1.6, Q, the maximal right quotient ring of R is semisimple so $IQ=Q$, hence there are $a_1, \ldots, a_n \in I$, $q_1, \ldots, q_n \in Q$ such that $\sum a_i q_i = 1$. For each i, $xa_i \in Z(A)$ so I_i, the annihilator of xa_i in R, is in $L(R_R)$. By the corollary to Proposition 1.2 $J=\{r \in R \mid q_i r \in I_i \text{ for each } i\} \in L(R_R)$. For $r \in J$, $xr = x(\sum a_i(q_i r)) = \sum xa_i(q_i r) =0$, so $x \in Z(A)$ and the theorem follows.

(2) Goldie [7] has shown this implication also.

This theorem raises the question of whether or not the condition $Z(A/Z(A))=0$ for every R-module is sufficient for R to possess a semisimple maximal ring of quotients.

2. **Rings with identity.** In this section R is a ring with unity 1, and all right R-modules are unitary.

LEMMA 2.1. *Let Q be a right quotient ring of R and A a right R-module, then if $a \otimes 1=0$ in $A \otimes_R Q$, there are finitely many $q_i \in Q$, $a_j \in A$, $a=a_1$, $\{r_{ij}\} \subseteq R$ such that*

$$\sum_i r_{ij} q_i = \delta_{1j} \quad (\textit{Kronecker delta})$$

and $\sum_j a_j r_{ij}=0$ for all i.

Proof. Let F be a free right R-module with basis $\{x_a : a \in A\}$, then the sequence

$$0 \to K \to F \to A \to 0$$

of right R-modules is exact, where $(F \to A)(x_a)=a$, $K=\operatorname{Ker}(F \to A)$. Tensoring over R with Q we have the exact sequence

$$K \otimes Q \to F \otimes Q \to A \otimes Q \to 0.$$

If $a \otimes 1=0$ in $A \otimes Q$, then $x_a \otimes 1$ is the image of an element from $K \otimes Q$. $x_a \otimes 1=\sum_i k_i \otimes q_i$. Since $k_i \in K \subseteq F$, $k_i=\sum_j x_{a_j}\lambda_{ij}$ for each i, a finite sum. Now $x_a \otimes 1=\sum_i (\sum_j x_{a_j}\lambda_{ij}) \otimes q_i=\sum_j x_{a_j} \otimes (\sum_i \lambda_{ij} q_i)$. Since representation in $F \otimes Q$ is unique with respect to basis elements $x_a=x_{a_j}$ for some j say $j=1$, hence $\sum_i \lambda_{ij} q_i=\delta_{1j}$ and from $k_i=\sum_j x_{a_j}\lambda_{ij}$, $0=\sum_j a_j \lambda_{ij}$ for all i, and the lemma follows.

PROPOSITION 2.2. *If Q is a right ring of quotients of R, A a right R-module, then the kernel of the map $A \otimes R \to A \otimes Q$ is contained in $Z(A_R)$.*

Proof. If $a \otimes 1=0$ in $A \otimes Q$, then by Lemma 1.3 there exist finitely many $\{q_i\} \subseteq Q$, $\{a_j\} \subseteq A$, $a_1=a$, $\{\lambda_{ij}\} \subseteq R$ such that

$$\sum_i \lambda_{ij} q_i = \delta_{1j}, \quad \sum_j a_j \lambda_{ij} = 0 \quad \text{for all } i.$$

Let $I=\{r \in R \mid q_i r \in R \text{ for each } i\}$, then I is a large right ideal of R by the corollary to Proposition 1.2 and for $\lambda \in I$

$$\begin{aligned} 0 &= \sum_i \Big(\sum_j a_j \lambda_{ij}\Big)(q_i \lambda) = \sum_{i,j} a_j \lambda_{ij}(q_i \lambda) \\ &= \sum_j a_j \Big(\sum \lambda_{ij}(q_i \lambda)\Big) = \sum_j a_j(\delta_{1j}\lambda) = a_1 \lambda = a\lambda, \end{aligned}$$

hence a is annihilated by I, so $a \in Z(A_R)$.

THEOREM 2.3. *If Q is the maximal right quotient ring of R, then* (a), (b), (c), (d), (e) *of Theorem* 1.6 *are all equivalent to* (f) $\operatorname{Ker}(A \otimes_R R \to A \otimes_R Q)=Z(A_R)$ *for every right R-module A.*

Proof. (a) implies (f). By Proposition 2.2, $\mathrm{Ker}(A \otimes R \to A \otimes Q) \subseteq Z(A_R)$. If $a \in Z(A_R)$ then I, the annihilator of a in R, is in $L(R_R)$ so $IQ = Q$, hence there are $a_1, \ldots, a_n \in I$, $q_1, \ldots, q_n \in Q$ such that $\sum a_i q_i = 1$. In $A \otimes Q$, $a \otimes 1 = a \otimes (\sum a_i q_i) = \sum aa_i \otimes q_i = 0$ so (f) follows.

(f) implies (a). If $I \in L(R_R)$, then from the exact sequence $I \otimes Q \to R \otimes Q \to R/I \otimes Q \to 0$ we have that $R/I \otimes Q$ is isomorphic to Q/IQ. Since $I \in L(R_R)$ $Z(R/I) = R/I$, so $R/I \otimes R \to R/I \otimes Q$ is the zero map hence $\bar{1} \otimes 1 = 0$ in $R/I \otimes Q$. However, $R/I \otimes Q \cong Q/IQ$ is a right Q-module generated by $\bar{1} \otimes 1$ so $Q/IQ = 0$, hence $I \otimes Q \to Q$ is onto. It is now clear that (a) follows since the image of $I \otimes Q \to Q$ in Q is IQ.

An immediate consequence of the notion of singular submodule is

Proposition 2.4. *If $E_R = \bigoplus_I E_i$, E_i right R-modules then $Z(E) = \bigoplus_I Z(E_i)$.*

If R is a commutative integral domain, then any direct sum of torsion free injective R-modules is injective, since it is torsion-free and divisible, hence injective, [1, Proposition VII.1.3]. A generalization holds.

Theorem 2.5. *If* dim R_R *is finite and E_R is the direct sum of injectives which have zero singular submodule, then E is injective.*

Proof. By Proposition 2.3 $Z(E) = 0$. It is sufficient to show that every R-homomorphism from a large right ideal of R into E can be extended to R. Let $f \in \mathrm{Hom}(I_R, E_R)$, $I_R \in L(R_R)$. By Theorem 1.3, there exist finitely many $a_1, \ldots, a_n \in I$ such that $J = \sum a_i R \in L(R_R)$. Let f' be the restriction of f to J. Since J is finitely generated, $f'(J)$ is contained in a finite direct sum of injectives, hence f' has an extension $f^* \in \mathrm{Hom}(R_R, E_R)$. The assertion is that f^* is an extension of f. Let $x \in I$, then $K = \{r \in R \mid xr \in J\} \in L(R_R)$. Now for $r \in K$, $(f(x) - f^*(x))r = f(xr) - f^*(xr) = f'(xr) - f'(xr) = 0$ so $f(x) - f^*(x) \in Z(E) = 0$, hence $f^*(x) = f(x)$.

It is known [3], that if $Z(R_R) = 0$, then Q the maximal right ring of R is injective as a right R-module.

Theorem 2.6. *If Q is the maximal right quotient ring of R, $Z(R_R) = 0$,* dim R_R *finite and A_R a right R-module such that $Z(A_R) = 0$, then the map $0 \to A \to A \otimes_R Q$ is an injective hull of A as a right R-module.*

Proof. The map is a monomorphism by Theorem 2.3.

Now $A \otimes_R Q$ is a right Q module, hence semisimple since Q is, so $A \otimes_R Q$ is a direct sum of direct summands of Q. Since $Z(Q_R) = 0$, $Z(A \otimes_R Q) = 0$ regarding $A \otimes Q$ as a right R-module and by Theorem 2.5 $A \otimes_R Q$ is injective as a right R-module.

If $0 \neq x = \sum a_i \otimes q_i \in A \otimes Q$, then $I = \{r \in R \mid q_i r \in R\} \in L(R_R)$. Now $0 \neq xI$ since $Z(A \otimes Q) = 0$ and $xI \subseteq \mathrm{Im}(A \to A \otimes Q)$ so $0 \to A \to A \otimes Q$ is an essential monomorphism, i.e., $\mathrm{Im}(A \to A \otimes Q)$ is a large right R submodule of $A \otimes Q$ and the theorem follows.

THEOREM 2.7. *If Q is a semisimple maximal right quotient ring of R, then Q is flat as a left R-module.*

Proof. It is sufficient to show that $I \otimes_R Q \to R \otimes_R Q$ is a monomorphism for every right ideal of R. As before $I \otimes_R Q$ is a right Q-module, then $Z(I \otimes Q)=0$ regarding $I \otimes Q$ as a right R-module. If $x=\sum a_i \otimes q_i \in \operatorname{Ker}(I \otimes Q \to R \otimes Q)$, then $I=\{r \in R \mid q_i r \in R\} \in L(R_R)$ and $\sum a_i q_i=0$ in Q. Clearly $xI=0$, so $x \in Z(I \otimes Q)=0$, so $x=0$, hence $I \otimes Q \to R \otimes Q$ is a monomorphism.

COROLLARY. *If Q is the maximal right quotient ring of R, $Z(R_R)=0$, $\dim R_R$ finite, then $\operatorname{Tor}_1^R(A, Q/R) \cong Z(A_R)$ for every right R-module A_R.*

Proof. It follows from the exact sequence $\operatorname{Tor}_1^R(A, Q) \to \operatorname{Tor}_1^R(A, Q/R) \to A \otimes_R R \to A \otimes_R Q$ since by the theorem $\operatorname{Tor}_1^R(A, Q)=0$ and $\operatorname{Ker}(A \otimes_R R \to A \otimes_R Q) \cong Z(A_R)$ by Theorem 2.3.

COROLLARY. *If Q is a semisimple maximal right quotient of R, then every left Q-module is flat as a left R-module.*

Proof. Every left Q-module is a direct sum of direct summands of Q, hence is flat as a left R-module since Tor_n^R commutes with direct sum s.

Matlis [12, Theorem 1.1] has shown that if R is a commutative integral domain and H an R-module, then the torsion submodule of H is a direct summand of H, if H is an epimorphic image of an injective R-module. This result is generalized and the proof does not appeal to the quotient ring of R.

First, the notion of a closed submodule of a module will be considered and some consequences. Johnson and Wong [11] considered the notion of a closed submodule.

DEFINITION 2.8. A submodule B of a module A is closed if B has no essential extension in A; i.e., C a submodule of A such that B is a large submodule of C implies $B=C$.

REMARK. If E is an injective R-module and A a submodule of E, then A is closed if and only if A is a direct summand of E. This follows from the fact that every submodule of E is a large submodule of its injective hull in E, which is a direct summand of E.

LEMMA 2.9. *If $f \in \operatorname{Hom}(M, A)$, B a submodule of A such that $Z(A/B)=0$, then $f^{-1}(B)$ is a closed submodule of M.*

Proof. Let D be a submodule of M containing $f^{-1}(B)$ as a large submodule. If $d \in D$, then $I=\{r \in R \mid dr \in f^{-1}(B)\} \in L(R_R)$. Now $f(d)I=f(dI) \subseteq B$, hence $f(d)+B=[f(d)]^- \in Z(A/B)=0$ so $f(d) \in B$ and $d \in f^{-1}(B)$ and the lemma follows.

THEOREM 2.10. *If $E \to^f H \to 0$ is an exact sequence of right R-module, E injective, $Z(H/Z(H))=0$, then $H=Z(H) \oplus F$ and F is injective.*

Proof. Since $Z(H/Z(H))=0$, by Lemma 2.9 $f^{-1}(Z(H))$ is closed in E, hence a direct summand of $E=f^{-1}(Z(H)) \oplus G$. Clearly $H=Z(H) \oplus f(G)$ and $f(G) \cong G$.

COROLLARY. *If R is a ring such that $Z(R_R)=0$,* dim R_R *finite, then every epimorphic image of an injective R-module has its singular submodule as a direct summand.*

Proof. An immediate consequence of the theorem and Theorem 1.8.

It is interesting to note that some of the propositions of [12] admit generalizations to noncommutative rings and their maximal right quotient rings, where torsion submodule is replaced with singular submodule and Q the maximal right quotient ring of R.

If in addition to the hypothesis of [12, Proposition 2.1] we assume $Z(R_R)=0$, dim R_R finite, then Proposition 2.1 is valid with the same proofs using the fact that $Z(Q_R)=0$ and a direct sum of copies of Q is injective by Theorem 2.5. Similarly a generalization of [12, Proposition 2.2] is valid in view of the corollary to Theorem 2.11, as well as [12, Corollary 2.3] of [12, Proposition 2.2].

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