ON A CLASS OF NODAL NONCOMMUTATIVE JORDAN ALGEBRAS

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1. A finite-dimensional power-associative algebra A with unity element 1 over a field F is a nodal algebra [6] if each element x of A may be written $x=\alpha 1+z$, where α is in F and z is nilpotent, and if A is not of the form $A=F1+N$ where N is a nil subalgebra of A. The class K, of p^s-dimensional nodal noncommutative Jordan algebras we will consider in this paper is described as follows: let

$$B_n = F[1, x_1, \dots, x_n] \quad \text{with } x_i^p = 0,\ x_i^0 = 1, \text{ for } i = 1, \dots, n,$$

be a commutative associative truncated polynomial algebra over the field F of characteristic $p \neq 2$. An algebra A in K of dimension p^n is taken to be the same vector space as B_n but with multiplication given by the product

$$fg = f\cdot g + \sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i}\cdot\frac{\partial g}{\partial x_j}\cdot c_{ij},$$

where $f\cdot g$ is the product of $f=f(x_1, \dots, x_n)$ and $g=g(x_1, \dots, x_n)$ in B_n and the $c_{ij}=\frac{1}{2}[x_i, x_j]=\frac{1}{2}(x_ix_j-x_jx_i)$ are arbitrary except for the proviso that at least one of them is nonsingular. That is, there must exist a $c_{ij}=\alpha_{ij}1+w_{ij}$ with $\alpha_{ij}\neq 0$. This implies that $n \geqq 2$.

The class K was constructed by L. Kokoris who proved [2], [3] that every simple nodal noncommutative Jordan algebra is in K. He also proved that not all the algebras of the class K are simple.

Two papers have appeared which were concerned with studying derivations of some of the algebras in K. The first of these by R. Schafer [7] described derivation algebras for the cases c_{ij} in F and $n=2$ and demonstrated relationships between certain ideals of these algebras and types of simple Lie algebras of characteristic p. He made use of two properties which were later generalized by R. Oehmke, namely that the algebras in K for the cases c_{ij} in F and $n=2$ are Lie-admissible and that, for $n=2$, the generators x_1 and x_2 could be chosen so that $c_{12}=1+\alpha x_1^{p-1}\cdot x_2^{p-1}$ for α in F.

In his paper [4], Oehmke determined the derivation algebras of all simple Lie-admissible algebras A of K. He proved that generators could be chosen for these A which satisfy a useful Lie-multiplication table, which is a generalization of

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Schafer's relation for the Lie product of two generators. We call this multiplication table an Oehmke multiplication table in this paper.

The theorems in our §§2 and 3, together with [4] show that existence of generators satisfying Oehmke-type multiplication relations is a necessary and sufficient condition for simplicity and Lie-admissibility of an algebra in K. The result of §3 concerning the Lie-admissibility of algebras of K having an Oehmke multiplication table is applied to the construction of the example of §4. This example is one of a simple algebra in K which is not Lie-admissible and which cannot have an Oehmke multiplication table. Thus, the converse to Theorem 1 is not true and something (Lie-admissibility here) must be added to provide a necessary and sufficient condition for simplicity in this context.

The author is grateful to the referee for his suggestions and for pointing out an index error which occurred in equations (2) below in an earlier draft and in [4].

2. We will say that an algebra $A = F1 + F[x_1, \ldots, x_n]$ (vector space direct sum) in K has an Oehmke multiplication table (OMT) if the generators $x_1, \ldots, x_n$ can be chosen such that for n even, $n = 2r$, we have:

$$
\begin{aligned}
&[x_i, x_{i+r}] = 1 + \alpha_i x_i^{p-1} \cdot x_{i+r}^{p-1} &&\text{for } i = 1, \ldots, r,\ \alpha_i \text{ in } F, \\
&[x_i, x_j] = 0 &&\text{for } j \neq i+r,
\end{aligned} \tag{1}
$$

and for n odd, $n = 2r+1$, we have (1) holding and

$$
\begin{aligned}
&[x_{2r+1}, x_j] = 0 &&\text{for } j \neq r,\ 2r, \\
&[x_{2r+1}, x_{2r}] = \alpha x_r^{p-1} \cdot (1 + \beta x_{2r+1}^{p-1}), && \\
&[x_{2r+1}, x_r] = x_{2r}^{p-1} \cdot (1 + \beta x_{2r+1}^{p-1}) &&\text{for } \alpha, \beta \text{ in } F.
\end{aligned} \tag{2}
$$

Here, as in the preceding section, we have used the notation, $[x, y]$, for the commutator $xy - yx$ of x and y. If we denote the mapping $x \to [x, y]$ by $D(y)$, i.e., $xD(y) = [x, y]$, then the fact that $D(y)$ is a derivation of A^+ for every y in A is a well-known consequence of the flexible identity.

In [4], Oehmke proved that a simple Lie-admissible algebra in K has an OMT.

We further remark here that B is an ideal of A if and only if B is an ideal of both A^+ and A^-.

THEOREM 1. *Let A be an algebra in K. If A has an* OMT, *then A is simple.*

Proof. Let B be a nonzero ideal of A, for A of dimension p^n. Because of the nature of (1) and (2) we must break up the proof into cases of even and odd numbers of generators.

We first consider an algebra A with an even number, $n = 2r$, of generators for which we can assert the following, due to Oehmke [4, pp. 425–426]. If m is any monomial in B and a generator x_i occurs (to any power) in m, then x_i is in B. We reproduce the proof of this claim here, because of typographical errors in [4].

Suppose $m=x_i^k\cdot q$ where q is independent of x_i. Since B is an ideal of A^-, $[m, x_{i+r}]$ is in B. (If $i>r$, Lie multiplication by x_{i-r} will give us the result.) But

$$[m, x_{i+r}] = mD(x_{i+r}) = kx_i^{k-1}\cdot x_iD(x_{i+r})\cdot q+x_i^k\cdot qD(x_{i+r})$$
$$= kx_i^{k-1}\cdot(1+\alpha_i x_i^{p-1}\cdot x_{i+r}^{p-1})\cdot q = kx_i^{k-1}\cdot q \quad \text{if} \quad k > 1.$$

Therefore, we might as well suppose that $k=1$ and that $m=x_i\cdot q$ is in B.

Using an induction argument and the above method, m may be assumed to be just a product of generators, each with exponent $=1$.

We may further assume that if the generator x_k appears in m for $k\leqq r$, then x_{k+r} does not appear. For, if $m=x_k\cdot x_{k+r}\cdot s$, then $mD(x_k)=-x_k\cdot s$ is in B.

The above considerations justify taking m to be of the form $m=x_i\cdot x_j\cdot \ldots \cdot x_k\cdot x_t$, where the subscript of each of the distinct generators is less than or equal to r.

Hence $mD(x_{t+r})=x_i\cdot x_j\cdot \ldots \cdot x_k\cdot x_tD(x_{t+r})$ is in B. But, B an ideal of A^+ and $x_tD(x_{t+r})$ nonsingular imply that $x_i\cdot x_j\cdot \ldots \cdot x_k$ is in B. Continuing this process, we find that $x_i\cdot x_j$ is in B. Thus,

$$[(x_i\cdot x_j)D(x_{j+r})]\cdot(x_jD(x_{j+r}))^{-1} = x_i$$

is in B, which proves our claim.

Now we introduce a result of A. A. Albert [1, Lemma 21]: the monomial $m=x_1^{p-1}\cdot \ldots \cdot x_{2r}^{p-1}$ is in B. By the claim then, B contains every generator, hence $B=A$. This proves Theorem 1 for $n=2r$.

We now turn to the proof of Theorem 1 for the case of $n=2r+1$ generators. Using the above quoted result of Albert, the maximal degree monomial

$$x_1^{p-1}\cdot \ldots \cdot x_{2r+1}^{p-1}$$

is in B. Making use of the multiplication table (1), it is clear that application of the operators $D(x_1), \ldots, D(x_{r-1}), D(x_{r+1}), \ldots, D(x_{2r-1})$ to this monomial will, just as in the even case above, guarantee that the monomial

$$m = x_1\cdot \ldots \cdot x_{r-1}\cdot x_r^{p-1}\cdot x_{r+1}\cdot \ldots \cdot x_{2r-1}\cdot x_{2r}^{p-1}\cdot x_{2r+1}^{p-1}$$

is in B.

Noting that $x_kD(x_{1+r})=0$ for $k\neq 1$, we then see that

$$mD(x_{1+r})\cdot[x_1, x_{1+r}]^{-1} = x_2\cdot \ldots \cdot x_{r-1}\cdot x_r^{p-1}\cdot x_{r+1}\cdot \ldots \cdot x_{2r-1}\cdot x_{2r}^{p-1}\cdot x_{2r+1}^{p-1}$$

is in B. The terms x_i, $i=2,\ldots,r-1, r+1,\ldots, 2r-1$ are eliminated in the same manner, thus showing that $x_r^{p-1}\cdot x_{2r}^{p-1}\cdot x_{2r+1}^{p-1}$ is in B.

The problem has now been reduced to the case of the ideal B containing a maximal degree monomial in the three generators x_r, x_{2r}, and x_{2r+1}. The technique for completion of the proof will be easier from the standpoint of both understanding and typography if we illustrate it by just showing that an algebra in K with three generators x, y, z satisfying (1) and (2) is simple.

Therefore, we will assume it is clear that the methods to come may be carried over to the general case and we will take $A=F1+F[x, y, z]$ in K satisfying

$$(3)\qquad \begin{aligned} [y, x] &= 1+\gamma x^{p-1}\cdot y^{p-1}, \qquad \gamma \text{ in } F,\\ [z, y] &= \alpha x^{p-1}\cdot(1+\beta z^{p-1}),\\ [z, x] &= y^{p-1}\cdot(1+\beta z^{p-1}),\end{aligned}$$

and prove that such an algebra A is simple.

We have $x^{p-1}\cdot y^{p-1}\cdot z^{p-1}$ is in B. Thus there is an integer $m=$the minimal exponent such that $x^{p-1}\cdot y^{p-1}\cdot z^m$ is in B.

Denote the sth iterate of the mapping $D(x)$ by $D^s(x)$. It may be verified, by virtue of (3), that the following two equations hold:

$$(4)\qquad (x^i\cdot y^j\cdot z^k)D^s(x) = P(j, s)x^i\cdot y^{j-s}\cdot z^k$$

and

$$(5)\qquad (x^i\cdot y^j\cdot z^k)D^t(y) = \delta P(i, t)x^{i-t}\cdot y^j\cdot z^k,$$

where $i, j>0$, $k=0, 1, \ldots, p-1$, $\delta=\pm 1$, and $P(h, q)=h!/(h-q)!$.

It is clear from (4) and (5) that $x^a\cdot y^b\cdot z^m$ is in B for any a, b such that $1\leqq a, b\leqq p-1$. Operating on $x\cdot y\cdot z^m$ by $D(x)$, we have $x\cdot z^m$ in B.

Suppose that $m\geqq 1$, and consider the calculation:

$$\begin{aligned}(x\cdot z^m)D(x^2) &= 2mx^2\cdot z^{m-1}\cdot zD(x)\\ &= 2mx^2\cdot y^{p-1}\cdot z^{m-1} \qquad \text{for } m \geqq 2\\ &= 2x^2\cdot y^{p-1}\cdot z^{m-1}+2\beta x^2\cdot y^{p-1}\cdot z^{p-1} \qquad \text{for } m = 1.\end{aligned}$$

Since $x^2\cdot y^{p-1}\cdot z^{p-1}$ is in B, either of the above two equations implies that $x^2\cdot y^{p-1}\cdot z^{m-1}$ is in B and hence that $x^{p-1}\cdot y^{p-1}\cdot z^{m-1}$ belongs to B for $m\geqq 1$, contrary to the minimality of m. Therefore $m=0$ and $x^{p-1}\cdot y^{p-1}$ is in B, thus $x\cdot y$ is in B.

Hence the nonsingular element $(x\cdot y)D(x)D(y)=xD(y)$ is in B and $B=A$. Therefore A is simple and the proof of Theorem 1 is complete. The remarks of the introduction yield the following

COROLLARY. *Let A be in K and suppose either all c_{ij} are in F or $n=2$. A is simple if and only if A has an* OMT.

3. We now examine the Lie-admissibility of algebras A in K having an OMT. The chief tool used in these considerations will be the criteria of R. Schafer [7, p. 318] which we state in the following manner.

An algebra A in K of dimension p^n is Lie-admissible if and only if

$$(6)\qquad \sum_{t=1}^{n} Q_t = 0$$

where

$$(7) \qquad Q_t = \partial c_{ij}/\partial x_t \cdot c_{tk} + \partial c_{jk}/\partial x_t \cdot c_{ti} + \partial c_{ki}/\partial x_t \cdot c_{tj}$$

with $2c_{ij} = [x_i, x_j]$ and $1 \leqq i < j < k \leqq n$.

Suppose A is in K and has an even number, $n = 2r$, of generators having an OMT. Since we intend to apply (6) and (7) to this A, we examine Q_t where we now must have

$$1 \leqq i < j < k \leqq 2r.$$

No generality is lost by assuming $t \leqq r$, since a symmetrical result holds for $t \geqq r+1$.

The only nonzero c_{tq} occurs when $q = t+r$. Suppose that $c_{tk} = c_{t,t+r}$. (Similar reasoning holds for $i = t+r$ and $j = t+r$.) Therefore $c_{ti} = c_{tj} = 0$. Hence, from (7), $Q_t = \partial c_{ij}/\partial x_t \cdot c_{t,t+r}$.

But $c_{ij} = 0$ unless $j = i+r$ (or $i = j-r$). Thus,

$$Q_t = \frac{\partial c_{i,i+r}}{\partial x_t} \cdot c_{t,t+r} = \frac{1}{2} \alpha_i \frac{\partial (x_i^{p-1} \cdot x_{i+r}^{p-1})}{\partial x_t} \cdot c_{t,t+r}.$$

However, we can assert that $t \neq i$, for otherwise we would have $k = j$, a contradiction, and also we have $t \neq i+r$, for if $t = i+r$, then $k = i+2r$, which is impossible. (Similarly, $t \neq j$ and $t \neq j-r$.)

Therefore the quantity being differentiated is not a function of x_t, which implies $Q_t = 0$. Thus, using (6), A^- is a Lie algebra and we have proved a portion of the following theorem.

THEOREM 2. *Let A be an algebra in K. If A has an* OMT, *then A is Lie-admissible.*

Proof. To deal with the second portion of this theorem, the case of $2r+1$ generators,

$$1 \leqq i < j < k \leqq 2r+1$$

must hold where we intend to apply (6) and (7). We see that $Q_t = 0$ for $t \neq r$, $2r$, or $2r+1$, since for t different from these values the generators concerned satisfy a multiplication of the form (1). The detailed reasoning involves considerations similar to the even case proof of Theorem 2.

The quantities Q_r, Q_{2r}, and Q_{2r+1} will now be considered separately.

We can exhaust the value possibilities for Q_r by considering the four mutually exclusive cases which arise when $k = 2r+1$ and either $i = r$ or $j = r$ or $i \neq r$ and $j \neq r$; or when $k \leqq 2r$.

Calculations will show that $Q_r = 0$ in all these cases; we detail some samples of the reasoning for purposes of illustration.

If $k = 2r+1$ and $i = r$, then using (7) we find that

$$Q_r = \partial c_{rj}/\partial x_r \cdot c_{r,2r+1} + \partial c_{2r+1,r}/\partial x_r \cdot c_{rj}.$$

Now $c_{2r+1,r}$ is not a function of x_r, hence

$$Q_r = \partial c_{rj}/\partial x_r \cdot c_{r,2r+1}.$$

The only nonzero possibility for Q_r would be for $j=2r$, since we cannot have $j=2r+1=k$. Making the computations for these index values we obtain

$$\begin{aligned} Q_r &= \frac{\partial}{\partial x_r}\,[\tfrac{1}{2}(1+\alpha_r x_r^{p-1}\cdot x_{2r}^{p-1})]\cdot(-\tfrac{1}{2}x_{2r}^{p-1}\cdot(1+\beta x_{2r+1}^{p-1})) \\ &= \tfrac{1}{4}\alpha_r x_r^{p-2}\cdot x_{2r}^{p-1}\cdot x_{2r}^{p-1}\cdot(1+\beta x_{2r+1}^{p-1}) = 0. \end{aligned}$$

As a last example, we consider the case in which $k=2r+1$, $i\neq r$, and $j\neq r$. Here we have

$$Q_r = \frac{\partial c_{ij}}{\partial x_r}\cdot c_{r,2r+1}+\frac{\partial c_{j,2r+1}}{\partial x_r}\cdot c_{ri}+\frac{\partial c_{2r+1,i}}{\partial x_r}\cdot c_{rj}.$$

Notice that $Q_r=0$ for $j=2r$, since we cannot have $i=2r=j$ and $i=r$ is disallowed by the case.

For $j<2r$, the above equation yields

$$Q_r = \partial c_{ij}/\partial x_r\cdot c_{r,2r+1},$$

and we note that c_{ij} cannot be a function of x_r in these circumstances. Thus $Q_r=0$ here.

We can demonstrate that $Q_{2r}=Q_{2r+1}=0$ in much the same manner. The cases which exhaust the value possibilities for Q_{2r} are $k=2r+1$ and either $i=r$ or $j=r$ or $i\neq r$ and $j\neq r$; or $k\leqq 2r$. We need only three mutually exclusive cases to represent all the possibilities for Q_{2r+1}. We could have $k=2r+1$ and either $j=2r$ or $j<2r$; or have $k\leqq 2r$.

The verification that $Q_{2r+1}=0$ is immediate by inspection in all but the first case: $k=2r+1$ and $j=2r$. Here we see

$$Q_{2r+1} = \frac{\partial c_{2r,2r+1}}{\partial x_{2r+1}}\cdot c_{2r+1,i}+\frac{\partial c_{2r+1,i}}{\partial x_{2r+1}}\cdot c_{2r+1,2r}$$

which may admit a nonzero possibility for $i=r$. However, we then find

$$\begin{aligned} 4Q_{2r+1} &= \frac{\partial(-\alpha x_r^{p-1}\cdot(1+\beta x_{2r+1}^{p-1}))}{\partial x_{2r+1}}\cdot x_{2r}^{p-1}\cdot(1+\beta x_{2r+1}^{p-1}) \\ &\quad+\frac{\partial(x_{2r}^{p-1}\cdot(1+\beta x_{2r+1}^{p-1}))}{\partial x_{2r+1}}\cdot\alpha x_r^{p-1}\cdot(1+\beta x_{2r+1}^{p-1}) \\ &= \alpha\beta x_r^{p-1}\cdot x_{2r+1}^{p-2}\cdot x_{2r}^{p-1}\cdot(1+\beta x_{2r+1}^{p-1}) \\ &\quad-\alpha\beta x_{2r}^{p-1}\cdot x_{2r+1}^{p-2}\cdot x_r^{p-1}\cdot(1+\beta x_{2r+1}^{p-1}) \\ &= 0. \end{aligned}$$

Therefore $Q_t=0$ for $1\leqq t\leqq 2r+1$ and Theorem 2 is proved.

4. We are now in a position to state the next theorem which makes it clear that simplicity alone is insufficient for an OMT.

THEOREM 3. *There exist simple algebras of K which are not Lie-admissible and which cannot have an* OMT.

Proof. Let $x_1, \ldots, x_{2R}$ for $R>1$, be $2R$ quantities and let F be a field of characteristic $p \neq 2$. Let M be the vector space over F with $x_1, \ldots, x_{2R}$ as basis and let $a=a(x, y)$ be a skew-symmetric bilinear form on M with $a(x_i, x_j)=\alpha_{ij}=-\alpha_{ji}$ whose rank is equal to $2R$.

Define A^+ by $A^+=F1+N^+$ where $N^+=F[x_1, \ldots, x_{2R}]$, with $x_i^p=0, i=1, \ldots, 2R$. For convenience write $x_i^0=1, i=1, \ldots, 2R$. That is, A^+ is a commutative associative truncated polynomial algebra with nilpotent generators. Now define $A=F1+N$ to be the same vector space as A^+ with generator products defined by $x_i x_j=\alpha_{ij}1+w_{ij}$, with $w_{ij}=2x_i \cdot x_j - w_{ji}$ in N for $i<j$. Further, define

$$fg = f \cdot g + \frac{1}{2} \sum_{i,j=1}^{2R} \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot [x_i, x_j]$$

for any elements f and g of A.

We note here that Kokoris [3, Theorems 5 and 6] has proved that the above conditions and construction of A are sufficient for A to be a simple nodal noncommutative Jordan algebra in the class K.

Finally, complete the definition of A by setting $w_{23}=w_{32}$, $w_{13}=w_{31}$, and choosing w_{12} such that $[w_{12}, x_3] \neq 2(\alpha_{13}x_2+\alpha_{23}x_1)$.

We shall show that A as defined above cannot be Lie-admissible, since the Jacobi identity in A^- is not satisfied. For consider the quantity

$$\begin{aligned} J &= [[x_1, x_2], x_3]+[[x_2, x_3], x_1]+[[x_3, x_1], x_2] \\ &= [w_{12}-w_{21}, x_3] = 2[w_{12}-x_1 \cdot x_2, x_3]. \end{aligned}$$

It has been previously noted that

$$[x_1 \cdot x_2, x_3] = [x_1, x_3] \cdot x_2 + x_1 \cdot [x_2, x_3] = 2(\alpha_{13}x_2+\alpha_{23}x_1).$$

Thus

$$J = 2[w_{12}, x_3] - 4(\alpha_{13}x_2+\alpha_{23}x_1) \neq 0$$

and A^- is not a Lie algebra.

Now suppose that there exists a set of generators $y_1, \ldots, y_{2R}$ for A satisfying the relations (1). By Theorem 2, we must have A Lie-admissible, which is a contradiction. Theorem 3 is proved.

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