ON HOMOGENEOUS SPACES AND REDUCTIVE SUBALGEBRAS OF SIMPLE LIE ALGEBRAS

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1. **Introduction.** Let G be a connected Lie group and H a closed subgroup. Then the homogeneous space $M=G/H$ is called *reductive* if in the Lie algebra $\mathfrak{g}$ of G there exists a subspace $\mathfrak{m}$ such that $\mathfrak{g}=\mathfrak{m}\dotplus\mathfrak{h}$ (subspace direct sum) and $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ where $\mathfrak{h}$ is the Lie algebra of H (see [4], [5]). In this case the pair $(\mathfrak{g}, \mathfrak{h})$ is called a *reductive pair* and the subspace $\mathfrak{m}$ can be made into an anti-commutative algebra as follows. For $X, Y\in\mathfrak{m}$ let $[X, Y]=XY+\mathfrak{h}(X, Y)$ where $XY=[X, Y]_{\mathfrak{m}}$ (resp. $\mathfrak{h}(X, Y)=[X, Y]_{\mathfrak{h}}$) is the projection of $[X, Y]$ in $\mathfrak{g}$ into $\mathfrak{m}$ (resp. $\mathfrak{h}$). This algebra is related to the canonical G-invariant connection ∇ of the first kind on G/H by $[\nabla_{X^*}(Y^*)]_{P_0}=\frac{1}{2}XY$ where $P_0=H\in M$ (see [5, Theorem 10.1]).

For a fixed decomposition $\mathfrak{g}=\mathfrak{m}\dotplus\mathfrak{h}$, the Lie algebra identities of $\mathfrak{g}$ yield the following identities for $\mathfrak{m}$ and $\mathfrak{h}$. For $X, Y, Z\in\mathfrak{m}$ and $U\in\mathfrak{h}$,

(1) $XY=-YX$ (bilinear);

(2) $\mathfrak{h}(X, Y)=-\mathfrak{h}(Y, X)$ (bilinear);

(3) $[Z, \mathfrak{h}(X, Y)]+[X, \mathfrak{h}(Y, Z)]+[Y, \mathfrak{h}(Z, X)]=J(X, Y, Z)\equiv(XY)Z+(YZ)X+(ZX)Y$.

(4) $\mathfrak{h}(XY, Z)+\mathfrak{h}(YZ, X)+\mathfrak{h}(ZX, Y)=0$;

(5) $\mathfrak{h}[(X, Y), U]=\mathfrak{h}([X, U], Y)+\mathfrak{h}(X, [Y, U])$;

(6) $[U, XY]=[U, X]Y+X[U, Y]$.

In particular (6) says the mappings $\mathrm{ad}_{\mathfrak{m}}\, U\colon \mathfrak{m}\to\mathfrak{m}\colon X\to[U, X]$ are derivations of the algebra $\mathfrak{m}$. Using these identities, there was established in [6] a correspondence between simple algebras $\mathfrak{m}$ and holonomy irreducible simply connected spaces G/H which are not symmetric ($\mathfrak{m}\mathfrak{m}=0$ if and only if G/H is a symmetric space); for example, if G/H is riemannian, then G/H is holonomy irreducible if and only if $\mathfrak{m}$ is a simple algebra.

In this paper, we consider pairs $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{g}$ is a simple Lie algebra over a field F of characteristic zero and $\mathfrak{h}$ is either semisimple, or regular and reductive (see [2]). In each case we show that the associated $\mathfrak{m}$ is either simple or abelian ($\mathfrak{m}^2=0$). This together with [6] shows in particular that if G is a simple connected Lie group and H a closed semisimple or regular reductive Lie subgroup of G such that G/H is simply connected, then either G/H is a symmetric space or G/H is holonomy irreducible. This is a reasonable account of the situation since it can be shown that

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if G/H is a holonomy irreducible pseudo-riemannian reductive space with G simple, then $\mathfrak{h}$ is a reductive subalgebra of $\mathfrak{g}$.

2. **The regular reductive case.**

LEMMA 1. *Let* $\mathfrak{a}$ *be a nonassociative algebra with derivation algebra* Der $\mathfrak{a}$. *Assume that* $\mathfrak{a}$ *has no proper ideal stable under* Der $\mathfrak{a}$. *Then either* $\mathfrak{a}$ *is simple or* $\mathfrak{a}^2=0$.

Proof. Assume $\mathfrak{a}^2\neq 0$ and let $\mathfrak{T}(\mathfrak{a})$ denote the associative algebra generated by the left and right multiplications of $\mathfrak{a}$ [3, p. 290]. Let R be the radical of $\mathfrak{T}(\mathfrak{a})$. Then $R\mathfrak{a}$ is an ideal of $\mathfrak{a}$ since $\mathfrak{T}(\mathfrak{a})(R\mathfrak{a})\subseteq(\mathfrak{T}(\mathfrak{a})R)\mathfrak{a}\subseteq R\mathfrak{a}$. If $D\in$ Der $\mathfrak{a}$, then $[D, \mathfrak{T}(\mathfrak{a})]\subseteq\mathfrak{T}(\mathfrak{a})$ since $\mathrm{ad}_{\mathrm{Hom}(\mathfrak{a},\mathfrak{a})} D$ stabilizes the set of right and left multiplications (e.g., $[D, L(A)]=L(D(A))$ where $L(B)$ denotes left multiplication by B in $\mathfrak{a}$). Thus $\mathrm{ad}_{\mathfrak{T}(\mathfrak{a})} D$ is a derivation of $\mathfrak{T}(\mathfrak{a})$ and it follows that $[D, R]\subseteq R$ [3, p. 30, exercise 22]. Thus $D(R\mathfrak{a})\subseteq[D, R]\mathfrak{a}+R(D\mathfrak{a})\subseteq R\mathfrak{a}$. Thus $R\mathfrak{a}$ is a Der $\mathfrak{a}$-stable ideal of $\mathfrak{a}$. By assumption, we must have $R\mathfrak{a}=\mathfrak{a}$ or $R\mathfrak{a}=0$. If $R\mathfrak{a}=\mathfrak{a}$, then for some i, $0=R^i\mathfrak{a}=R^{i-1}\mathfrak{a}=\cdots=R\mathfrak{a}=\mathfrak{a}$ and $\mathfrak{a}=0$. Thus we may assume that $R\mathfrak{a}=0$. Then $R=0$ and $\mathfrak{T}(\mathfrak{a})$ is completely reducible on $\mathfrak{a}$. $\mathfrak{a}^2$ is clearly Der $\mathfrak{a}$-stable. Assuming that $\mathfrak{a}^2\neq 0$, we must have $\mathfrak{a}^2=\mathfrak{a}$ by hypothesis. We claim that $\mathfrak{a}^2=\mathfrak{a}$ implies that $\mathfrak{a}$ is simple. For if $\mathfrak{b}$ were a proper ideal of $\mathfrak{a}$, then $\mathfrak{b}$ would be $\mathfrak{T}(\mathfrak{a})$-stable and hence $\mathfrak{a}=\mathfrak{b}\oplus\mathfrak{b}'$ for some $\mathfrak{T}(\mathfrak{a})$-stable $\mathfrak{b}'$. This $\mathfrak{b}'$ would be an ideal and $\mathfrak{a}=\mathfrak{a}^2=\mathfrak{b}^2+(\mathfrak{b}')^2$ shows that $\mathfrak{b}^2=\mathfrak{b}$. But then $\mathfrak{b}=\mathfrak{b}^2$ would be Der $\mathfrak{a}$-stable since for B_1, B_2 in $\mathfrak{b}$, $D(B_1B_2)=(DB_1)B_2+B_1(DB_2)\in\mathfrak{b}$. Thus $\mathfrak{a}$ is simple.

We now consider reductive pairs $(\mathfrak{g}, \mathfrak{h})$. Thus let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{g}$, $\mathfrak{m}$ a complementary subspace of $\mathfrak{h}$ in $\mathfrak{g}$ such that $[\mathfrak{m}\mathfrak{h}]\subseteq\mathfrak{m}$. For $X, Y\in\mathfrak{m}$ we define XY in $\mathfrak{m}$ and $\mathfrak{h}(X, Y)$ in $\mathfrak{h}$ by requiring that $[XY]=XY+\mathfrak{h}(X, Y)$. We regard $\mathfrak{m}$ as a nonassociative algebra with respect to the product XY. Then $\mathfrak{m}$ is clearly anti-commutative and $\mathrm{ad}_{\mathfrak{m}}\, U$ is a derivation of $\mathfrak{m}$ for $U\in\mathfrak{h}$ (by (6)).

LEMMA 2. *Let* $\mathfrak{n}$ *be an* ad $\mathfrak{h}$*-stable ideal of* $\mathfrak{m}$. *Let* $\mathfrak{q}=\mathfrak{n}+\mathfrak{h}(\mathfrak{n}, \mathfrak{n})$. *If* $[\mathfrak{n}, \mathfrak{n}']\subseteq\mathfrak{q}$ *for some complementary subspace* $\mathfrak{n}'$ *of* $\mathfrak{n}$ *in* $\mathfrak{m}$, *then* $\mathfrak{q}$ *is an ideal of* $\mathfrak{g}$.

Proof. $[\mathfrak{q}, \mathfrak{n}]\subseteq[\mathfrak{n}, \mathfrak{n}]+[\mathfrak{h}(\mathfrak{n}, \mathfrak{n}), \mathfrak{n}]\subseteq\mathfrak{n}\mathfrak{n}+\mathfrak{h}(\mathfrak{n}, \mathfrak{n})+\mathfrak{n}$ by (3) since $\mathfrak{n}$ is ad $\mathfrak{h}$-stable. Thus $[\mathfrak{q}, \mathfrak{n}]\subseteq\mathfrak{q}$. And $[\mathfrak{q}, \mathfrak{h}]\subseteq\mathfrak{q}$ since $\mathfrak{n}$ is ad $\mathfrak{h}$-stable and $\mathfrak{q}=\mathfrak{n}+[\mathfrak{n}, \mathfrak{n}]$. It remains to show that $[\mathfrak{q}, \mathfrak{n}']\subseteq\mathfrak{q}$. But we have

$$\begin{aligned}[\mathfrak{q}, \mathfrak{n}'] &\subseteq \mathfrak{n}\mathfrak{n}'+\mathfrak{h}(\mathfrak{n}, \mathfrak{n}')+[\mathfrak{h}(\mathfrak{n}, \mathfrak{n}), \mathfrak{n}'],\\ [\mathfrak{h}(\mathfrak{n}, \mathfrak{n}), \mathfrak{n}'] &\subseteq [\mathfrak{n}\mathfrak{n}, \mathfrak{n}']+[[\mathfrak{n}, \mathfrak{n}], \mathfrak{n}']\subseteq[\mathfrak{n}, \mathfrak{n}']+[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}']],\\ \mathfrak{h}(\mathfrak{n}, \mathfrak{n}') &\subseteq \mathfrak{n}\mathfrak{n}'+[\mathfrak{n}, \mathfrak{n}'].\end{aligned}$$

But since $[\mathfrak{n}, \mathfrak{n}']\subseteq\mathfrak{q}$ by hypothesis, $\mathfrak{q}$ contains $[\mathfrak{h}(\mathfrak{n}, \mathfrak{n}), \mathfrak{n}']$ (using (3)) and $\mathfrak{h}(\mathfrak{n}, \mathfrak{n}')$. Since $\mathfrak{n}\mathfrak{n}'\subseteq\mathfrak{n}$ ($\mathfrak{n}$ is an ideal of $\mathfrak{m}$), $[\mathfrak{q}, \mathfrak{n}']\subseteq\mathfrak{q}$. Thus $\mathfrak{q}$ is an ideal of $\mathfrak{g}$.

LEMMA 3. *Suppose that the Killing form* $B(\ ,\ )$ *of* $\mathfrak{g}$ *is nondegenerate and that* $B(\mathfrak{m}, \mathfrak{h})=0$. *Then* $B(\ ,\ )|\mathfrak{m}$ *is nondegenerate and invariant, i.e.,* $B(XY, Z)=B(X, YZ)$. *Moreover every* ad $\mathfrak{h}$*-stable ideal* $\mathfrak{n}$ *of* $\mathfrak{m}$ *satisfies* $[\mathfrak{n}, \mathfrak{n}^\perp]=0$ *where* $\mathfrak{n}^\perp=\{X\in\mathfrak{m}\mid B(X, \mathfrak{n})=0\}$.

Proof. For $X, Y, Z \in \mathfrak{m}$ we have:

$$B(XY, Z) = B([X, Y]-\mathfrak{h}(X, Y), Z) = B([X, Y], Z) = B(X, [Y, Z]) = B(X, YZ+\mathfrak{h}(Y, Z)) = B(X, YZ).$$

Now $B(\mathfrak{n}^{\perp}, \mathfrak{n})=0$ implies that $0=B(\mathfrak{n}^{\perp}, \mathfrak{n}\mathfrak{m})=B(\mathfrak{n}\mathfrak{n}^{\perp}, \mathfrak{m})$. And $B(\mathfrak{m}, \mathfrak{h})=0$ implies that $B(\mathfrak{n}\mathfrak{n}^{\perp}, \mathfrak{h})=0$. Thus $B(\mathfrak{n}\mathfrak{n}^{\perp}, \mathfrak{g})=0$ and $\mathfrak{n}\mathfrak{n}^{\perp}=0$. Consequently $[\mathfrak{n}, \mathfrak{n}^{\perp}]=\mathfrak{h}(\mathfrak{n}, \mathfrak{n}^{\perp}) \subseteq \mathfrak{h}$ and $B([\mathfrak{n}, \mathfrak{n}^{\perp}], \mathfrak{m})=0$. But we also have $B([\mathfrak{n}, \mathfrak{n}^{\perp}], \mathfrak{h})=B(\mathfrak{n}^{\perp}, [\mathfrak{n}\mathfrak{h}])=B(\mathfrak{n}^{\perp}, \mathfrak{n})=0$. Thus $B([\mathfrak{n}, \mathfrak{n}^{\perp}], \mathfrak{g})=0$ and $0=[\mathfrak{n}, \mathfrak{n}^{\perp}]=\mathfrak{h}(\mathfrak{n}, \mathfrak{n}^{\perp})$.

THEOREM 1. *Let* $\mathfrak{g}$ *be a split simple Lie algebra. Let* $\mathfrak{h}$ *be a reductive subalgebra of* $\mathfrak{g}$ *which is normalized by a split Cartan subalgebra* $\mathfrak{c}$ *of* $\mathfrak{g}$ (*i.e.*, $\mathfrak{h}$ *is reductive and regular* [2]). *Then* $\mathfrak{h}$ *has an* $\operatorname{ad}(\mathfrak{c}+\mathfrak{h})$-*stable complement* $\mathfrak{m}$. *Such an* $\mathfrak{m}$ *is either simple or abelian* ($\mathfrak{m}^2=0$).

Proof. We first show that $\mathfrak{c}+\mathfrak{h}$ is reductive. Letting $\mathfrak{g}=\mathfrak{g}_0+\sum \mathfrak{g}_\alpha$ be the root space decomposition of $\mathfrak{g}$, it suffices to show that for $\alpha\neq 0$, $\mathfrak{g}_\alpha\subseteq\mathfrak{c}+\mathfrak{h}$ implies $\mathfrak{g}_{-\alpha}\subseteq\mathfrak{c}+\mathfrak{h}$ [7, p. 669]. Since $[\mathfrak{c}, \mathfrak{h}]\subseteq\mathfrak{h}$ we have $[\mathfrak{c}, \mathfrak{d}]\subseteq\mathfrak{d}$ where $\mathfrak{d}$ is the center of $\mathfrak{h}$. Thus $\mathfrak{c}+\mathfrak{d}$ is solvable. Thus $\operatorname{ad}(\mathfrak{c}+\mathfrak{d})$ is triangulizable and $0=[\operatorname{ad}\mathfrak{c}, \operatorname{ad}\mathfrak{d}]=\operatorname{ad}[\mathfrak{c}, \mathfrak{d}]$ since $\operatorname{ad}[\mathfrak{c}, \mathfrak{d}]\subseteq\operatorname{ad}\mathfrak{d}$ and $\operatorname{ad}\mathfrak{d}$ consists of semisimple transformations. Thus $[\mathfrak{c}, \mathfrak{d}]=0$ and $\mathfrak{d}\subseteq\mathfrak{c}=\mathfrak{g}_0$. Now $\mathfrak{h}=\mathfrak{d}\oplus\mathfrak{h}^{(1)}$ with $\mathfrak{h}^{(1)}$ semisimple, since $\mathfrak{h}$ is reductive. Let α be a nonzero root such that $\mathfrak{g}_\alpha\subseteq\mathfrak{c}+\mathfrak{h}$. Then since $\mathfrak{h}^{(1)}$ is $\operatorname{ad}\mathfrak{c}$-stable and $\mathfrak{c}+\mathfrak{h}=\mathfrak{g}_0+\mathfrak{d}+\mathfrak{h}^{(1)}=\mathfrak{g}_0+\mathfrak{h}^{(1)}$, we have $\mathfrak{g}_\alpha\subseteq\mathfrak{h}^{(1)}$. Now the restriction of the Killing form $B(\ ,\ )$ of $\mathfrak{g}$ to $\mathfrak{h}^{(1)}$ is nondegenerate since it is the trace form of a faithful representation of the semisimple Lie algebra $\mathfrak{h}^{(1)}$ (see [3, p. 69]). Thus $B(\mathfrak{g}_\alpha, \mathfrak{h}^{(1)})\neq 0$. Since $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta)=0$ for $\alpha+\beta\neq 0$, it follows $\mathfrak{g}_{-\alpha}\subseteq\mathfrak{h}^{(1)}$. Thus $\mathfrak{g}_\alpha\subseteq\mathfrak{c}+\mathfrak{h}$ implies $\mathfrak{g}_{-\alpha}\subseteq\mathfrak{c}+\mathfrak{h}$ and $\mathfrak{c}+\mathfrak{h}$ is reductive.

It follows that $\mathfrak{h}$ has a complement $\mathfrak{m}$ stable under $\operatorname{ad}(\mathfrak{c}+\mathfrak{h})$. Any complement $\mathfrak{m}$ is the sum of $\mathfrak{m}\cap\mathfrak{g}_0$ and those root spaces $\mathfrak{g}_\beta$ not occurring in $\mathfrak{h}$. In particular, $\mathfrak{g}_\alpha\subseteq\mathfrak{m}$ implies $\mathfrak{g}_{-\alpha}\subseteq\mathfrak{m}$.

We now show that such an $\mathfrak{m}$ is either simple or abelian. Assume that $\mathfrak{m}^2\neq 0$ and $\mathfrak{m}$ not simple. Then by Lemma 1, $\mathfrak{m}$ has a proper $\operatorname{Der}\mathfrak{m}$-stable ideal. Since $\mathfrak{m}$ is $\operatorname{ad}(\mathfrak{c}+\mathfrak{h})$-stable, $\operatorname{ad}(\mathfrak{c}+\mathfrak{h})$ consists of derivations of $\mathfrak{m}$. Thus $\mathfrak{m}$ has a proper ideal $\mathfrak{n}$ stable under $\operatorname{ad}(\mathfrak{c}+\mathfrak{h})$.

Let σ be an automorphism of $\mathfrak{g}$ such that $\sigma|\mathfrak{c}=-id_{\mathfrak{c}}$ and $\mathfrak{g}_\alpha^\sigma=\mathfrak{g}_{-\alpha}$ for all α (see [3, p. 127]). Then the above discussion shows that $\mathfrak{m}$ and $\mathfrak{h}$ are σ-stable. It follows that $(XY)^\sigma=X^\sigma Y^\sigma$ and $(\mathfrak{h}(X, Y))^\sigma=\mathfrak{h}(X^\sigma, Y^\sigma)$. Thus $\sigma|\mathfrak{m}$ is an automorphism of $\mathfrak{m}$ and $\mathfrak{n}^\sigma$ is an ideal of $\mathfrak{m}$. Since $[\mathfrak{n}^\sigma, \mathfrak{c}+\mathfrak{h}]=[\mathfrak{n}^\sigma, (\mathfrak{c}+\mathfrak{h})^\sigma]=[\mathfrak{n}, \mathfrak{c}+\mathfrak{h}]^\sigma\subseteq\mathfrak{n}^\sigma$, $\mathfrak{n}^\sigma$ is also $\operatorname{ad}(\mathfrak{c}+\mathfrak{h})$-stable.

Suppose that one of the ideals $\mathfrak{n}\cap\mathfrak{n}^\sigma$, $\mathfrak{n}+\mathfrak{n}^\sigma$ is proper in $\mathfrak{m}$. Call it $\mathfrak{p}$. Then $\mathfrak{p}$ is the sum of $\mathfrak{p}\cap\mathfrak{g}_0$ and root spaces $\mathfrak{g}_\alpha$. Moreover $\mathfrak{g}_\alpha\subseteq\mathfrak{p}$ implies $\mathfrak{g}_{-\alpha}\subseteq\mathfrak{p}$. It follows that $\mathfrak{m}=\mathfrak{m}\cap\mathfrak{g}_0+\mathfrak{p}+\mathfrak{p}^\perp$ where $\mathfrak{p}^\perp=\{X\in\mathfrak{m} \mid B(X, \mathfrak{p})=0\}$ (thus $\mathfrak{g}_\alpha\subseteq\mathfrak{m}-\mathfrak{g}_0$ and $\mathfrak{g}_\alpha\nsubseteq\mathfrak{p}$ implies $\mathfrak{g}_{-\alpha}\nsubseteq\mathfrak{p}$ which implies $B(\mathfrak{g}_\alpha, \mathfrak{p})=0$). We use this to show that $\mathfrak{q}=\mathfrak{p}+\mathfrak{h}(\mathfrak{p}, \mathfrak{p})$ is an ideal of $\mathfrak{g}$. By Lemma 2 it suffices to show that $[\mathfrak{p}, \mathfrak{p}']\subseteq\mathfrak{q}$ where $\mathfrak{p}'=\mathfrak{p}^\perp+\mathfrak{m}\cap\mathfrak{g}_0$. But $[\mathfrak{p}, \mathfrak{m}\cap\mathfrak{g}_0]\subseteq[\mathfrak{p}, \mathfrak{c}]\subseteq\mathfrak{p}$. Thus it suffices to show that $[\mathfrak{p}, \mathfrak{p}^\perp]$

$\subseteq \mathfrak{q}$. But $B([\mathfrak{p}, \mathfrak{p}^\perp], \mathfrak{c}+\mathfrak{h})=B(\mathfrak{p}^\perp, [\mathfrak{p}, \mathfrak{c}+\mathfrak{h}])=B(\mathfrak{p}^\perp, \mathfrak{p})=0$ and $[\mathfrak{p}, \mathfrak{p}^\perp]\subseteq(\mathfrak{c}+\mathfrak{h})^\perp\subseteq\mathfrak{m}$. Thus $\mathfrak{h}(\mathfrak{p}, \mathfrak{p}^\perp)\subseteq[\mathfrak{p}, \mathfrak{p}^\perp]+\mathfrak{p}\mathfrak{p}^\perp\subseteq\mathfrak{m}$ and $\mathfrak{h}(\mathfrak{p}, \mathfrak{p}^\perp)=0$. Thus $[\mathfrak{p}, \mathfrak{p}^\perp]=\mathfrak{p}\mathfrak{p}^\perp\subseteq\mathfrak{p}\subseteq\mathfrak{q}$ and $\mathfrak{q}$ is an ideal of $\mathfrak{g}$. Thus $\mathfrak{q}=\mathfrak{g}$ and $\mathfrak{n}$ cannot be proper in $\mathfrak{m}$, a contradiction.

Thus we have $\mathfrak{n}\cap\mathfrak{n}^\sigma=0$ and $\mathfrak{n}+\mathfrak{n}^\sigma=\mathfrak{m}$. Thus $\mathfrak{n}\cap\mathfrak{g}_0=(\mathfrak{n}\cap\mathfrak{g}_0)^\sigma=0$ (since $\sigma|\mathfrak{g}_0=-\mathrm{id}_{\mathfrak{g}_0}$). Thus $\mathfrak{m}\cap\mathfrak{g}_0=\mathfrak{n}\cap\mathfrak{g}_0+(\mathfrak{n}\cap\mathfrak{g}_0)^\sigma=0$. It follows that $B(\mathfrak{m}, \mathfrak{h})=0$ (e.g., $\mathfrak{m}=\sum_{\alpha\in S}\mathfrak{g}_\alpha$ for some set S of nonzero roots, and $\alpha\in S$ implies $-\alpha\in S$ which implies $\mathfrak{g}_{-\alpha}\not\subseteq\mathfrak{h}$ and therefore $B(\mathfrak{g}_\alpha, \mathfrak{h})=0$). Also $B(\mathfrak{n}, \mathfrak{n})=0$ (e.g., $\mathfrak{n}=\sum_{\alpha\in T}\mathfrak{g}_\alpha$ for some set T of nonzero roots, and $\alpha\in T$ implies $-\alpha\notin T$ which implies $B(\mathfrak{g}_\alpha, \mathfrak{n})=0$). It follows from Lemma 3 that $[\mathfrak{n}, \mathfrak{n}]=\mathfrak{n}\mathfrak{n}=\mathfrak{h}(\mathfrak{n}, \mathfrak{n})=0$. Thus $\mathfrak{n}^\sigma\mathfrak{n}^\sigma=0$. Finally $\mathfrak{m}^2=(\mathfrak{n}+\mathfrak{n}^\sigma)^2=\mathfrak{n}^2+\mathfrak{n}\mathfrak{n}^\sigma+(\mathfrak{n}^\sigma)^2\subseteq 0+\mathfrak{n}\cap\mathfrak{n}^\sigma+0=0$, a contradiction.

3. **The semisimple case.** We now consider the reductive pair $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{g}$ is a simple Lie algebra and $\mathfrak{h}$ is a semisimple Lie subalgebra. We note that the Killing form $B(\ ,\ )$ of $\mathfrak{g}$ restricted to $\mathfrak{h}$ is nondegenerate. For if $U, V\in\mathfrak{h}$, then $B(U, V)=\mathrm{tr}\,\mathrm{ad}_\mathfrak{g} U\,\mathrm{ad}_\mathfrak{g} V$ is the trace form of the representation ad $\mathfrak{h}$ in $\mathfrak{g}$, and is nondegenerate by Cartan's criterion [3, p. 69]. (Note that $\mathrm{ad}_\mathfrak{g} U=0$ implies UF is a one-dimensional ideal in the simple algebra $\mathfrak{g}$ so that $U=0$.) Thus if $\mathfrak{h}^\perp=\{X\in\mathfrak{g} \mid B(X, \mathfrak{h})=0\}$, then $\mathfrak{h}\cap\mathfrak{h}^\perp=0$ and therefore $\mathfrak{g}=\mathfrak{h}^\perp+\mathfrak{h}$. And $B([\mathfrak{h}^\perp, \mathfrak{h}], \mathfrak{h})=B(\mathfrak{h}^\perp, [\mathfrak{h}, \mathfrak{h}])=0$ so that for $\mathfrak{m}=\mathfrak{h}^\perp$, $(\mathfrak{g}, \mathfrak{h})$ is a reductive pair with (fixed) decomposition $\mathfrak{g}=\mathfrak{m}\dot{+}\mathfrak{h}$. Note that since $\mathfrak{m}=\mathfrak{h}^\perp$, the Killing form B, restricted to $\mathfrak{m}$, is a nondegenerate invariant form, i.e., $B(XY, Z)=B(X, YZ)$.

THEOREM 2. *Let* $\mathfrak{g}$ *be a simple Lie algebra and* $\mathfrak{h}$ *a semisimple subalgebra. Then* $(\mathfrak{g}, \mathfrak{h})$ *is a reductive pair with* $\mathfrak{m}=\mathfrak{h}^\perp$. *Furthermore* $\mathfrak{m}^2=0$ *or* $\mathfrak{m}$ *is simple.*

Proof. Assume $\mathfrak{m}^2\neq 0$. Then we have from Lemma 1 that $\mathfrak{m}$ has a minimal proper ad $\mathfrak{h}$-stable ideal $\mathfrak{n}$. Then since B is a nondegenerate invariant form on $\mathfrak{m}$ and $B([XU], Y)=B(X, [UY])$ for $X, Y\in\mathfrak{m}$, $U\in\mathfrak{h}$, we have $\mathfrak{n}^\perp=\{X\in\mathfrak{m} \mid B(X, \mathfrak{n})=0\}$ is an ad $\mathfrak{h}$-stable ideal of $\mathfrak{m}$. Thus $\mathfrak{n}\cap\mathfrak{n}^\perp$ is an ad $\mathfrak{h}$-stable ideal of $\mathfrak{m}$; and since $\mathfrak{n}$ is minimal, either $\mathfrak{n}\cap\mathfrak{n}^\perp=0$ or $\mathfrak{n}\cap\mathfrak{n}^\perp=\mathfrak{n}$.

In case $\mathfrak{n}\cap\mathfrak{n}^\perp=0$ we have $\mathfrak{m}=\mathfrak{n}\oplus\mathfrak{n}^\perp$. And we know from Lemma 3 that $[\mathfrak{n}, \mathfrak{n}^\perp]=0$. Thus $\mathfrak{q}=\mathfrak{n}\dot{+}\mathfrak{h}(\mathfrak{n}, \mathfrak{n})$ is a proper ideal of $\mathfrak{g}$ by Lemma 2. This contradiction shows we must have $\mathfrak{n}\cap\mathfrak{n}^\perp=\mathfrak{n}$.

In the case $\mathfrak{n}\cap\mathfrak{n}^\perp=\mathfrak{n}$ we can find an ad $\mathfrak{h}$-stable complement, $\mathfrak{n}'$ (since ad $\mathfrak{h}$ is semisimple and therefore completely reducible); and we write $\mathfrak{m}=\mathfrak{n}\dot{+}\mathfrak{n}'$. Thus since $B(\mathfrak{n}, \mathfrak{n})=0$, to show that $\mathfrak{n}=0$ it suffices to show $B(\mathfrak{n}, \mathfrak{n}')=0$.

To find a formula for $B(X, Y)$ with $X, Y\in\mathfrak{m}$, define $\varepsilon(X)$ and $\delta(X)$ by

$$\varepsilon(X)\colon \mathfrak{m}\to\mathfrak{h}\colon \qquad Y\to\mathfrak{h}(X, Y)\equiv\varepsilon(X)(Y),$$
$$\delta(X)\colon \mathfrak{h}\to\mathfrak{m}\colon \qquad U\to[X, U]\equiv\delta(X)(U),$$

where $U\in\mathfrak{h}$. Using these maps we have for any $Z, X\in\mathfrak{m}$, $U\in\mathfrak{h}$ that

$$\begin{aligned}(\mathrm{ad}_\mathfrak{g} Z)(X) &= [Z, X] = ZX+\mathfrak{h}(Z, X)\\ &= (L(Z)+\varepsilon(Z))(X)\\ (\mathrm{ad}_\mathfrak{g} Z)(U) &= [Z, U] = \delta(Z)(U)\end{aligned}$$

and therefore

$$\operatorname{ad}_{\mathfrak{g}} Z = \begin{pmatrix} L(Z) & \varepsilon(Z) \\ \delta(Z) & 0 \end{pmatrix}.$$

From this, note that since $\mathfrak{g}$ is simple $0 = \operatorname{tr} \operatorname{ad}_{\mathfrak{g}} Z = \operatorname{tr} L(Z)$. Also since $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}]$ is semisimple, and since $\mathfrak{h} \to \operatorname{ad}_{\mathfrak{m}} \mathfrak{h}$: $U \to \operatorname{ad}_{\mathfrak{m}} U$ and $\mathfrak{h} \to \operatorname{ad}_{\mathfrak{h}} \mathfrak{h}$: $U \to \operatorname{ad}_{\mathfrak{h}} U$ are representations of $\mathfrak{h}$, we have $\operatorname{tr} \operatorname{ad}_{\mathfrak{m}} U = \operatorname{tr} \operatorname{ad}_{\mathfrak{h}} U = 0$ for all $U \in \mathfrak{h}$.

Next for $X, Y \in \mathfrak{m}$ define the linear transformation $\sigma(X, Y)\colon \mathfrak{m} \to \mathfrak{m}$ by $\sigma(X, Y) = \delta(X)\varepsilon(Y)$, that is, $\sigma(X, Y)Z = [X, \mathfrak{h}(Y, Z)]$ $(= [\mathfrak{h}(Z, Y), X])$. From (3) we have the identity

$$\operatorname{ad}_{\mathfrak{m}} \mathfrak{h}(X, Y) - \sigma(X, Y) + \sigma(Y, X) = [L(X), L(Y)] - L(XY)$$

and therefore $\operatorname{tr} \sigma(X, Y) = \operatorname{tr} \sigma(Y, X)$. From this and the matrix for $\operatorname{ad}_{\mathfrak{g}} Z$ we obtain for $X, Y \in \mathfrak{m}$ that

$$\begin{aligned} B(X, Y) &= \operatorname{tr} \operatorname{ad}_{\mathfrak{g}} X \operatorname{ad}_{\mathfrak{g}} Y \\ &= \operatorname{tr} L(X)L(Y) + \operatorname{tr} \varepsilon(X)\delta(Y) + \operatorname{tr} \delta(X)\varepsilon(Y) \\ &= \operatorname{tr} L(X)L(Y) + \operatorname{tr} \delta(Y)\varepsilon(X) + \operatorname{tr} \delta(X)\varepsilon(Y) \\ &= \operatorname{tr} L(X)L(Y) + \operatorname{tr} \sigma(Y, X) + \operatorname{tr} \sigma(X, Y) \\ &= \operatorname{tr} L(X)L(Y) + 2 \operatorname{tr} \sigma(X, Y), \end{aligned}$$

using for the third equality that if $S \in \operatorname{Hom}(V, W)$ and $T \in \operatorname{Hom}(W, V)$ for vector spaces V and W, then $\operatorname{tr} ST = \operatorname{tr} TS$.

Now recall that in the decomposition $\mathfrak{m} = \mathfrak{n} + \mathfrak{n}'$ we must show $B(\mathfrak{n}, \mathfrak{n}') = 0$. Thus for $X \in \mathfrak{n}$, $Y \in \mathfrak{n}'$ we have (from the fact that $\mathfrak{n}$ is an ideal and $\mathfrak{n}\mathfrak{n} = 0$) the matrices

$$L(X) = \begin{pmatrix} 0 & 0 \\ X_{21} & 0 \end{pmatrix} \quad \text{and} \quad L(Y) = \begin{pmatrix} Y_{11} & 0 \\ Y_{21} & Y_{22} \end{pmatrix}$$

and therefore $\operatorname{tr} L(X)L(Y) = 0$ and $B(X, Y) = 2 \operatorname{tr} \sigma(X, Y)$.

To find the matrix for $\sigma(X, Y)$ (with $X \in \mathfrak{n}$, $Y \in \mathfrak{n}'$) let $Z \in \mathfrak{n}$, $Z' \in \mathfrak{n}'$. Then

$$\begin{aligned} \sigma(X, Y)Z &= [\mathfrak{h}(Z, Y), X] \in \mathfrak{n}, \\ \sigma(X, Y)Z' &= [\mathfrak{h}(Z', Y), X] \in \mathfrak{n}. \end{aligned}$$

Therefore

$$\sigma(X, Y) = \begin{pmatrix} \sigma_{11} & 0 \\ \sigma_{21} & 0 \end{pmatrix}$$

and $\operatorname{tr} \sigma(X, Y) = \operatorname{tr} \sigma_{11} = \operatorname{tr}_{\mathfrak{n}} \sigma(X, Y)$. To find the action of $\sigma(X, Y)$ on $\mathfrak{n}$ again let $Z \in \mathfrak{n}$. Then since $\mathfrak{n}$ is an ideal, $\mathfrak{n}\mathfrak{n} = 0$ and $\mathfrak{h}(\mathfrak{n}, \mathfrak{n}) = 0$, we have from (3) that

$$\begin{aligned} 0 = J(Z, X, Y) &= [Z, \mathfrak{h}(X, Y)] + [X, \mathfrak{h}(Y, Z)] \\ &= [-\operatorname{ad}_{\mathfrak{n}} \mathfrak{h}(X, Y) + \sigma(X, Y)]Z. \end{aligned}$$

Therefore on $\mathfrak{n}$ we have $\sigma(X, Y) = \operatorname{ad}_{\mathfrak{n}} \mathfrak{h}(X, Y)$ and since $U \to \operatorname{ad}_{\mathfrak{n}} U$ is a representation of the semisimple Lie algebra $\mathfrak{h}$, $0 = \operatorname{tr} \operatorname{ad}_{\mathfrak{n}} \mathfrak{h}(X, Y) = \operatorname{tr}_{\mathfrak{n}} \sigma(X, Y)$. Thus $B(\mathfrak{n}, \mathfrak{n}') = 0$ and $\mathfrak{m}$ is simple, a contradiction. Thus either $\mathfrak{m}^2 = 0$ or $\mathfrak{m}$ is simple.

4. **Remarks.** (i) The above discussion for $\mathfrak{h}$ semisimple holds for $\mathfrak{h}$ reductive in $\mathfrak{g}$ except for the assertion that $\operatorname{tr} \operatorname{ad}_{\mathfrak{n}} \mathfrak{h}(X, Y)=0$ and its consequences. The authors do not know whether the theorem holds for all reductive $\mathfrak{h}$.

(ii) If $\mathfrak{h}$ is the zero-space of a derivation of $\mathfrak{g}$ or the one-space of an automorphism of $\mathfrak{g}$, then $\mathfrak{h}$ is reductive and contains a regular element of $\mathfrak{g}$ [1]. Thus if $\mathfrak{g}$ is simple and the underlying field algebraically closed, the associated $\mathfrak{m}$ is simple or abelian by Theorem 1.

(iii) It would be of value to determine all pairs $(\mathfrak{g}, \mathfrak{h})$ with $\mathfrak{g}$ semisimple for which an associated $\mathfrak{m}$ is simple. We now give an example of one nontrivial such pair $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{g}$ is not simple. Thus let $\mathfrak{g}=\mathfrak{g}_1 \oplus \mathfrak{g}_2$ (direct) where the $\mathfrak{g}_i$ $(i=1, 2)$ are real compact simple Lie algebras. Suppose that $\mathfrak{b}$ is a simple subalgebra of $\mathfrak{g}_1$, $\mathfrak{b}'$ a simple subalgebra of $\mathfrak{g}_2$, $B \to B'$ an isomorphism from $\mathfrak{b}$ onto $\mathfrak{b}'$. Let $\mathfrak{h} =\{B+B' \mid B \in \mathfrak{b}\}$ and $m=\mathfrak{h}^{\perp}$. Then $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{b}$, and $\mathfrak{b}'$ can easily be chosen such that $\mathfrak{m}^2 \neq 0$. We claim that for any such choice, $\mathfrak{m}$ is simple. By Lemma 1, it suffices to show that $\mathfrak{m}$ has no proper ad $\mathfrak{h}$-stable ideal. If $\mathfrak{n}$ were such an ideal, then since the Killing form is negative definite on $\mathfrak{g}$, $\mathfrak{m}=\mathfrak{n} \oplus \mathfrak{n}^{\perp}$. It is now clear that $\mathfrak{n}+\mathfrak{h}(\mathfrak{n}, \mathfrak{n})$ is an ideal of $\mathfrak{g}$ by Lemma 2, since $[\mathfrak{n}, \mathfrak{n}^{\perp}]=0$ by Lemma 3. But then $\mathfrak{n}+\mathfrak{h}(\mathfrak{n}, \mathfrak{n})=\mathfrak{g}_1$ or $\mathfrak{g}_2$. But by construction, $\mathfrak{h} \cap \mathfrak{g}_1=\mathfrak{h} \cap \mathfrak{g}_2=0$. Thus $\mathfrak{n}=\mathfrak{g}_1$ or $\mathfrak{g}_2$. This is impossible since $B(\mathfrak{n}, \mathfrak{h})=0$ whereas $B(\mathfrak{g}_i, \mathfrak{h}) \neq 0$ for $i=1, 2$.

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