ON HOMOGENEOUS SPACES AND REDUCTIVE SUBALGEBRAS OF SIMPLE LIE ALGEBRAS

BY

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1. Introduction. Let $G$ be a connected Lie group and $H$ a closed subgroup. Then the homogeneous space $M = G/H$ is called reductive if in the Lie algebra $\mathfrak{g}$ of $G$ there exists a subspace $\mathfrak{m}$ such that $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ (subspace direct sum) and $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$ where $\mathfrak{h}$ is the Lie algebra of $H$ (see [4], [5]). In this case the pair $(\mathfrak{g}, \mathfrak{h})$ is called a reductive pair and the subspace $\mathfrak{m}$ can be made into an anti-commutative algebra as follows. For $X, Y \in \mathfrak{m}$ let $[X, Y] = XY + \mathfrak{h}(X, Y)$ where $XY = [X, Y]_{\mathfrak{m}}$ (resp. $\mathfrak{h}(X, Y) = [X, Y]_{\mathfrak{h}}$) is the projection of $[X, Y]$ in $\mathfrak{g}$ into $\mathfrak{m}$ (resp. $\mathfrak{h}$). This algebra is related to the canonical $G$-invariant connection $\nabla$ of the first kind on $G/H$ by $\nabla_{\mathfrak{m}(Y^*)} = \frac{1}{2} XY$ where $P_0 = H \in M$ (see [5, Theorem 10.1]).

For a fixed decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, the Lie algebra identities of $\mathfrak{g}$ yield the following identities for $\mathfrak{m}$ and $\mathfrak{h}$. For $X, Y, Z \in \mathfrak{m}$ and $U \in \mathfrak{h}$,

1. $XY = -YX$ (bilinear);
2. $\mathfrak{h}(X, Y) = -\mathfrak{h}(Y, X)$ (bilinear);
3. $[Z, \mathfrak{h}(X, Y)] + [X, \mathfrak{h}(Y, Z)] + [Y, \mathfrak{h}(Z, X)] = J(X, Y, Z) = (XY)Z + (YZ)X + (ZX)Y$;
4. $\mathfrak{h}(XY, Z) + \mathfrak{h}(YZ, X) + \mathfrak{h}(ZX, Y) = 0$;
5. $\mathfrak{h}([X, Y, U]) = \mathfrak{h}([X, U], Y) + \mathfrak{h}(X, [Y, U]);$

In particular (6) says the mappings $\text{ad}_m U : m \to m : X \to [U, X]$ are derivations of the algebra $m$. Using these identities, there was established in [6] a correspondence between simple algebras $m$ and holonomy irreducible simply connected spaces $G/H$ which are not symmetric ($m^2 = 0$ if and only if $G/H$ is a symmetric space); for example, if $G/H$ is riemannian, then $G/H$ is holonomy irreducible if and only if $m$ is a simple algebra.

In this paper, we consider pairs $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{g}$ is a simple Lie algebra over a field $F$ of characteristic zero and $\mathfrak{h}$ is either semisimple, or regular and reductive (see [2]). In each case we show that the associated $m$ is either simple or abelian ($m^2 = 0$). This together with [6] shows in particular that if $G$ is a simple connected Lie group and $H$ a closed semisimple or regular reductive Lie subgroup of $G$ such that $G/H$ is simply connected, then either $G/H$ is a symmetric space or $G/H$ is holonomy irreducible. This is a reasonable account of the situation since it can be shown that

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if $G/H$ is a holonomy irreducible pseudo-riemannian reductive space with $G$ simple, then $\mathfrak{h}$ is a reductive subalgebra of $\mathfrak{g}$.

2. The regular reductive case.

**Lemma 1.** Let $\mathfrak{a}$ be a nonassociative algebra with derivation algebra $\text{Der} \, \mathfrak{a}$. Assume that $\mathfrak{a}$ has no proper ideal stable under $\text{Der} \, \mathfrak{a}$. Then either $\mathfrak{a}$ is simple or $\mathfrak{a}^2 = 0$.

**Proof.** Assume $\mathfrak{a}^2 \neq 0$ and let $\mathfrak{X}(\mathfrak{a})$ denote the associative algebra generated by the left and right multiplications of $\mathfrak{a}$ [3, p. 290]. Let $R$ be the radical of $\mathfrak{X}(\mathfrak{a})$. Then $Ra$ is an ideal of $\mathfrak{a}$ since $\mathfrak{X}(\mathfrak{a})(Ra) \subseteq (\mathfrak{X}(\mathfrak{a})R)a \subseteq Ra$. If $D \in \text{Der} \, \mathfrak{a}$, then $[D, \mathfrak{X}(\mathfrak{a})] \subseteq \mathfrak{X}(\mathfrak{a})$ since $ad_{\text{Hom}(\mathfrak{a}, \mathfrak{a})} D$ stabilizes the set of right and left multiplications (e.g., $[D, L(A)] = L([D, A])$ where $L(B)$ denotes left multiplication by $B$ in $\mathfrak{a}$). Thus $ad_{\text{Hom}(\mathfrak{a}, \mathfrak{a})} D$ is a derivation of $\mathfrak{X}(\mathfrak{a})$ and it follows that $[D, R] \subseteq R$ [3, p. 30, exercise 22]. Thus $D(Ra) \subseteq [D, R]a + R(Da) \subseteq Ra$. Thus $Ra$ is a $\text{Der} \, \mathfrak{a}$-stable ideal of $\mathfrak{a}$. By assumption, we must have $Ra = \mathfrak{a}$ or $Ra = 0$. If $Ra = \mathfrak{a}$, then for some $i$, $0 = Ra = R^{-1}a = \ldots = Ra = a$ and $a = 0$. Thus we may assume that $Ra = 0$. Then $R = 0$ and $\mathfrak{X}(\mathfrak{a})$ is completely reducible on $\mathfrak{a}$. $\mathfrak{a}^2$ is clearly $\text{Der} \, \mathfrak{a}$-stable. Assuming that $\mathfrak{a}^2 \neq 0$, we must have $\mathfrak{a}^2 = \mathfrak{a}$ by hypothesis. We claim that $\mathfrak{a}^2 = \mathfrak{a}$ implies that $\mathfrak{a}$ is simple. For if $\mathfrak{b}$ were a proper ideal of $\mathfrak{a}$, then $\mathfrak{b}$ would be $\mathfrak{X}(\mathfrak{a})$-stable and hence $\mathfrak{a} = \mathfrak{b} \oplus \mathfrak{b}'$ for some $\mathfrak{X}(\mathfrak{a})$-stable $\mathfrak{b}'$. This $\mathfrak{b}'$ would be an ideal and $\mathfrak{a} = \mathfrak{a}^2 = \mathfrak{b}^2 + (\mathfrak{b}')^2$ shows that $\mathfrak{b}^2 = 0$. But then $\mathfrak{b} = \mathfrak{b}^2$ would be $\text{Der} \, \mathfrak{a}$-stable since for $B_1, B_2$ in $\mathfrak{b}$, $D(B_1B_2) = (DB_1)B_2 + B_1(DB_2)$ e $\mathfrak{b}$. Thus $\mathfrak{a}$ is simple.

We now consider reductive pairs $(\mathfrak{g}, \mathfrak{h})$. Thus let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{g}$, $\mathfrak{m}$ a complementary subspace of $\mathfrak{h}$ in $\mathfrak{g}$ such that $[\mathfrak{m}, \mathfrak{h}] \subseteq \mathfrak{m}$. For $X, Y \in \mathfrak{m}$ we define $XY$ in $\mathfrak{m}$ and $\mathfrak{h}(X, Y)$ in $\mathfrak{h}$ by requiring that $[XY] = XY + \mathfrak{h}(X, Y)$. We regard $\mathfrak{m}$ as a nonassociative algebra with respect to the product $XY$. Then $\mathfrak{m}$ is clearly anti-commutative and $ad_{\mathfrak{m}} U$ is a derivation of $\mathfrak{m}$ for $U \in \mathfrak{h}$ (by (6)).

**Lemma 2.** Let $\mathfrak{m}$ be an $ad_{\mathfrak{h}}$-stable ideal of $\mathfrak{m}$. Let $\mathfrak{q} = \mathfrak{m} + \mathfrak{h}(\mathfrak{n}, \mathfrak{n})$. If $[\mathfrak{n}, \mathfrak{n}'] \subseteq \mathfrak{q}$ for some complementary subspace $\mathfrak{n}'$ of $\mathfrak{n}$ in $\mathfrak{m}$, then $\mathfrak{q}$ is an ideal of $\mathfrak{g}$.

**Proof.** $[\mathfrak{q}, \mathfrak{n}] \subseteq [\mathfrak{n}, \mathfrak{n}] + [\mathfrak{h}(\mathfrak{n}, \mathfrak{n}), \mathfrak{n}] \subseteq \mathfrak{m} + \mathfrak{h}(\mathfrak{n}, \mathfrak{n}) + \mathfrak{n}$ by (3) since $\mathfrak{n}$ is ad $\mathfrak{h}$-stable. Thus $[\mathfrak{q}, \mathfrak{n}] \subseteq \mathfrak{q}$. And $[\mathfrak{q}, \mathfrak{h}] \subseteq \mathfrak{q}$ since $\mathfrak{n}$ is ad $\mathfrak{h}$-stable and $\mathfrak{q} = \mathfrak{m} + [\mathfrak{n}, \mathfrak{n}]$. It remains to show that $[\mathfrak{q}, \mathfrak{n}'] \subseteq \mathfrak{q}$. But we have $[\mathfrak{q}, \mathfrak{n}'] \subseteq \mathfrak{m} + \mathfrak{h}(\mathfrak{n}, \mathfrak{n}') + [\mathfrak{h}(\mathfrak{n}, \mathfrak{n}), \mathfrak{n}']$, $[\mathfrak{h}(\mathfrak{n}, \mathfrak{n}), \mathfrak{n}'] \subseteq [\mathfrak{m}, \mathfrak{n}'] + [[\mathfrak{n}, \mathfrak{n}], \mathfrak{n}'] \subseteq [\mathfrak{n}, \mathfrak{n}'] + [\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}']]$, $\mathfrak{h}(\mathfrak{n}, \mathfrak{n}') \subseteq \mathfrak{m} + [\mathfrak{n}, \mathfrak{n}']$. But since $[\mathfrak{n}, \mathfrak{n}'] \subseteq \mathfrak{q}$ by hypothesis, $\mathfrak{q}$ contains $[\mathfrak{h}(\mathfrak{n}, \mathfrak{n}), \mathfrak{n}']$ (using (3)) and $\mathfrak{h}(\mathfrak{n}, \mathfrak{n}')$. Since $\mathfrak{m} \subseteq \mathfrak{n} (\mathfrak{n}$ is an ideal of $\mathfrak{m}$), $[\mathfrak{q}, \mathfrak{n}'] \subseteq \mathfrak{q}$. Thus $\mathfrak{q}$ is an ideal of $\mathfrak{g}$.

**Lemma 3.** Suppose that the Killing form $B(\ , \ )$ of $\mathfrak{g}$ is nondegenerate and that $B(\mathfrak{m}, \mathfrak{h}) = 0$. Then $B(\ , \ )|\mathfrak{m}$ is nondegenerate and invariant, i.e., $B(XY, Z) = B(X, YZ)$. Moreover every $ad_{\mathfrak{h}}$-stable ideal $\mathfrak{n}$ of $\mathfrak{m}$ satisfies $[\mathfrak{n}, \mathfrak{n}'] = 0$ where $\mathfrak{n}' = \{X \in \mathfrak{m} \mid B(X, \mathfrak{n}) = 0\}$. 
Proof. For $X, Y, Z \in m$ we have:

$$B(XY, Z) = B([X, Y] - \h(X, Y), Z) = B([X, Y], Z) = B(X, [Y, Z]) = B(X, YZ).$$

Now $B(n^\perp, n) = 0$ implies that $0 = B(n^\perp, nm) = B(nn^\perp, m)$. And $B(m, \h) = 0$ implies that $B(nm^\perp, \h) = 0$. Thus $B(nm^\perp, g) = 0$ and $nm^\perp = 0$. Consequently $[n, n^\perp] = \h(n, n^\perp) \subseteq \h$ and $B([n, n^\perp], m) = 0$. But we also have $B([n, n^\perp], \h) = B(n^\perp, [\h, n]) = B(n^\perp, n) = 0$. Thus $B([n, n^\perp], g) = 0$ and $0 = [n, n^\perp] = \h(n, n^\perp)$.

Theorem 1. Let $g$ be a split simple Lie algebra. Let $\h$ be a reductive subalgebra of $g$ which is normalized by a split Cartan subalgebra $c$ of $g$ (i.e., $\h$ is reductive and regular [2]). Then $\h$ has an $\text{ad}(c + \h)$-stable complement $m$. Such an $m$ is either simple or abelian ($m^2 = 0$).

Proof. We first show that $c + \h$ is reductive. Letting $g = g_0 + \sum g_{\alpha}$ be the root space decomposition of $g$, it suffices to show that for $\alpha \neq 0$, $g_\alpha \subseteq c + \h$ implies $g_{-\alpha} \subseteq c + \h$ [7, p. 669]. Since $[c, \h] \subseteq \h$ we have $[c, b] \subseteq b$ where $b$ is the center of $\h$. Thus $c + b$ is solvable. Thus $\text{ad}(c + b)$ is triangulizable and $0 = [\text{ad} c, \text{ad} b] = \text{ad}[c, b]$ since $\text{ad}[c, b] \subseteq \text{ad} b$ and $\text{ad} b$ consists of semisimple transformations. Thus $[c, b] = 0$ and $b \subseteq c = g_0$. Now $\h = b \oplus \h(1)$ with $\h(1)$ semisimple, since $\h$ is reductive. Let $\alpha$ be a nonzero root such that $g_\alpha \subseteq c + \h$. Then since $\h(1)$ is $\text{ad} c$-stable and $c + \h = g_0 + b + \h(1) = g_0 + \h(1)$, we have $g_{-\alpha} \subseteq \h(1)$. Now the restriction of the Killing form $B( , )$ of $g$ to $\h(1)$ is nondegenerate since it is the trace form of a faithful representation of the semisimple Lie algebra $\h(1)$ (see [3, p. 69]). Thus $B(gh, \h(1)) = 0$. Since $B(gh, \h(1)) = 0$ for $\alpha + \beta \neq 0$, it follows $g_{-\alpha} \subseteq \h(1)$. Thus $g_{-\alpha} \subseteq c + \h$ implies $g_{-\alpha} \subseteq c + \h$ and $c + \h$ is reductive.

It follows that $\h$ has a complement $m$ stable under $\text{ad}(c + \h)$. Any complement $m$ is the sum of $m \cap g_0$ and those root spaces $g_{\alpha}$ not occurring in $\h$. In particular, $g_{-\alpha} \subseteq m$ implies $g_{-\alpha} \subseteq m$.

We now show that such an $m$ is either simple or abelian. Assume that $m^2 \neq 0$ and $m$ not simple. Then by Lemma 1, $m$ has a proper $\text{Der} m$-stable ideal. Since $m$ is $\text{ad}(c + \h)$-stable, $\text{ad}(c + \h)$ consists of derivations of $m$. Thus $m$ has a proper ideal $n$ stable under $\text{ad}(c + \h)$.

Let $\sigma$ be an automorphism of $g$ such that $\sigma|c = -id_c$ and $g_{-\alpha} = g_{-\alpha}$ for all $\alpha$ (see [3, p. 127]). Then the above discussion shows that $m$ and $\h$ are $\sigma$-stable. It follows that $(XY)^\sigma = X^\sigma Y^\sigma$ and $(\h(X, Y))^{\sigma} = \h(X^\sigma, Y^\sigma)$. Thus $\sigma|m$ is an automorphism of $m$ and $n^\sigma$ is an ideal of $m$. Since $[n^\sigma, c + \h] = [n^\sigma, (c + \h)^0] = [n, c + \h]^0 \subseteq n^\sigma$, $n^\sigma$ is also $\text{ad}(c + \h)$-stable.

Suppose that one of the ideals $n \cap n^\sigma$, $n + n^\sigma$ is proper in $m$. Call it $\nu$. Then $\nu$ is the sum of $\nu \cap g_0$ and root spaces $g_\alpha$. Moreover $g_{-\alpha} \subseteq \nu$ implies $g_{-\alpha} \subseteq \nu$. It follows that $m = m \cap g_0 + \nu + n^\perp$ where $\nu^\perp = \{X \in m \mid B(X, \nu) = 0\}$ (thus $g_{-\alpha} \subseteq m - g_0$ and $g_{-\alpha} \subseteq \nu$ implies $g_{-\alpha} \subseteq \nu$ which implies $B(g_{-\alpha}, \nu) = 0$). We use this to show that $g = \nu + \h(\nu, \nu)$ is an ideal of $g$. By Lemma 2 it suffices to show that $[\nu, \nu'] \subseteq \nu$ where $\nu' = \nu^\perp + m \cap g_0$. But $[\nu, m \cap g_0] \subseteq [\nu, c] \subseteq \nu$. Thus it suffices to show that $[\nu, \nu^\perp]$. 


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\[ B([p, p], c + h) = B(p, c + h) = B(p, p) = 0 \]
Thus \( h(v, v) \in \mathfrak{g} \) and \( h(p, p) = 0 \). Thus \( [p, p] = p^2 \leq q \) and \( q \)

is an ideal of \( g \). Thus \( q = g \) and \( n \) cannot be proper in \( m \), a contradiction.

Thus we have \( n \cap n^2 = 0 \) and \( n + n^2 = m \). Thus \( n \cap g_0 = (n \cap g_0)^2 = 0 \) (since \( \sigma|g_0 = -id_{g_0} \)). Thus \( m \cap g_0 = n \cap g_0 + (n \cap g_0)^2 = 0 \). It follows that \( B(m, m) = 0 \) (e.g., \( m = \sum_{\alpha \in S} g_\alpha \) for some set \( S \) of nonzero roots, and \( \alpha \in S \) implies \( -\alpha \in S \) which implies \( g_\alpha \not= g_\alpha \) and therefore \( B(g_\alpha, h) = 0 \). Also \( B(n, n) = 0 \) (e.g., \( n = \sum_{T \in T} g_\alpha \) for some set \( T \) of nonzero roots, and \( \alpha \in T \) implies \( -\alpha \in T \) which implies \( B(g_\alpha, n) = 0 \)).

It follows from Lemma 3 that \( [n, n] = n \oplus b(n, n) = 0 \). Thus \( n^2 = 0 \). Finally \( m^2 = (n + n^2)^2 = n^2 + n^3 + (n^2)^2 \leq 0 + n \cap n^2 + 0 = 0 \), a contradiction.

3. The semisimple case. We now consider the reductive pair \((g, h)\) where \( g \) is a simple Lie algebra and \( h \) is a semisimple Lie subalgebra. We note that the Killing form \( B(\ , \ ) \) of \( g \) restricted to \( h \) is nondegenerate. For if \( U, V \in h \), then \( B(U, V) = \text{tr} \, \text{ad}_h \, U \, \text{ad}_h \, V \) is the trace form of the representation \( \text{ad}_h \) in \( g \), and is nondegenerate by Cartan's criterion [3, p. 69]. (Note that \( \text{ad}_h \, U = 0 \) implies \( UF \) is a one-dimensional ideal in the simple algebra \( g \) so that \( U = 0 \).) Thus if \( h^1 = \{ X \in g \mid B(X, h) = 0 \} \), then \( h \cap h^1 = 0 \) and therefore \( g = h^1 + h \). And \( B([h^1, h], h) = B(h^1, [h, h]) = 0 \) so that for \( m = h^1 \), \( (g, h) \) is a reductive pair with (fixed) decomposition \( g = m \oplus h \). Note that since \( m = h^1 \), the Killing form \( B \), restricted to \( m \), is a nondegenerate invariant form, i.e., \( B(XY, Z) = B(X, YZ) \).

Theorem 2. Let \( g \) be a simple Lie algebra and \( h \) a semisimple subalgebra. Then \( (g, h) \) is a reductive pair with \( m = h^1 \). Furthermore \( m^2 = 0 \) or \( m \) is simple.

Proof. Assume \( m^2 \neq 0 \). Then we have from Lemma 1 that \( m \) has a minimal proper ad \( h \)-stable ideal \( n \). Then since \( B \) is a nondegenerate invariant form on \( m \) and \( B([XU], Y) = B(X, [UY]) \) for \( X, Y \in m, U \in h \), we have \( n^2 = \{ X \in m \mid B(X, n) = 0 \} \) is an ad \( h \)-stable ideal of \( m \). Thus \( n \cap n^2 \) is an ad \( h \)-stable ideal of \( m \); and since \( n \) is minimal, either \( n \cap n^2 = 0 \) or \( n \cap n^2 = n \).

In case \( n \cap n^2 = 0 \) we have \( m = n \oplus n^2 \). And we know from Lemma 3 that \([n, n^2] = 0\). Thus \( q = n \oplus h(n, n) \) is a proper ideal of \( g \) by Lemma 2. This contradiction shows we must have \( n \cap n^2 = n \).

In the case \( n \cap n^2 = n \) we can find an ad \( h \)-stable complement, \( n' \) (since ad \( h \) is semisimple and therefore completely reducible); and we write \( m = n \oplus n' \). Thus since \( B(n, n) = 0 \), to show that \( n = 0 \) it suffices to show \( B(n, n') = 0 \).

To find a formula for \( B(X, Y) \) with \( X, Y \in m \), define \( \epsilon(X) \) and \( \delta(X) \) by
\[
\epsilon(X) : m \to h; \quad Y \to \delta(X)(Y) = \epsilon(X)(Y), \\
\delta(X) : h \to m; \quad U \to [X, U] = \delta(X)(U),
\]
where \( U \in h \). Using these maps we have for any \( Z, X \in m, U \in h \) that
\[
(\text{ad}_h Z)(X) = [Z, X] = ZX + h(Z, X) = (L(Z) + \epsilon(Z))(X), \\
(\text{ad}_h Z)(U) = [Z, U] = \delta(Z)(U)
\]
and therefore

\[ \text{ad}_g Z = \begin{pmatrix} L(Z) & e(Z) \\ \delta(Z) & 0 \end{pmatrix}. \]

From this, note that since \( g \) is simple \( 0 = \text{tr} \, \text{ad}_g Z = \text{tr} \, L(Z) \). Also since \( \mathfrak{h} = [\mathfrak{h}, \mathfrak{h}] \) is semisimple, and since \( \mathfrak{h} \to \text{ad}_m \mathfrak{h}: U \to \text{ad}_m U \) and \( \mathfrak{h} \to \text{ad}_g \mathfrak{h}: U \to \text{ad}_g U \) are representations of \( \mathfrak{h} \), we have \( \text{tr} \, \text{ad}_m U = \text{tr} \, \text{ad}_g U = 0 \) for all \( U \in \mathfrak{h} \).

Next for \( X, Y \in m \) define the linear transformation \( \sigma(X, Y): m \to m \) by \( \sigma(X, Y) = \delta(X)e(Y) \), that is, \( \sigma(X, Y)Z = [X, \delta(Y, Z)] (=[\mathfrak{h}(Z, Y), X]) \). From (3) we have the identity

\[ \text{ad}_m \mathfrak{h}(X, Y) - \sigma(X, Y) + \sigma(Y, X) = [L(X), L(Y)] - L(XY) \]

and therefore \( \text{tr} \, \sigma(X, Y) = \text{tr} \, \sigma(Y, X) \). From this and the matrix for \( \text{ad}_g Z \) we obtain for \( X, Y \in m \) that

\[ B(X, Y) = \text{tr} \, \text{ad}_g X \text{ad}_g Y \]

\[ = \text{tr} \, L(X)L(Y) + \text{tr} \, e(X)\delta(Y) + \text{tr} \, \delta(X)e(Y) \]

\[ = \text{tr} \, L(X)L(Y) + \text{tr} \, \delta(Y)e(X) + \text{tr} \, \delta(X)e(Y) \]

\[ = \text{tr} \, L(X)L(Y) + \text{tr} \, \sigma(Y, X) + \text{tr} \, \sigma(X, Y) \]

\[ = \text{tr} \, L(X)L(Y) + 2 \text{tr} \, \sigma(X, Y), \]

using for the third equality that if \( S \in \text{Hom}(V, W) \) and \( T \in \text{Hom}(W, V) \) for vector spaces \( V \) and \( W \), then \( \text{tr} \, ST = \text{tr} \, TS \).

Now recall that in the decomposition \( m = n + n' \) we must show \( B(n, n') = 0 \). Thus for \( X \in n, \ Y \in n' \) we have (from the fact that \( n \) is an ideal and \( nn = 0 \)) the matrices

\[ L(X) = \begin{pmatrix} 0 & 0 \\ X_{21} & 0 \end{pmatrix} \quad \text{and} \quad L(Y) = \begin{pmatrix} Y_{11} & 0 \\ Y_{21} & Y_{22} \end{pmatrix} \]

and therefore \( \text{tr} \, L(X)L(Y) = 0 \) and \( B(X, Y) = 2 \text{tr} \, \sigma(X, Y) \).

To find the matrix for \( \sigma(X, Y) \) (with \( X \in n, \ Y \in n' \)) let \( Z \in n, \ Z' \in n' \). Then

\[ \sigma(X, Y)Z = [\mathfrak{h}(Z, Y), X] \in n, \]

\[ \sigma(X, Y)Z' = [\mathfrak{h}(Z', Y), X] \in n. \]

Therefore

\[ \sigma(X, Y) = \begin{pmatrix} \sigma_{11} & 0 \\ \sigma_{21} & 0 \end{pmatrix} \]

and \( \text{tr} \, \sigma(X, Y) = \text{tr} \, \sigma_{11} = \text{tr}_n \sigma(X, Y) \). To find the action of \( \sigma(X, Y) \) on \( n \) again let \( Z \in n \). Then since \( n \) is an ideal, \( nn = 0 \) and \( \mathfrak{h}(n, n) = 0 \), we have from (3) that

\[ 0 = J(Z, X, Y) = [Z, \mathfrak{h}(X, Y)] + [X, \mathfrak{h}(Y, Z)] \]

\[ = [-\text{ad}_n \mathfrak{h}(X, Y) + \sigma(X, Y)]Z. \]

Therefore on \( n \) we have \( \sigma(X, Y) = \text{ad}_n \mathfrak{h}(X, Y) \) and since \( U \to \text{ad}_n U \) is a representation of the semisimple Lie algebra \( \mathfrak{h} \), \( 0 = \text{tr} \, \text{ad}_n \mathfrak{h}(X, Y) = \text{tr}_n \sigma(X, Y) \). Thus \( B(n, n') = 0 \) and \( m \) is simple, a contradiction. Thus either \( m^2 = 0 \) or \( m \) is simple.
4. Remarks. (i) The above discussion for \( \mathfrak{h} \) semisimple holds for \( \mathfrak{h} \) reductive in \( \mathfrak{g} \) except for the assertion that \( \text{tr} \text{ ad}_n \mathfrak{h}(X, Y) = 0 \) and its consequences. The authors do not know whether the theorem holds for all reductive \( \mathfrak{h} \).

(ii) If \( \mathfrak{h} \) is the zero-space of a derivation of \( \mathfrak{g} \) or the one-space of an automorphism of \( \mathfrak{g} \), then \( \mathfrak{h} \) is reductive and contains a regular element of \( \mathfrak{g} \) [1]. Thus if \( \mathfrak{g} \) is simple and the underlying field algebraically closed, the associated \( \mathfrak{m} \) is simple or abelian by Theorem 1.

(iii) It would be of value to determine all pairs \( (\mathfrak{g}, \mathfrak{h}) \) with \( \mathfrak{g} \) semisimple for which an associated \( \mathfrak{m} \) is simple. We now give an example of one nontrivial such pair \( (\mathfrak{g}, \mathfrak{h}) \) where \( \mathfrak{g} \) is not simple. Thus let \( \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \) (direct) where the \( \mathfrak{g}_i \) \((i=1, 2)\) are real compact simple Lie algebras. Suppose that \( \mathfrak{b} \) is a simple subalgebra of \( \mathfrak{g}_1 \), \( \mathfrak{b}' \) a simple subalgebra of \( \mathfrak{g}_2 \), \( B \to B' \) an isomorphism from \( \mathfrak{b} \) onto \( \mathfrak{b}' \). Let \( \mathfrak{h} = \{B + B' \mid B \in \mathfrak{b} \} \) and \( \mathfrak{m} = \mathfrak{h}^\perp \). Then \( \mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{b}, \text{ and } \mathfrak{b}' \) can easily be chosen such that \( \mathfrak{m}^2 \neq 0 \). We claim that for any such choice, \( \mathfrak{m} \) is simple. By Lemma 1, it suffices to show that \( \mathfrak{m} \) has no proper \( \text{ad} \mathfrak{h} \)-stable ideal. If \( \mathfrak{n} \) were such an ideal, then since the Killing form is negative definite on \( \mathfrak{g} \), \( \mathfrak{m} = \mathfrak{n} \oplus \mathfrak{n}^\perp \). It is now clear that \( \mathfrak{n} + \mathfrak{h}(\mathfrak{n}, \mathfrak{n}) \) is an ideal of \( \mathfrak{g} \) by Lemma 2, since \( [\mathfrak{n}, \mathfrak{n}^\perp] = 0 \) by Lemma 3. But then \( \mathfrak{n} + \mathfrak{h}(\mathfrak{n}, \mathfrak{n}) = \mathfrak{g}_1 \) or \( \mathfrak{g}_2 \). But by construction, \( \mathfrak{h} \cap \mathfrak{g}_1 = \mathfrak{h} \cap \mathfrak{g}_2 = 0 \). Thus \( \mathfrak{n} = \mathfrak{g}_1 \) or \( \mathfrak{g}_2 \). This is impossible since \( B(\mathfrak{n}, \mathfrak{h}) = 0 \) whereas \( B(\mathfrak{g}_i, \mathfrak{h}) \neq 0 \) for \( i = 1, 2 \).

Bibliography


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