

# ON A CONSTRUCTIVE DEFINITION OF THE RESTRICTED DENJOY INTEGRAL

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1. **Introduction.** Let  $I = [a, b]$  and  $F$  be a function of the closed subintervals of  $I$ . One says that  $F$  has Burkill integral  $\int_J F$  over the closed interval  $J \subseteq I$  if

$$\int_J F = \lim_{P \in \mathcal{P}, |P| \rightarrow 0} \sum_{I' \in P} F(I'),$$

where  $\mathcal{P}$  is the set of partitions of  $J$  and  $|P|$  is the norm of  $P$ . The Burkill integral has been employed in a more general setting [1] to give a descriptive definition of the restricted Denjoy integral of point functions  $f$ . In this paper we show how this integral can be used to give a constructive definition of the restricted Denjoy integral and compare the classical construction with ours. We adopt the convention that  $I$  and  $J$ , with or without subscripts or superscripts, always denote a closed interval.

Before we begin our discussion, let us recall the classical constructive definition ([2], 255–259) of the restricted Denjoy integral.

Let  $T$  be a real-valued function whose domain,  $\text{dom } T$ , is a set of ordered pairs  $\{(f, J)\}$ , where  $f$  is a real-valued point function defined on  $J$ . The set

$$\{f: (f, J) \in \text{dom } T\}$$

will be denoted by  $\text{dom}_J T$ .

$T$  is called an *integral* if

(i)  $f \in \text{dom}_J T$  implies  $f \in \text{dom}_{J'} T$  for all closed subintervals  $J' \subseteq J$ , and  $T(f, J')$  is an additive, continuous function of  $J'$ ;

(ii) if  $f \in \text{dom}_{I_i} T$ ,  $i = 1, 2$ , where  $I_1$  and  $I_2$  are abutting, then  $f \in \text{dom}_{I_1 \cup I_2} T$ ;

(iii) if  $f \equiv 0$  on  $I$ , then  $f \in \text{dom}_I T$  and  $T(f, I) = 0$ .

One says that  $f$  is  $T$ -integrable on  $I$  if  $(f, I) \in \text{dom } T$ .

Two integrals  $T_1$  and  $T_2$  are *compatible* if  $T_1(f, I') = T_2(f, I')$  whenever they both exist. We say  $T_1 \subseteq T_2$  if  $T_1$  and  $T_2$  are compatible and  $\text{dom } T_1 \subseteq \text{dom } T_2$ .

Given a function  $f$  defined on  $I' \subseteq I$  and an integral  $T$ , one says that a point  $x \in I'$  is a  $T$ -singular point of  $f$  in  $I'$  if there exists  $\{I_n\}$ , with  $I_n \subseteq I'$ ,  $|I_n| \rightarrow 0$ ,  $x \in I_n$ , such that  $(f, I_n) \notin \text{dom } T$ .

If  $\mathcal{S}$  is the set of  $T$ -singular points in  $I$ , clearly  $\mathcal{S}$  is closed and  $(f, I') \in \text{dom } T$  for all  $I' \subseteq I$  such that  $I' \cap \mathcal{S} = \emptyset$ .

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Let  $T$  be an integral. One defines  $\text{dom}_I T^C$  by the following conditions:

(c<sub>1</sub>)  $\mathcal{S} \cap I$  is finite or void;

(c<sub>2</sub>) there exists a continuous, additive  $F$  such that  $F(I') = T(f, I')$  whenever  $(f, I') \in \text{dom } T, I' \subseteq I$ .

Define  $T^C(f, I) = F(I)$ . It is clear that  $T^C$  is an integral.

Let  $E \subseteq I$ . Let  $f_E = f\chi_E$ , where  $\chi_E$  is the characteristic function of  $E$ . One says  $f$  is  $T$ -integrable on  $E$  if  $f_E$  is  $T$ -integrable on  $I$ .

One defines  $\text{dom}_I T^H$  by the following conditions:

(h<sub>1</sub>)  $f$  is  $T$ -integrable on  $\mathcal{S}$  and on each of the intervals  $I_k$  contiguous to  $\mathcal{S} \cup \{a, b\}$ .

(h<sub>2</sub>)  $\sum O(T; f; I_k) < \infty$ , where  $O(T; f; J) = \sup_{J' \subseteq J} |T(f, J')|$ .

Define  $T^H(f, I) = T(f, \mathcal{S}) + \sum T(f, I_k)$ . Clearly  $T^H$  is also an integral.

Let  $\{T^\alpha\}$  be a sequence of integrals, in general transfinite, such that  $T^\alpha \subseteq T^\beta$  whenever  $\alpha < \beta$ . Define  $\text{dom}_{\sum_{\beta < \alpha} T^\beta} = \bigcup_{\beta < \alpha} \text{dom } T^\beta$  and if  $(f, I') \in \text{dom}_{\sum_{\beta < \alpha} T^\beta}$ , define  $(\sum_{\beta < \alpha} T^\beta)(f, I') = T^{\beta_0}(f, I')$ , where  $\beta_0$  is the least ordinal such that  $(f, I') \in \text{dom } T^{\beta_0}$ .

Write  $T^{CH} = (T^C)^H$ . We define a transfinite sequence  $\{D^\alpha\}$  of integrals as follows: let  $\mathcal{L}$  be the Lebesgue integral,

$$D^0 = \mathcal{L},$$

$$D^\alpha = \left( \sum_{\beta < \alpha} D^\beta \right)^{CH}$$

if  $\alpha > 0$ . Let  $\Omega$  be the first uncountable ordinal. Then it is well known ([2], 258) that if  $D_*$  is the restricted Denjoy integral,

$$D_* = \sum_{\alpha < \Omega} D^\alpha.$$

**2. A constructive definition using the Burkill integral.** We define  $\text{dom}_I T^{H*}$  by the following conditions:

(h<sub>1</sub><sup>\*</sup>) there exists a closed set  $W \subseteq I$  such that  $f$  is  $T$ -integrable on  $W$  and on each  $I' \subseteq I$  with  $I' \cap W = \emptyset$ ;

(h<sub>2</sub><sup>\*</sup>) if we define

$$\begin{aligned} \psi(I') &= T(f, I') \quad \text{if } I' \cap W = \emptyset, \\ &= 0 \quad \text{if } I' \cap W \neq \emptyset, \end{aligned}$$

then  $\int_I \psi$  exists and  $\int \psi$  is continuous (note that if  $\int_I F$  exists, then  $\int_{I'} F$  exists for all  $I' \subseteq I$  and  $\int_I F$  is additive).

Define  $T^{H*}(f, I) = T(f_w, I) + \int_I \psi$ . We note that  $T^{H*}$  is an integral. For suppose that  $f \in \text{dom}_I T^{H*}$ . Let  $J' \subseteq J$ . If  $J' \cap W = \emptyset$ , then  $f \in \text{dom}_{J'} T$  so that, taking  $W' = J'$  in the definition of  $\text{dom}_{J'} T^{H*}$ , we see that  $f \in \text{dom}_{J'} T^{H*}$ . If  $J' \cap W \neq \emptyset$ , then since  $f_w \in \text{dom}_I T$  and  $T$  is an integral,  $f_w \in \text{dom}_{J'} T$ . Now, since  $\int_I \psi$  exists, so does  $\int_{J'} \psi$  ([1], p. 70). Recall that continuity of  $\int \psi$  is assumed in the definition

of  $\text{dom}_J T^{H^*}$ . Therefore, in either case  $f \in \text{dom}_J T^{H^*}$ . Since  $T$  is an integral,  $T(f_W, J')$  is an additive, continuous function of the  $J' \subseteq J$ ; it is known ([1], p. 70) that  $\int \psi$  is additive. Therefore  $T^{H^*}(f, J')$  is an additive, continuous function of the  $J' \subseteq J$ , and condition (i) is satisfied. Now suppose  $f \in \text{dom}_{I_i} T^{H^*}$ ,  $i=1, 2$ , where  $I_1$  and  $I_2$  are abutting. With no loss in generality we may assume  $I_1$  is to the left of  $I_2$ . Choose  $W_i \subseteq I_i$ ,  $i=1, 2$ , closed sets satisfying the requirements of the definition of  $f \in \text{dom}_{I_i} T^{H^*}$ , and let  $\psi_i$  be the functions corresponding to  $T^{H^*}$ ,  $f$  and  $W_i$ ,  $i=1, 2$ , in this definition. Set  $a_1 = \sup W_1$ ,  $b_1 = \inf W_2$ . If  $a_1 = b_1$ , choose

$$W = W_1 \cup W_2.$$

Then  $W$  is closed and, since  $T$  is an integral, by condition (ii),  $f_W \in \text{dom}_{I_1 \cup I_2} T$ . Let  $\epsilon > 0$  and choose  $\eta_i = \eta_i(\epsilon) > 0$ ,  $i=1, 2$ , so that if  $P_i$  is a partition of  $I_i$ ,  $i=1, 2$ , with  $|P_i| < \eta_i$ , then  $|\sum_{I' \in P_i} \psi_i(I') - \int_{I_i} \psi_i| < \epsilon/8$ , and also so that if  $\{J_1, \dots, J_p\}$  is any finite sequence of nonoverlapping subintervals of  $I_1$  with  $\text{Max } |J_k| < \eta_1$ , then  $\sum_1^p |\psi_1(J_k) - \int_{J_k} \psi_1| < \epsilon/8$  (see [1], p. 70). Also, since  $\int \psi_i$  is additive and continuous on  $I_k$ ,  $i=1, 2$ , we may choose  $\eta$ ,  $0 < \eta < \min(\eta_1, \eta_2)$ , so that  $J \subseteq I_i$ ,  $|J| < \eta$  imply that  $|\int_J \psi_i| < \epsilon/8$ . Now let  $P$  be any partition of  $I_1 \cup I_2$  with  $|P| < \eta$ . Then

$$\left| \sum_{I' \in P} \psi(I') - \left( \int_{I_1} \psi_1 + \int_{I_2} \psi_2 \right) \right| < \epsilon.$$

It therefore follows that  $\int_{I_1 \cup I_2} \psi = \int_{I_1} \psi_1 + \int_{I_2} \psi_2$ . It is immediate that  $\int \psi$  is continuous on  $I_1 \cup I_2$ . If  $a_1 < b_1$ , choose  $W = W_1 \cup W_2 \cup [a_1, b_1]$ . A similar argument shows that, again, condition (ii) is satisfied. Finally, if  $f \equiv 0$  on  $I$ , then since  $T$  is an integral,  $f \in \text{dom}_I T$  and  $T(f, I) = 0$ . Choosing  $W = I$ , we see that  $f \in \text{dom}_I T^{H^*}$  and  $T^{H^*}(f, I) = 0$ . Therefore  $T^{H^*}$  is an integral. By a known result ([1], 90), if  $T \subseteq D_*$ , then  $T^{H^*} \subseteq D_*$ .

Romanovski ([1], 91) has defined the following sequence  $\{K_\alpha\}$  of classes of  $D_*$ -integrable functions defined on  $I$ . Put  $f \in K_0$  if  $f$  is Lebesgue integrable on  $I$ . Let  $\alpha > 0$ . Then we put  $f \in K_\alpha$  if one of the following four conditions is satisfied:

- (1) there is a partition  $P$  such that  $f$  is in a class  $K_\beta$ , with  $\beta < \alpha$ , on each member of  $P$ ;
- (2)  $f$  can be extended to a function which belongs to a class  $K_\beta$ , with  $\beta < \alpha$ , on some  $I' \supseteq I$ ;
- (3)  $f$  belongs to a class  $K_\beta$ , with  $\beta < \alpha$ , on each closed sub-interval of  $I^0 = (a, b)$ ;
- (4) there is a closed set  $W$  such that  $f_W$  is Lebesgue integrable on  $I$ , and such that, on each  $I' \subseteq I$  with  $I' \cap W = \emptyset$ ,  $f$  belongs to a class  $K_\beta$ , with  $\beta < \alpha$ ; in addition, if

$$\begin{aligned} \psi(I') &= (D_*) \int_{I'} f dx && \text{if } I' \cap W = \emptyset, \\ &= 0 && \text{if } I' \cap W \neq \emptyset, \end{aligned}$$

then  $\int_I \psi$  exists and  $\int \psi$  is continuous. It is known ([1], 91) that if  $f$  is  $D_*$ -integrable, then  $f$  belongs to some class  $K_\alpha$ .

We define a transfinite sequence  $\{D_*^\alpha\}$  of integrals as follows:

$$D_*^0 = \mathcal{L},$$

$$D_*^\alpha = \left( \sum_{\beta < \alpha} D_*^\beta \right)^{H^*}$$

if  $\alpha > 0$ .

**THEOREM 1.**  $K_\alpha \subseteq \text{dom}_I D_*^{\alpha+1}$ .

**Proof.** Clearly  $\text{dom}_I D_*^0 = K_0$ . Suppose the theorem is true for all  $\beta < \alpha$ , for some  $\alpha > 0$  and that  $f \in K_\alpha$ . If condition (1) of the definition of  $K_\alpha$  is satisfied, then there is a partition  $P$  of  $I$  such that  $f \in \text{dom}_{I'} D_*^{\beta+1}$  for all  $I' \in P$ , where  $\beta < \alpha$ . By condition (ii) for an integral,  $f \in \text{dom}_I D_*^{\beta+1}$ . Since  $\beta + 1 < \alpha + 1$ ,  $f \in \text{dom}_I D_*^{\alpha+1}$ . If condition (2) of the definition of  $K_\alpha$  is satisfied, then there exists  $I' \supseteq I$  such that  $f \in K_\beta$  on  $I'$ ,  $\beta < \alpha$ . But then by the induction hypothesis,  $f \in \text{dom}_{I'} D_*^{\beta+1}$  and by condition (i) for integrals,  $f \in \text{dom}_I D_*^{\beta+1}$ . Since  $\beta + 1 < \alpha + 1$ , again  $f \in \text{dom}_I D_*^{\alpha+1}$ . Now suppose condition (4) of the definition of  $K_\alpha$  is satisfied. Then there is a closed set  $W \subseteq I$  such that  $f_W \in \text{dom}_I \mathcal{L}$ , and such that, on each  $I' \subseteq I$  with  $I' \cap W = \emptyset$ ,  $f \in \text{dom}_{I'} D_*^{\beta+1}$ , where  $\beta < \alpha$ ; in addition,  $\int_I \psi$  exists and  $\int \psi$  is continuous. It follows that  $f \in \text{dom}_I D_*^{\alpha+1}$ .

We finally show that if condition (3) of the definition is satisfied, then  $f$  is  $D_*^{\alpha+1}$ -integrable. Now  $f \in \text{dom}_{I'} D_*^\beta$ , with  $\beta < \alpha + 1$ , for all  $I' \subseteq (a, b)$ . Let

$$F = (D_*) \int f dx.$$

Then  $F$  is continuous on  $I$ . In  $(h_1^*)$  and  $(h_2^*)$  choose  $T = D_*^\alpha$ . Note that  $F(I') = T(f, I')$  for all  $I' \subseteq I^0$ . Let  $W = \{a, b\}$  and  $\epsilon > 0$ . Choose  $\delta > 0$  such that  $I' \subseteq I$ ,  $|I - I'| < \delta$  imply  $|F(I) - F(I')| < \epsilon$ . Let  $P$  be any partition of  $I$  with  $|P| < \delta/2$ . Then

$$\left| \sum_{I' \in P} \psi(I') - F(I) \right| = \left| \sum_{I' \in P'} F(I') - F(I) \right| < \epsilon,$$

where  $P'$  is the set of those members of  $P$  which contain neither  $a$  nor  $b$ . It follows that  $\int_I \psi = F(I)$ . Clearly  $\int \psi$  is continuous. Therefore

$$K_\alpha \subseteq \text{dom } D_*^{\alpha+1}.$$

**THEOREM 2.**  $D_* = \sum_{\alpha < \Omega} D_*^\alpha$ .

**Proof.** Let  $f$  be  $D_*$ -integrable on  $I$ . Define  $f = 0$  outside of  $I$ . Let  $S^\alpha$  denote the set of  $D_*^\alpha$ -singular points of  $f$  in  $I$ . Then  $\{S^\alpha\}$  is decreasing and, thus, stationary (i.e., there is an  $\alpha_0 < \Omega$  such that  $S^\alpha = S^{\alpha_0}$  for all  $\alpha > \alpha_0$ ; see, e.g., [2], 258). However,  $f \in K_\beta$  for some  $\beta$  and, therefore,  $f \in \text{dom}_I D_*^{\beta+1}$ . Thus  $S^{\beta+1} = \emptyset$ , which implies  $S^{\alpha_0} = \emptyset$  and  $f \in \text{dom}_I D_*^{\alpha_0}$ . We have observed above that  $T \subseteq D_*$  implies  $T^{H^*} \subseteq D_*$ . To show that  $D_* \supseteq \sum_{\alpha < \Omega} D_*^\alpha$  it is easy to see that if  $D_*^\beta \subseteq D_*$  for all  $\beta < \alpha$ , then  $\sum_{\beta < \alpha} D_*^\beta \subseteq D_*$ , and that therefore  $(\sum_{\beta < \alpha} D_*^\beta)^{H^*} \subseteq D_*$ . Since  $D_*^0 = \mathcal{L} \subseteq D_*$ , we have the desired containment relation by applying transfinite induction.

3. **Comparison of our construction with the classical construction.** In the following theorem we restrict our attention to those integrals  $T$  which have the property that  $f \in \text{dom}_I T$  whenever  $I \cap \{f \neq 0\}$  is finite.

**THEOREM 3.**  $T^c \subseteq T^{H^*}$ .

**Proof.** Let  $f$  be  $T^c$ -integrable on  $I$ . Let  $\mathcal{S}$  be the set of  $T$ -singular points in  $I$  and  $\{I_1, \dots, I_k\}$  be the set of intervals contiguous to  $\mathcal{S} \cup \{a, b\} = E$ . Let  $F(I') = T^c(f, I')$  for all  $I' \subseteq I$ . Let  $\varepsilon > 0$ . Choose  $\delta > 0$  such that for each  $i$ ,  $I' \subseteq I_i$  and  $|I_i - I'| < \delta$  imply  $|F(I_i) - F(I')| < \varepsilon/k$ . Let  $S_I = \{J_1, \dots, J_m\}$  be any partition of  $I$  with  $|S_I| < \delta/2$ . Define

$$\begin{aligned} \psi(I') &= F(I') \quad \text{if } I' \cap E = \emptyset, \\ &= 0 \quad \quad \text{if } I' \cap E \neq \emptyset, \end{aligned}$$

for all  $I' \subseteq I$ . If  $S \subseteq S_I$  is the subset of  $S_I$  consisting of those  $J_i$  such that  $J_i \cap E = \emptyset$ , we see that, since  $F$  is additive,

$$\left| \sum_{i=1}^m \psi(J_i) - F(I) \right| = \left| \sum_{J_i \in S} \psi(J_i) - \sum F(J_i) \right| < \varepsilon.$$

It follows that  $\int_I \psi = F(I)$ .

We shall show below that the containment guaranteed by Theorem 3 is proper in general, i.e., there are  $T^{H^*}$ -integrable functions which are not  $T^c$ -integrable.

We note that  $(h_1^*)$  and  $(h_2^*)$  do not require that  $f$  be  $T$ -integrable on the intervals  $I_j$  contiguous to  $W$ . Since  $\int_I \psi$  exists,  $\int \psi$  is continuous, and  $T(f, I')$  is an additive function of the closed subintervals of  $I_j^0$  for all  $j$ , it follows that  $\int_{I'} \psi = T(f, I')$  for all such  $I'$ , and therefore  $f$  is  $T^c$ -integrable on  $I_j$  for all  $j$ . Moreover,

$$\int_{I_j} \psi = \lim_{|I_j - I'| \rightarrow 0} T(f, I').$$

The following theorem will immediately establish the inclusion relationships between  $T^H$ ,  $T^{H^*}$ , and  $T^{CH}$ .

**THEOREM 4.** Suppose  $E \subseteq I$  is closed. Let  $\{I_k\}$  be the sequence of intervals contiguous to  $E \cup \{a, b\}$ . Consider the following two conditions:

- (1) for every  $k$ ,  $f \in \text{dom}_{I'} T$  for all  $I' \subseteq I_k^0$  and  $f \in \text{dom}_{I_k} T^c$ ; moreover,  $\sum O(T^c; f; I_k) < \infty$ ;
- (2)  $f$  is  $T$ -integrable on all  $I' \subseteq I$  such that  $I' \cap E = \emptyset$ ; moreover, if

$$\begin{aligned} \psi(I') &= T(f, I') \quad \text{if } I' \cap E = \emptyset, \\ &= 0 \quad \quad \text{if } I' \cap E \neq \emptyset, \end{aligned}$$

then  $\int_I \psi$  exists and  $\int \psi$  is continuous. Then (1) and (2) are equivalent.

**COROLLARY 5.**  $T^H \subseteq T^{H^*}$ .

**COROLLARY 6.**  $T^{H^*} \subseteq T^{CH}$ .

These corollaries follow from a comparison of the conditions in the definitions of  $T^H$ ,  $T^{H^*}$ , and  $T^{CH}$  with conditions (1) and (2) of Theorem 4.

**Proof of Theorem 4.** We show first that (1) implies (2). Let  $\epsilon > 0$ . Choose  $N = N(\epsilon)$  such that

$$\sum_{N+1}^{\infty} O(T^C; f; I_k) < \epsilon/3.$$

Let  $I_k = [a_k, b_k]$  for all  $k$ . Consider  $I_j$  with  $j \leq N$ . By continuity of  $T^C(f, I')$  there is a  $\delta_j = \delta_j(\epsilon) > 0$  such that for  $I' \subseteq I_j$ ,  $|T^C(f, I_j) - T^C(f, I')| < \epsilon/6N$  whenever  $|I_j - I'| < \delta_j$ . In particular, then, if  $J' = [a_j, b']$  and  $J'' = [a'', b_j]$  are contained in  $I_j$ , and if  $b' - a_j < \delta_j$  and  $b_j - a'' < \delta_j$ , then

$$|T^C(f, J')| + |T^C(f, J'')| < \epsilon/3N.$$

Note, also, that since  $T^C(f, I)$  is additive, if  $J' \subseteq I_j$  and  $\{J'_1, \dots, J'_p\}$  is a partition of  $J'$ , then  $\sum T^C(f, J'_i) = T^C(f, J')$ .

Now choose  $\delta = \min_{j=1, \dots, N} \delta_j$ . Suppose  $S = \{J_1, \dots, J_p\}$  is a partition of  $I$  with  $|S| < \delta$ . The only members  $J_i$  of  $S$  for which it is possible that  $\psi(J_i) \neq 0$  are those for which  $J_i \cap E = \emptyset$ . Consider  $I_k$  with  $k \leq N$ . Let  $\{J_1^k, \dots, J_{n(k)}^k\}$  be those  $J_i$  which are contained in  $I_k$ , such that  $J_m^k \cap E = \emptyset$ . Then by the above discussion we see that

$$\left| \sum_{m=1}^{n(k)} \psi(J_m^k) - T^C(f, I_k) \right| < \epsilon/3N.$$

For  $k > N$ , clearly  $|\sum_{m=1}^{n(k)} \psi(J_m^k)| \leq O(T^C; f; I_k)$ . Thus,

$$\begin{aligned} & \left| \sum_{k=1}^p \psi(J_k) - \sum_{k=1}^{\infty} T^C(f, I_k) \right| \\ &= \left| \sum_{k=1}^{\infty} \sum_{m=1}^{n(k)} \psi(J_m^k) - \sum_{k=1}^N T^C(f, I_k) - \sum_{k=N+1}^{\infty} T^C(f, I_k) \right| \\ &\leq \left| \sum_{k=1}^N \left[ \sum_{m=1}^{n(k)} \psi(J_m^k) - T^C(f, I_k) \right] \right| + \sum_{k=N+1}^{\infty} \left| \sum_{m=1}^{n(k)} \psi(J_m^k) \right| + \sum_{k=N+1}^{\infty} |T^C(f, I_k)| \\ &< N\epsilon/3N + 2 \sum_{N+1}^{\infty} O(T^C; f; I_k) < \epsilon. \end{aligned}$$

Thus  $\int_I \psi$  exists and  $\int_I \psi = \sum_1^{\infty} T^C(f, I_k)$ . The continuity of  $\int \psi$  is clear.

We now show that (2) implies (1). Let  $\delta = \delta(1) > 0$  be such that

$$\left| \sum_{I' \in S_I} \psi(I') - \int_I \psi \right| < 1$$

whenever  $S_I$  is a partition of  $I$  with  $|S_I| < \delta$ , and also such that

$$\left| \sum_{I' \in S} \psi(I') - \sum_{I' \in S} \int_{I'} \psi \right| < 1$$

whenever  $S$  is a finite family of nonoverlapping subintervals of  $I$  with  $\sup_{I' \in S} |I'| < \delta$  (see [1], 70).

Choose  $N=N(\delta)$  such that  $\sum_{N+1}^{\infty} |I_k| < \delta/2$ . We shall divide  $\{I_k\}_{N+1}^{\infty}$  into two classes,  $\mathcal{J}^+$  and  $\mathcal{J}^-$ , as follows: put  $I_k$  in  $\mathcal{J}^+$  if there exists  $\{I_j^k\}$ , with  $I_j^k \subseteq I_k$ , such that  $T^c(f, I_j^k) \rightarrow O(T^c; f; I_k)$  and put  $I_k$  in  $\mathcal{J}^-$  otherwise. Write  $\mathcal{J}^+ = \{I_{n_i}\} = \{J_i\}$ . Choose  $J'_i$  with  $J'_i \subseteq J_i^0$ , such that  $O(T^c; f; J_i) - T^c(f, J'_i) < 1/2^i$ . We shall show that  $\sum O(T^c; f; J_i) < \infty$ . A similar argument yields the same inequality for the sum of the oscillations over the members of  $\mathcal{J}^-$ . The choice of  $J'_i$  induces a partition of  $J_i$ ,  $\{J'_i, J''_i, J'''_i\}$ . Append to the family  $\{J_i\}$  the family  $\{I_1, \dots, I_n\}$ . For each  $k=1, \dots, N$ , choose a partition  $\{I_1^k, \dots, I_{n(k)}^k\}$  (enumerated from left to right) of  $I_k$  such that  $|I_j^k| < \delta$  for all  $j$  and such that

$$\left| T^c(f, I_k) - \sum_2^{n(k)-1} T^c(f, I_j^k) \right| < 1/2N.$$

This can be done since  $T^c(f, I)$  is additive and continuous on  $I_k$  for all  $k$ . For any positive integer  $p$ , consider

$$H = I - \left[ \left( \bigcup_1^N I_j \right) \cup \left( \bigcup_1^p J_i \right) \right].$$

If  $H$  is nonvoid, it consists of a finite number of intervals. We shall now form a partition  $S$  of  $I$ . Put  $\{J'_i, J''_i, J'''_i\}_{i=1}^p \cup \{I_1^k, \dots, I_{n(k)}^k\}_{k=1}^N$  into  $S$ . Suppose  $I' = [a', b']$  and  $I'' = [a'', b'']$  are two of these intervals with  $b' < a''$  such that none of the above intervals intersects  $(b', a'')$ . Then, since the sum of the lengths of the contiguous intervals contained in  $H$  is  $< \delta/2$ , it follows that we can partition  $(b', a'')$  by a sequence of intervals all of whose end-points are in  $E$ , and all of which have length  $< \delta/2$ . We put this partition into  $S$ . We do this for all pairs  $I'$  and  $I''$  satisfying the above conditions. We thereby obtain a partition  $S$  of  $I$  such that  $|S| < \delta$ . Then

$$\left| \int_I \psi - \sum_{I' \in S} \psi(I') \right| < 1.$$

But

$$\sum_{I' \in S} \psi(I') = \sum_{j=1}^N \sum_{k=2}^{n(j)-1} T(f, I_k^j) + \sum_{i=1}^p T(f, J'_i).$$

Thus

$$\begin{aligned} \sum_{i=1}^p T(f, J'_i) &\leq \left| \sum_{j=1}^N \sum_{k=2}^{n(j)-1} T(f, I_k^j) \right| + 1 + \left| \int_I \psi \right| \\ &\leq 1 + \left| \int_I \psi \right| + N/2N + \sum_1^N |T^c(f, I_k)| < 2 + \left| \int_I \psi \right| + \sum_1^N |T^c(f, I_k)|. \end{aligned}$$

Since  $O(T^c; f; J_i) - T(f, J'_i) < 1/2^i$  for all  $i$ ,

$$\begin{aligned} \sum_{i=1}^p O(T^c; f; J_i) &\leq \sum_{i=1}^p 1/2^i + 2 + \left| \int_I \psi \right| + \sum_1^N |T^c(f, I_k)| \\ &< 3 + \left| \int_I \psi \right| + \sum_1^N |T^c(f, I_k)|. \end{aligned}$$

Since  $3 + \int_I |\psi| + \sum_1^N |T^C(f, I_k)|$  is an absolute finite constant and  $p$  was arbitrary,  $\sum_1^\infty O(T^C; f; J_i) < \infty$ . As we noted above, a similar argument can be used to establish the result for  $\mathcal{J}^-$ . Thus (2) implies (1).

The following example illustrates the fact that

$$T^H \not\subseteq T^{H^*} \text{ and } T^C \not\subseteq T^{H^*}.$$

Let

$$\{I_n\} = \left\{ \left[ \frac{1}{n+1}, \frac{1}{n} \right] \right\} n = 1, 2, \dots$$

Define

$$\begin{aligned} f(x) &= x^2 \sin \frac{1}{x^2} && \text{if } 0 < x \leq \frac{1}{2}, \\ &= (\frac{1}{2} \sin 4)(1-x) && \text{if } \frac{1}{2} \leq x \leq 1. \end{aligned}$$

Extend  $f$  by the formula

$$f(x+n) = f(x)$$

for all integers  $n$ . Define

$$\begin{aligned} f_n(x) &= \frac{1}{2^{n+1}} f(n(n+1)x) && \text{if } x \in I_n, \\ &= 0 && \text{if } x \notin I_n, \end{aligned}$$

and

$$F(x) = \sum_1^\infty f_n(x)$$

for all  $x \in I$ .  $F(x)$  is clearly continuous on  $I$  and differentiable a.e. Let  $g(x) = F'(x)$ . Let  $E = \{0\} \cup \{1/n\}$ ,  $n = 2, 3, \dots$ . Clearly  $g$  is  $\mathcal{L}^C$ -integrable on  $I_n$  for all  $n$ . By Theorem 4, to show  $g$  is  $\mathcal{L}^{H^*}$ -integrable on  $I$  we need only show that  $\sum O(F; I_k) < \infty$ . Since  $O(F; I_k) < 1/2^k$ , this is immediate. Thus  $g$  is  $\mathcal{L}^{H^*}$ -integrable. However, since  $E$  is the set of  $\mathcal{L}$ -singular points of  $g$ , and since  $g$  is not  $\mathcal{L}$ -integrable on  $I_n$  for any  $n$ , it follows that  $g$  is not  $\mathcal{L}^H$ -integrable. It is clear that  $g$  is not  $\mathcal{L}^C$ -integrable.

Finally we shall show that  $T^{H^*} \not\subseteq T^{CH}$ . Let  $E$  be the Cantor set. Let  $\{I_n\} = \{[a_n, b_n]\}$  be the sequence of intervals contiguous to  $E$ . Let  $f(x)$  be as in the previous example and

$$\begin{aligned} F_n(x) &= \frac{1}{2^{n+1}} f\left(\frac{2^{n+1}(x-a_n)}{|I_n|}\right) && \text{if } x \in I_n, \\ &= 0 && \text{otherwise,} \end{aligned}$$

$n = 1, 2, \dots$  Define

$$F(x) = \sum F_n(x).$$



Clearly  $g(x) = F'(x)$  exists a.e. Moreover, each point of  $E$  is an  $\mathcal{L}^c$ -singular point of  $g$ , and no other point is singular. In addition,  $O(F; I_n) < 1/2^n$ . Thus  $\sum O(F; I_n) < \infty$ , implying that  $g$  is  $\mathcal{L}^{CH}$ -integrable. However,  $I_n$  contains  $2^{n+1} + 1$   $\mathcal{L}$ -singular points, namely  $\{a_n + j(|I_n|/2^{n+1})\} j=0, 1, \dots, 2^{n+1}$ . In addition,

$$O\left(F; \left[ a_n + j \frac{|I_n|}{2^{n+1}}, a_n + (j+1) \frac{|I_n|}{2^{n+1}} \right] \right) = \frac{1}{2^{n+1}} O(f; [0, 1])$$

for all  $j$  and  $n$ . Then, if  $H = E \cup [\bigcup_{n=1}^{\infty} \bigcup_{j=0}^{2^{n+1}-1} \{a_n + j(|I_n|/2^{n+1})\}]$ , we note that  $H$  is closed and that  $\sum O(F; J_k) = \infty$ , where  $\{J_k\}$  is the sequence of intervals contiguous to  $H$ . Moreover, if  $P \subseteq I$  is a closed set such that  $g$  is  $\mathcal{L}$ -integrable on  $P$  and on all  $I' \subseteq I$  with  $I' \cap P = \emptyset$ , then  $\sum O(F; I'_j) = \infty$ , where  $\{I'_j\}$  is the sequence of intervals contiguous to  $P$ . For, let

$$I_{nj} = [a_n + j(|I_n|/2^{n+1}), a_n + (j+1)(|I_n|/2^{n+1})],$$

$j=0, 1, \dots, 2^{n+1} - 1$ , and write  $a_{nj} = a_n + j(|I_n|/2^{n+1})$ . Let  $c_{nj}$  be the midpoint of  $I_{nj}$ . Note that  $F$  is monotone increasing on  $[c_{nj}, a_{n,j+1}]$  and that  $g$  is positive on  $[c_{nj}, a_{n,j+1}]$ . Let  $\{I_{nj}^k\}$  be the sequence of intervals contiguous to  $P \cup \{c_{nj}, a_{n,j+1}\}$  in  $[c_{nj}, a_{n,j+1}]$ . Then either

$$\int_{P \cap [c_{nj}, a_{n,j+1}]} g \, dx \geq \frac{1}{8} |\sin 4|$$

or

$$\sum_k O(F; I_{nj}^k) \geq \frac{1}{8} |\sin 4|.$$

If

$$\int_{P \cap [c_{nj}, a_{n,j+1}]} g \, dx \geq \frac{1}{8} |\sin 4|$$

for infinitely many pairs  $n, j$ , then  $\int_P |g| \, dx = \infty$  contradicting the assumption that  $g$  is  $\mathcal{L}$ -integrable on  $P$ . Thus, assuming that  $g$  is  $\mathcal{L}$ -integrable on  $P$ ,

$$\sum_k O(F; I_{nj}^k) \geq |\sin 4|/8$$

for infinitely many pairs  $n, j$ , implying  $\sum O(F; I'_j) = \infty$ . Thus  $g$  is not  $\mathcal{L}^{H^*}$ -integrable although it is  $\mathcal{L}^{CH}$ -integrable.

REFERENCES

1. P. Romanovski, *Intégrale de Denjoy dans les espaces abstraits*, Mat. Sb. **9(51)** (1941), 67-120.
2. S. Saks, *Theory of the integral*, Hafner, New York, 1937.

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