CONVOLUTION MEASURE ALGEBRAS WITH GROUP MAXIMAL IDEAL SPACES(1)

BY
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Let G denote a locally compact abelian topological group (an l.c.a. group) with dual group $G^\wedge$. We will denote by $M(G)$ the Banach algebra of bounded regular Borel measures on G under convolution multiplication and by $L(G)$ the algebra of bounded measures absolutely continuous with respect to Haar measure on G (for discussions of these Banach algebras cf. [1], [2], and [5]).

In this paper we shall be concerned with closed subalgebras $\mathfrak{M}$ of $M(G)$ with the following two properties:

(1) if $\mu \in \mathfrak{M}$ and ν is absolutely continuous with respect to μ, then $\nu \in \mathfrak{M}$;

(2) the maximal ideal space of $\mathfrak{M}$ is $G^\wedge$, where the Gelfand transform $\mu^\wedge$ of $\mu \in \mathfrak{M}$ is given by $\mu^\wedge(\chi) = \int \bar{\chi}\, d\mu$ for $\chi \in G^\wedge$; i.e., the Gelfand transform coincides with the Fourier-Stieltjes transform on $\mathfrak{M}$.

Any closed subalgebra of $M(G)$ satisfying (1) will be called an L-subalgebra of $M(G)$. It is well known that $L(G)$ satisfies (1) and (2) (cf. [5, Chapter 1]). In Theorem 1 we show that any L-subalgebra $\mathfrak{M}$ of $M(G)$, with $L(G) \subset \mathfrak{M} \subset (L(G))^{1/2}$, also satisfies (2), where $(L(G))^{1/2}$ is the intersection of all maximal ideals of $M(G)$ containing $L(G)$. We conjecture that the converse is also true; i.e., if $\mathfrak{M}$ satisfies (1) and (2) then $L(G) \subset \mathfrak{M} \subset (L(G))^{1/2}$. In Theorem 2 we prove that this is true provided G contains no copy of R^n for $n > 1$.

The problem arises in the following way: in [6] we define the concept of abstract convolution measure algebra and prove that any such algebra $\mathfrak{M}$, provided it is commutative and semisimple, may be represented as an L-subalgebra of $M(S)$, where S is a compact topological semigroup called the structure semigroup of $\mathfrak{M}$. $M(G)$ and $L(G)$ are convolution measure algebras as is any L-subalgebra of the measure algebra on a semigroup. The map $\mu \to \mu_S$ which embeds $\mathfrak{M}$ in $M(S)$ is an isometry as well as an algebraic isomorphism and it preserves the order theoretic properties of $\mathfrak{M}$ as a space of measures. The maximal ideal space of $\mathfrak{M}$ may be identified as $S^\wedge$, the set of all semicharacters of S, where the Gelfand transform of $\mu \in \mathfrak{M}$ is given by $\mu^\wedge(f) = \int f\, d\mu_S$ for $f \in S^\wedge$ (a semicharacter of S is a continuous homomorphism of S into the unit disc which is not identically zero). Under pointwise multiplication $S^\wedge$ is a semigroup provided $\mathfrak{M}$ has an approximate identity.

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Now, given this representation of $\mathfrak{M}$, it is natural to try to relate the algebraic structures of S and $S^\wedge$ to the structure of $\mathfrak{M}$. In particular, if S and $S^\wedge$ are groups, what can be said about $\mathfrak{M}$? More generally, if G_1 is a maximal group contained in S, then its dual group $G_1^\wedge$ is embedded in $S^\wedge$ (cf. [6, §3]) and, if G_1 is not a set of measure zero for $\mathfrak{M}$, then $G_1^\wedge$ is locally compact in the Gelfand topology of $S^\wedge$ and, with this topology, is the maximal ideal space of $\mathfrak{M}_1$, the algebra of measures in $\mathfrak{M}$ which are concentrated on G_1. It follows that $G_1^\wedge$, with the Gelfand topology (as opposed to the discrete topology it inherits from the compact group G_1), is the dual group $G^\wedge$ of some l.c.a. group G (with Bohr compactification G_1) and that $\mathfrak{M}_1$ may be embedded as an L-subalgebra of $M(G)$. Thus $\mathfrak{M}_1$ is a subalgebra of $M(G)$ satisfying (1) and (2); i.e., by studying the relationship between maximal groups of S and the structure of $\mathfrak{M}$ we are led to the problem posed in this paper.

Our problem may also be viewed as an attempt to extend results of Rieffel in [4]. Rieffel proves that if A is a commutative semisimple Banach algebra which is Tauberian and in which every multiplicative linear functional is L'-inducing, then A is $L(G)$ for some l.c.a. group G. The hypothesis that every multiplicative linear functional is L'-inducing is equivalent to the existence of an l.c.a. group G such that A is a closed subalgebra of $M(G)$ satisfying (1) and (2). The Tauberian hypothesis is extremely strong; it means that those elements of A whose transforms have compact support on $G^\wedge$ are dense in A. If our conjecture is true, then without Rieffel's Tauberian hypothesis we may still conclude that $L(G)\subset A\subset(L(G))^{1/2}$. In Theorem 5.1 of [4] Rieffel mentions an example which shows that $L(G)$ and $(L(G))^{1/2}$ may be distinct.

We now proceed with our development of Theorems 1 and 2. If B is any commutative Banach algebra and I is a closed ideal of B, then $I^{1/2}$ denotes the intersection of all maximal ideals of B which contain I. Under the natural map $x\to\xi_x$ from B to B/I, $I^{1/2}$ is the inverse image of $(0)^{1/2}$; but $\xi_x\in(0)^{1/2}$ if and only if $\lim_n\|\xi_x^n\|^{1/n}=0$ (cf. [2, 24B]). Hence, $x\in I^{1/2}$ if and only if $\lim_n\inf_{y\in I}\|x^n+y\|^{1/n}=0$. In Lemma 1 we use this fact to obtain a measure theoretic characterization of the radical of an L-ideal, where an L-ideal $\mathfrak{N}$ of a convolution measure algebra $\mathfrak{M}$ is an ideal such that whenever $\mu\in\mathfrak{N}$ and $\nu\in\mathfrak{M}$ with ν absolutely continuous with respect to μ, then $\nu\in\mathfrak{N}$.

If two measures μ and ν are mutually singular, then we write $\mu\perp\nu$. If $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are sets of measures, then $\mathfrak{N}_1\perp\mathfrak{N}_2$ means $\mu\perp\nu$ for every $\mu\in\mathfrak{N}_1$ and $\nu\in\mathfrak{N}_2$. By $|\mu|$ we mean the measure defined by the total variation of μ.

Lemma 1. *If $\mathfrak{N}$ is an L-ideal in a convolution measure algebra $\mathfrak{M}$, then $\mathfrak{N}^{1/2}$ is also an L-ideal and $\mu\in\mathfrak{N}^{1/2}$ if and only if $\mu\perp\{\nu\in\mathfrak{M} : |\nu|^n\perp\mathfrak{N}$ for $n=1,2,\ldots\}$.*

Proof. Let S be the structure semigroup of $\mathfrak{M}$ as in [6] and $\mu\to\mu_S$ the embedding of $\mathfrak{M}$ into $M(S)$. Let A be the smallest closed subset of S on which each μ_S for $\mu\in\mathfrak{N}$ is concentrated. A maximal regular ideal of $\mathfrak{M}$ containing $\mathfrak{N}$ corresponds to a semicharacter $f\in S^\wedge$ such that $\int f\,d\mu_S=0$ for each $\mu\in\mathfrak{N}$. Since $\mathfrak{N}$ is an L-ideal it

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follows that such an f must be identically zero on A. Let J be the set of all such semicharacters f together with the identically zero function on S; then J is closed under conjugation and pointwise multiplication. Let $A' = \{s \in S : f(s) = 0$ for all $f \in J\}$. Since $S^\wedge$ separates points in S (cf. [6, Theorem 2.2]), J separates points in $S \backslash A'$ and, thus, the Stone-Weierstrass theorem implies that the closed linear span of J in $C(S)$ consists of those continuous functions which vanish on A'. Now $\mathfrak{N}^{1/2}$ consists of those measures μ in $\mathfrak{M}$ for which $\int f\, d\mu_S = 0$ for all $f \in J$; i.e., those measures $\mu \in \mathfrak{M}$ for which μ_S is concentrated on A'. It follows that $\mathfrak{N}^{1/2}$ is an L-ideal.

Since $\mathfrak{N}^{1/2}$ is an L-ideal, $\mu \in \mathfrak{N}^{1/2}$ if and only if $|\mu| \in \mathfrak{N}^{1/2}$. Therefore, it suffices to prove the second part of Lemma 1 for positive μ. Let $\mu \in \mathfrak{M}$, $\mu \geqq 0$, and $\nu \in \mathfrak{M}$ such that $|\nu|^n \perp \mathfrak{N}$ for $n = 1, 2, \ldots$; then if μ and ν are not mutually singular, we may write $\mu = \mu_1 + \mu_2$ where $\mu_1, \mu_2 \geqq 0$ and $0 \neq \mu_2 \leqq |\nu|$. Then $\|\omega + \mu^n\| \geqq \|\mu_2\|^n$ for every $\omega \in \mathfrak{N}$, since $\mu_2^n \leqq |\nu|^n \perp \mathfrak{N}$. Hence $\lim_n \inf_{\omega \in \mathfrak{N}} \|\omega + \mu^n\|^{1/n} \geqq \|\mu_2\| \neq 0$, and $\mu \notin \mathfrak{N}^{1/2}$. Thus if $\mu \in \mathfrak{N}^{1/2}$ then $\mu \perp \nu$ whenever $\nu \in \mathfrak{M}$ and $|\nu|^n \perp \mathfrak{N}$ for $n = 1, 2, \ldots$.

To see the converse, note that if $\mu \notin \mathfrak{N}^{1/2}$ then there is a semicharacter f in the set J described above for which $\int f\, d\mu_S \neq 0$. We may write $\mu = \omega + \nu$ where ω_S is concentrated on the set where f is zero and ν_S is concentrated on the set where f is nonzero. Since f is multiplicative, ν_S^n is also concentrated on the set where f is nonzero and, hence, $|\nu|^n \perp \mathfrak{N}$ for each n. This completes the proof, since μ and ν are not mutually singular.

Theorem 1. *If $\mathfrak{M}$ is an L-subalgebra of $M(G)$ and $L(G) \subset \mathfrak{M} \subset (L(G))^{1/2}$ then the maximal ideal space of $\mathfrak{M}$ is $G^\wedge$; i.e., $\mathfrak{M}$ satisfies* (2).

Proof. It follows from the characterization of the radical in Lemma 1, that if $\mathfrak{M}_2$ is an L-subalgebra of a convolution measure algebra $\mathfrak{M}_1$ and if $\mathfrak{N}$ is an L-ideal of both $\mathfrak{M}_1$ and $\mathfrak{M}_2$, then the radical of $\mathfrak{N}$ in $\mathfrak{M}_2$ is the intersection with $\mathfrak{M}_2$ of the radical of $\mathfrak{N}$ in $\mathfrak{M}_1$. Thus, if $L(G) \subset \mathfrak{M} \subset (L(G))^{1/2}$, then the radical of $L(G)$ in $\mathfrak{M}$ is $\mathfrak{M}$. It follows that $L(G)$ and $\mathfrak{M}$ have the same maximal ideal space, namely $G^\wedge$.

Before considering the converse problem we require a lemma concerning closed semigroups in l.c.a. groups.

Lemma 2. *If T is a closed proper subsemigroup of the* l.c.a. *group G which generates G (i.e., the closed subgroup generated by T is G), then there is a continuous homomorphism γ of G into the additive real numbers R such that $0 \neq \gamma(T) \subset R^+$, the set of nonnegative reals.*

Proof. If T has an interior point x such that $-x \notin T$, then the lemma follows from results of Rieffel (cf. [4, Theorem 6.4]). We shall prove the lemma directly in the case where $G = K \times R^n$ for a compact group K, and use this to reduce the general case to Rieffel's result.

We prove the lemma for G of the form $K \times R^n$ by induction on n. If $n = 0$ then G is compact and the lemma holds vacuously, since a closed subsemigroup of a

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compact group must be a group (cf. [7, p. 99]) and, thus, T generates G would imply $T=G$. We now assume the lemma holds for $n=p-1$ and let $G=K\times R^p$ for some compact group K. Since T is not a group, there is some $x\in T$ for which $-x\notin T$. Since K is compact, $T\cap K$ is a group; hence, $x\notin K$. Note that without loss of generality we may assume that $0\notin T$ for, if $0\in T$, we may replace T by $T'=x+T$; then $0\notin T'$ and if γ is a homomorphism of G into R such that $\gamma(T')\subset R^+$, then $\gamma(T)\subset R^+-\gamma(x)$; but $\gamma(T)$ is a semigroup, so $\gamma(T)\subset R^+$.

In view of the above considerations, we may, without loss of generality, write G as $H\times R$, where $H=K\times R^{p-1}$, x is the element $(0, 1)\in H\times R$, and $(0, n)\in T$ for positive integers n but not for nonpositive integers n.

Let $G^+=\{(h, r)\in H\times R : r\geqq 0\}$ and $G^-=\{(h, r)\in H\times R : r\leqq 0\}$. If $T\subset G^+$ then the projection of $H\times R$ on R is the required γ. If $T\not\subset G^+$ we set $T^-=T\cap G^-$, $Z=\{(0, n) : n=0, \pm 1, \pm 2, \ldots\}$, and let ϕ be the natural map from G to G/Z. Note that $(0, 1)\in T$ implies that $\phi(T^-)=\phi(T_0^-)$, where

$$T_0^- = \{(h, r)\in H\times R : -1 \leqq r \leqq 0\}\cap T^-,$$

and since $\phi(T_0^-)$ is closed, so is $\phi(T^-)$. Also, $Z\cap T^-$ is empty so $0\notin\phi(T^-)$ and $\phi(T^-)$ is a closed proper subsemigroup of G/Z. To apply the induction hypothesis we need only show that $\phi(T^-)$ generates G/Z. If not, then $\phi(T^-)\subset J$ where J is a proper closed subgroup of G; if $y\in T$ and $w\in T^-$, then $y+w^n\in T^-\subset\phi^{-1}(J)$ for sufficiently large n, and, hence, $y\in\phi^{-1}(J)$; i.e., $T\subset\phi^{-1}(J)$ which contradicts the assumption that T generates G. Note that if $p=1$ then T must be contained in G^+, since G/Z is compact. Thus the lemma is proven in this case.

Now if $p>1$ the noncompact part of G/Z has dimension $p-1\neq 0$, and we may apply the induction hypothesis and conclude that there is a continuous homomorphism f of G/Z into R such that $0\neq f(\phi(T^-))\subset R^+$. We now define a map g of G into $R\times R$ by $g(h, r)=(f(\phi(h, r)), r)$; then g is a continuous homomorphism and $g(T)\cap\{(t, r) : t<0, r<0\}$ is empty; i.e., $g(T)$ misses the open third quadrant. From this and the fact that $g(T)$ is a semigroup, it follows readily that the convex hull of $g(T)$ also misses the third quadrant and may thus be separated from it by a straight line l passing through the origin. The line l may contain $g(T)$, in which case we have reduced the problem to the one-dimensional case and may apply the last comment of the previous paragraph. If $g(T)\not\subset l$, then there is a continuous homomorphism α of $R\times R$ onto R, with kernel l, such that $0\neq\alpha(g(T))\subset R^+$. Then α composed with g is the required map γ. This completes the induction.

To prove the lemma for G a general l.c.a. group, we use the structure theorem for l.c.a. groups (cf. [5, Theorem 2.4.1]) which says that G contains an open subgroup G_1 of the form $K\times R^n$ for a compact group K. Let β be the natural map from G to G/G_1. If $\beta(T)$ is proper in G/G_1, we may apply the above mentioned result of Rieffel to obtain γ, since G/G_1 is discrete. If $\beta(T)=G/G_1$ then every coset of G_1 in G contains an element of T. Then $T\cap G_1$ must be a proper generating subsemigroup of G_1 if T is to be a proper generating subsemigroup of G. Thus, we

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may apply the result of the previous paragraph to obtain a continuous homomorphism γ_1 of G_1 into R such that $0 \neq \gamma(T \cap G_1) \subset R^+$. We write

$$G_1^+ = \{x \in G_1 : \gamma_1(x) \geqq 0\}.$$

Then $G_1^+ + T$ is proper in G, for if $y \in G_1$ and $y = x + t$ with $x \in G_1^+$ and $t \in T$, then $y - x \in G_1 \cap T$ and $\gamma_1(y-x) = \gamma_1(y) - \gamma_1(x) \geqq 0$ so that $y \in G_1^+$ also. Clearly, $G_1^+ + T$ is closed, generates G, and has an interior point x such that $-x \notin G_1^+ + T$ (e.g., any $x \in G_1$ for which $\gamma_1(x) > 0$). We again apply Rieffel's result and the proof is complete.

Note that Lemma 2 implies that on a semigroup T of the above type there is a proper semicharacter, i.e., a semicharacter f for which $|f| \not\equiv 1$; in fact, $f(x) = e^{-s\gamma(x)}$ for $s > 0$ and $x \in T$ is such a semicharacter.

In what follows, $\mathfrak{M}$ will denote a closed subalgebra of $M(G)$ which satisfies (1) and (2). Lemmas 3 and 4 hold for arbitrary l.c.a. groups G, while the remainder of the proof of Theorem 2 requires special assumptions on G.

LEMMA 3. *$\mathfrak{M}$ is weak-$*$ dense in $M(G)$ and $\mathfrak{M}^\wedge$ is uniformly dense in $C_0(G^\wedge)$.*

Proof. Let $T = \text{carrier }(\mathfrak{M})$ be the smallest closed subset of G on which each $\mu \in \mathfrak{M}$ is concentrated. Since $\mathfrak{M}$ is an L-subalgebra of $M(G)$, T is a closed subsemigroup of G, and $\mathfrak{M}$ is weak-$*$ dense in $M(G)$ if and only if $T = G$. Since $\mathfrak{M}$ separates points in $G^\wedge$, T cannot be contained in any closed proper subgroup of G; i.e., T generates G. If $T \neq G$ we may apply Lemma 2 and obtain a continuous homomorphism γ of G into R such that $0 \neq \gamma(T) \subset R^+$. Then $\mu \to \int e^{-\gamma(x)}\, d\mu(x)$ is a complex homomorphism of $\mathfrak{M}$ which clearly does not correspond to any character of G. This contradicts the fact that $\mathfrak{M}$ satisfies (2). We conclude that $T = G$ and $\mathfrak{M}$ is weak-$*$ dense in $M(G)$.

Let $\lambda \in M(G^\wedge)$; then $\lambda^\wedge(x) = \int \bar{\chi}(x)\, d\lambda(x)$ is continuous on G, so there exists $\mu \in \mathfrak{M}$ for which $\int \lambda^\wedge\, d\mu = \int \mu^\wedge\, d\lambda \neq 0$. Hence $\mathfrak{M}^\wedge$ is a dense subspace of $C_0(G^\wedge)$. This completes the proof.

LEMMA 4. *If $\mathfrak{M} \cap L(G) \neq (0)$ then $L(G) \subset \mathfrak{M} \subset (L(G))^{1/2}$.*

Proof. $\mathfrak{M} \cap L(G)$ is an L-subalgebra of $L(G)$ and an L-ideal of $\mathfrak{M}$. If $\nu \in L(G)$ then Lemma 3 implies that ν is the weak-$*$ limit in $M(G)$ of a net $\{\mu_\alpha\}$ of elements of $\mathfrak{M}$. If $f \in L_\infty(G)$ and $\omega \in \mathfrak{M} \cap L(G)$, then $g(x) = \int f(x+y)\, d\omega(y) \in C(G)$ and, hence, $\int f\, d\omega \cdot \mu_\alpha$ converges to $\int g\, d\nu = \int f\, d\omega \cdot \nu$; i.e., $\{\omega \cdot \mu_\alpha\}$ is a net in $\mathfrak{M} \cap L(G)$ which converges weakly to $\omega \cdot \nu$ and so $\omega \cdot \nu \in \mathfrak{M} \cap L(G)$. It follows that $\mathfrak{M} \cap L(G)$ is an L-ideal of $L(G)$. However, the only L-ideals of $L(G)$ are (0) and $L(G)$. Hence $L(G) \subset \mathfrak{M}$. Also $\mathfrak{M} \subset (L(G))^{1/2}$, otherwise there would be a complex homomorphism of $M(G)$ which was zero on $L(G)$ and nonzero on $\mathfrak{M}$, contradicting the fact that the maximal ideal space of $\mathfrak{M}$ is $G^\wedge$.

LEMMA 5. *If G contains no copy of R, then $L(G) \subset \mathfrak{M} \subset (L(G))^{1/2}$.*

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Proof. Under this hypothesis, G and $G^\wedge$ each contain an open-compact subgroup, by the principal structure theorem (cf. [5, Theorem 2.4.1]). Let K be an open-compact subgroup of $G^\wedge$. Since $G^\wedge$ is the maximal ideal space of $\mathfrak{M}$, Shilov's theorem (cf. [3, Theorem 3.6.3]) implies the existence of $\mu \in \mathfrak{M}$ such that $\mu^\wedge(\chi)=0$ for $\chi \notin K$ and $\mu^\wedge(\chi)=1$ for $\chi \in K$. Then μ is Haar measure on an open-compact subgroup H of G ($H=\{x \in G : \chi(x)=1 \text{ for } x \in K\}$). Thus $\mu \in L(G)$ and the conclusion follows from Lemma 4.

Lemma 6. *If G is of the form $K \times R$ where K is compact, and*

$$G^+ = \{(k, r) \in K \times R : r > 0\},$$

then there exists a nonnegative measure $\mu \in \mathfrak{M}$ concentrated on G^+ with compact support and $\|\mu\| > 1$ such that $\mu^\wedge(\chi) \neq 1$ for each $\chi \in G^\wedge$.

Proof. It follows from Lemma 3 that it is enough to show the existence of a $\nu \in L(G)$ with the above properties. An example of such a measure ν is $\nu = 2(\rho \times \omega)$, where ρ is Haar measure on K and ω is Lebesgue measure on [0, 1] in R.

Theorem 2. *If G contains no copy of R^n for $n > 1$, then $L(G) \subset \mathfrak{M} \subset (L(G))^{1/2}$.*

Proof. If G contains no copy of R, then Lemma 5 gives us the result; otherwise, the hypothesis, together with the structure theorem for l.c.a. groups, implies that G has an open subgroup G_1 of the form $K \times R$ with K compact. Let $\mathfrak{M}_1$ be the subalgebra of $\mathfrak{M}$ consisting of all measures in $\mathfrak{M}$ which are concentrated on G_1. It is easily seen that $\mathfrak{M}_1$ is an L-subalgebra of $M(G_1)$ with maximal ideal space $\hat{G_1}$. Thus the preceding lemmas hold for $\mathfrak{M}_1$ as a subalgebra of $M(G_1)$.

Now let μ be the element of $\mathfrak{M}_1$ given by Lemma 6. Since $\mu^\wedge$ never assumes the value one, μ has an adverse (cf. [2, §21]); i.e., there is an element $\nu \in \mathfrak{M}_1$ such that $\mu\nu = \mu + \nu$. If μ had norm less than one, its adverse would be $\omega = -\sum_{n=1}^\infty \mu^n$. Since μ is concentrated on G_1^+, the series $\sum \mu^n$ converges on each compact subset of G_1, but since $\|\mu\| > 1$ and $\mu \geqq 0$, it converges to an unbounded measure.

The Laplace-Stieltjes transform, $\tilde{\rho}$, for a measure ρ on $G_1 = K \times R$ is defined by $\tilde{\rho}(\chi, z) = \int_{K \times R} e^{-zt}\, \bar{\chi}(k)\, d\rho(k, t)$, for $\chi \in K^\wedge$ and z a complex number, whenever this integral converges. Note that since μ has compact support, $\tilde{\mu}$ exists and is analytic in z for all χ, and $\mu^\wedge(\chi, y) = \tilde{\mu}(\chi, iy)$, for y real, is the Fourier-Stieltjes transform of μ. Since ν is a bounded measure, $\tilde{\nu}$ exists for z imaginary and

$$\tilde{\nu}(\chi, iy) = \nu^\wedge(\chi, y) = \mu^\wedge(\chi, y)[\mu^\wedge(\chi, y) - 1]^{-1}.$$

Also, for sufficiently large positive r, $\int e^{-rt}\, d\mu(k, t) < 1$ and $\int e^{-rt}\, d\omega(k, t)$ is finite. Thus, for $x \geqq r$, $\tilde{\omega}(\chi, x+iy)$ exists and equals $\tilde{\mu}(\chi, x+iy)[\tilde{\mu}(\chi, x+iy) - 1]^{-1}$.

The function $f(\chi, z) = \tilde{\mu}(\chi, z)[\tilde{\mu}(\chi, z) - 1]^{-1}$ is analytic in z except where $\tilde{\mu}(\chi, z) = 1$, and at such points it has simple poles. Also, since $\tilde{\mu}(\chi, z)$ approaches zero at infinity for $\mathrm{Re}(z) \geqq 0$, there can be at most finitely many poles of f in the region $\mathrm{Re}(z) \geqq 0$.

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Let $\{\rho_\alpha\}$ be a weak-$*$ approximate identity for $M(G)$ consisting of elements of $L(G_1)$ with compact support and norm one, whose Laplace transforms $\tilde{\rho}_\alpha$ have the property that $\tilde{\rho}_\alpha(\chi, a+iy)$ is integrable in (χ, y) for each $a \in R$. Let $g_\alpha(k, t)$ be the Radon-Nikodym derivative of ρ_α for each α; then, the inversion formula (cf. [5], Theorem 1.5.1) implies that

$$g_\alpha * \nu(k, t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left[\sum_{\chi\in K^\wedge} e^{zt}\chi(k)f(\chi, z)\hat{\rho_\alpha}(\chi, z)\right] dz$$

and

$$g_\alpha * \omega(k, t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}\left[\sum_{\chi\in K^\wedge} e^{zt}\chi(k)f(\chi, z)\hat{\rho_\alpha}(\chi, z)\right] dz,$$

for $x \geqq r$. It follows that

$$g_\alpha * (\omega-\nu)(k, t) = \frac{1}{2\pi i}\sum_{\chi\in P}\int_\Gamma e^{zt}\chi(k)f(\chi, z)\hat{\rho_\alpha}(\chi, z)\, dz$$

where P is the finite set of points $\chi \in K^\wedge$ for which $f(\chi, z)$ has a pole in the region $\mathrm{Re}(z) \geqq 0$, and Γ is any simple closed curve in $\mathrm{Re}(z) \geqq 0$ enclosing all such poles. Then $g_\alpha * (\omega-\nu)$ converges uniformly on compact subsets of G_1 to the function

$$\phi(k, t) = \frac{1}{2\pi i}\sum_{\chi\in P}\int_\Gamma e^{zt}\chi(k)f(\chi, z)\, dz,$$

while $\rho_\alpha \cdot (\omega-\nu)$ converges weakly, relative to continuous functions with compact support, to the measure $\omega-\nu$. It follows that $\omega-\nu$ is absolutely continuous on each compact set with Radon-Nikodym derivative ϕ. Thus, if λ is the restriction of $\omega-\nu$ to any compact subset of G_1, we have $\lambda \in L(G_1) \cap \mathfrak{M}_1 \subset L(G) \cap \mathfrak{M}$. In view of Lemma 4, the proof is complete.

Added in proof. We have recently obtained a proof of Theorem 2 without the condition on G. Thus, the conjecture mentioned at the beginning of this paper is true.

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