

# OUTER DERIVATIONS OF LIE ALGEBRAS

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1. **Introduction.** It is known that certain types of Lie algebras have actually outer derivations. Schenkman and Jacobson [3] have shown that every nonzero nilpotent Lie algebra over a field of arbitrary characteristic has an outer derivation. Leger [4] has shown that if a Lie algebra with nonzero center, over a field of characteristic 0, has no outer derivations, it is not solvable and its radical is nilpotent. Recently it has been shown by Satô [5] that every nonzero nilpotent Lie algebra over a field of characteristic 0 admits an outer derivation in the radical of its derivation algebra. The purpose of this paper is to generalize and sharpen these results and make a more detailed study of outer derivations of Lie algebras.

We shall consider the class  $\mathfrak{D}$  of all the Lie algebras  $L$  over a field of arbitrary characteristic such that  $L \neq [L, L]$  and the center  $Z(L) \neq (0)$ . All nonzero nilpotent Lie algebras belong to  $\mathfrak{D}$ . In §3, we shall show that, if  $L \in \mathfrak{D}$  is not the direct sum of the 1-dimensional ideal and an ideal  $L_1$  such that  $L_1 = [L_1, L_1]$  and  $Z(L_1) = (0)$ , it has a nilpotent outer derivation  $D$  with  $D^2 = 0$ , and otherwise it has a semisimple outer derivation. In order to generalize the above result of Satô, we introduce the notion of Lie algebras of type  $(T)$ , which will be defined as Lie algebras in  $\mathfrak{D}$  satisfying some additional conditions. §4 will be devoted to the study of this special type of Lie algebras. In §5 we shall show that, if  $L \in \mathfrak{D}$  is not of type  $(T)$ , it admits an outer derivation in an abelian ideal of its derivation algebra  $\mathfrak{D}(L)$ , and if  $L \in \mathfrak{D}$  is of type  $(T)$  and  $L^{(1)} \neq L^{(2)}$ , it admits a semisimple outer derivation in the radical of  $\mathfrak{D}(L)$ . Any solvable Lie algebra  $L$  of type  $(T)$  satisfies the condition  $L^{(1)} \neq L^{(2)}$ . Hence every solvable Lie algebra with nonzero center admits an outer derivation in the radical of its derivation algebra. In §6 we shall show that besides the class  $\mathfrak{D}$  there is another class of nonsolvable Lie algebras which have semisimple outer derivations.

2. **Preliminaries and notations.** Throughout the paper we denote by  $\Phi$  a field of arbitrary characteristic unless otherwise stated and by  $L$  a finite dimensional Lie algebra over  $\Phi$ . As usual we put  $L^2 = L^{(1)} = [L, L]$  and  $L^{(k)} = [L^{(k-1)}, L^{(k-1)}]$  for  $k \geq 2$ . We denote by  $Z(L)$  the center of  $L$ . The 2-dimensional nonabelian solvable Lie algebra has no outer derivations and its center is  $(0)$ . Every semisimple Lie algebra  $L$  over a field of characteristic 0 has no outer derivations and  $L = L^2$ . Therefore, to investigate the outer derivations of Lie algebras, in this paper we shall mainly consider the following class of Lie algebras.

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Received by the editors April 13, 1966.

<sup>(1)</sup> This work was partially supported by the National Science Foundation.

$\mathfrak{D}$ : The class of all the Lie algebras  $L$  over a field  $\Phi$  such that  $L \neq L^2$  and  $Z(L) \neq (0)$ .

Every nonzero nilpotent Lie algebra and every solvable Lie algebra with nonzero center belong to this class  $\mathfrak{D}$ . We denote by  $\text{ad } x$  the inner derivation associated to  $x \in L$ . If a subalgebra  $H$  of  $L$  is stable under  $\text{ad } x$ , then  $\text{ad } x$  induces the derivation of  $H$  which we denote by  $\text{ad}_H x$ . We furthermore employ the following notations.

$\mathfrak{D}(L)$ : The derivation algebra of  $L$ , that is, the Lie algebra of all the derivations of  $L$ .

$\mathfrak{I}(L)$ : The ideal of  $\mathfrak{D}(L)$  consisting of all the inner derivations of  $L$ .

$\mathfrak{C}(L)$ : The ideal of  $\mathfrak{D}(L)$  consisting of all the central derivations of  $L$ , that is, the derivations of  $L$  which map  $L$  into  $Z(L)$ .

3. Lie algebras in the class  $\mathfrak{D}$ . We begin with the following

LEMMA 1. *Let  $L$  be a Lie algebra over a field  $\Phi$  such that  $Z(L) \neq (0)$  and  $M$  be an ideal of  $L$  of codimension 1 containing  $Z(L)$ . Then  $[L, Z(M)] \subset Z(M)$  but  $[L, Z(M)] \neq Z(M)$ .*

**Proof.** Choose an element  $e$  of  $L$  such that  $L = \Phi e + M$ . Then  $[L, Z(M)] = [e, Z(M)]$ . From the fact that  $M$  is an ideal of  $L$ , it follows that  $Z(M)$  is stable under  $\text{ad } e$ . Hence  $\text{ad}_{Z(M)} e$  is a derivation of  $Z(M)$ . Since  $Z(L) \subset M$ ,  $Z(L) \subset Z(M)$ . Hence the kernel of  $\text{ad}_{Z(M)} e$  is  $Z(L)$  and therefore is  $\neq (0)$ . Consequently,  $\dim Z(M) > \dim [e, Z(M)]$ , from which it follows that  $[L, Z(M)] \subset Z(M)$  but  $[L, Z(M)] \neq Z(M)$ .

THEOREM 1. *Every Lie algebra  $L \in \mathfrak{D}$  has an outer derivation. More precisely, if  $L \in \mathfrak{D}$  is not the direct sum of the 1-dimensional ideal and of an ideal  $L_1$  such that  $L_1 = L_1^2$  and  $Z(L_1) = (0)$ ,  $L$  has a nilpotent outer derivation  $D$  with  $D^2 = 0$ , and otherwise  $L$  has a semisimple outer derivation.*

**Proof.** The case where  $L$  is not abelian and has no nonzero abelian direct summands: Since  $L \neq L^2$ , there exists a subspace  $M$  of  $L$  of codimension 1 containing  $L^2$ . Then  $M$  is an ideal of  $L$ . Since  $Z(L) \subset L^2 \subset M$ , it follows from Lemma 1 that  $[L, Z(M)] \subset Z(M)$  but  $[L, Z(M)] \neq Z(M)$ . Choose an element  $e$  of  $L$  such that  $L = \Phi e + M$  and an element  $z$  of  $Z(M) \setminus [L, Z(M)]$ . Define the endomorphism  $D$  of  $L$  in such a way that  $De = z$  and  $DM = (0)$ . Then  $D$  is a derivation of  $L$  such that  $D^2 = 0$ . Furthermore  $D$  is not inner. In fact, suppose that  $D$  is inner. Then  $D = \text{ad}(\lambda e + x)$  with  $\lambda \in \Phi$  and  $x \in M$ . It follows that  $0 = Dx = [\lambda e + x, x] = \lambda [e, x]$ . If  $\lambda \neq 0$ ,  $[e, x] = 0$  and therefore  $z = De = [\lambda e + x, e] = 0$ , contradicting the choice of  $z$ . Therefore  $\lambda = 0$ . Hence  $D = \text{ad } x$  and therefore  $x$  is in  $Z(M)$ . It follows that  $z = De = [x, e] \in [L, Z(M)]$ . This again contradicts the choice of  $z$ . Thus  $D$  is not inner.

The case where  $L$  is not abelian and has a nonzero abelian direct summand: In this case,  $Z(L) \not\subset L^2$ . Take a subspace  $L_0$  of  $Z(L)$  complementary to  $Z(L) \cap L^2$ , and choose a subspace  $L_1$  of  $L$  complementary to  $L_0$  and containing  $L^2$ . Then  $L$  is the

direct sum of the nonzero ideals  $L_0$  and  $L_1$ ,  $L_0$  is central and  $L_1$  has no nonzero abelian direct summands. If  $\dim L_0 > 1$ , take an arbitrary nonzero endomorphism  $D_0$  of  $L_0$  and let  $D$  be the trivial extension to  $L$ . Then  $D$  is an outer derivation of  $L$ . Especially by taking  $D_0$  in such a way that  $D_0^2 = 0$ , we have a  $D$  such that  $D^2 = 0$ . Now assume that  $\dim L_0 = 1$ . In the case where  $L_1 \neq L_1^2$ , an arbitrary nonzero endomorphism  $D$  of  $L$  satisfying the conditions  $DL_1 \subset L_0$  and  $D(L_0 + L_1^2) = (0)$  is an outer derivation of  $L$  such that  $D^2 = 0$ . In the case where  $Z(L_1) \neq (0)$ , an arbitrary nonzero endomorphism  $D$  of  $L$  satisfying the conditions  $DL_0 \subset Z(L_1)$  and  $DL_1 = (0)$  is an outer derivation of  $L$  such that  $D^2 = 0$ . Finally in the case where  $L_1 = L_1^2$  and  $Z(L_1) = (0)$ , take a nonzero endomorphism  $D_0$  of  $L_0$  and let  $D$  be the trivial extension to  $L$ . Then  $D$  is a semisimple outer derivation of  $L$ .

The case where  $L$  is abelian: Every nonzero endomorphism  $D$  of  $L$  is an outer derivation. If  $\dim L > 1$ , there is a  $D$  such that  $D^2 = 0$ . If  $\dim L = 1$ , every  $D$  is semisimple.

**COROLLARY 1.** *Every solvable Lie algebra whose center is  $\neq (0)$  and every nilpotent Lie algebra, of dimension  $> 1$  over a field  $\Phi$ , have nilpotent outer derivations  $D$  such that  $D^2 = 0$ .*

**Proof.** A solvable Lie algebra has no nonzero direct summands  $L_1$  such that  $L_1 = L_1^2$ . Hence the statement follows from Theorem 1.

As another consequence of Theorem 1, we have the following result which generalizes and sharpens a result of Leger [4, p. 642].

**COROLLARY 2.** *Let  $L$  be a Lie algebra over a field of characteristic 0 such that  $Z(L) \neq (0)$ . Let  $R$  be the radical of  $L$ . Then:*

(1) *If  $L$  has no outer derivations  $D$  such that  $D^2 = 0$ , either  $L$  is 1-dimensional, or  $L$  is not solvable and  $R$  is  $[L, R]$  or the direct sum of  $[L, R]$  and of the center  $Z(L)$  which is 1-dimensional. And then  $R$  is nilpotent.*

(2) *If  $L$  has no outer derivations,  $L$  is not solvable and  $R = [L, R]$ .*

**Proof.** Since the basic field of  $L$  is of characteristic 0, we take a Levi decomposition  $L = S + R$  of  $L$ . Then  $L^2 = S + [L, R]$  and therefore  $[L, R]$  is the radical of  $L^2$ . Furthermore by using Lie's theorem we see that  $[L, R]$  is nilpotent. With these in mind we prove

(1) Assume that  $L$  has no outer derivations  $D$  such that  $D^2 = 0$ . If  $L$  is not 1-dimensional, by Theorem 1 either  $L = L^2$  or  $L$  is the direct sum of a 1-dimensional ideal  $L_0$  and a nonzero ideal  $L_1$  such that  $L_1 = L_1^2$  and  $Z(L_1) = (0)$ . Hence  $L$  is not solvable. In the first case,  $R = [L, R]$ . In the second case,  $L_0 = Z(L)$  and the radical  $R_1$  of  $L_1$  is  $[L_1, R_1] = [L, R]$ . Hence  $R$  is the direct sum of  $Z(L)$  and  $[L, R]$ . In both cases  $R$  is nilpotent.

(2) If  $L$  has no outer derivations  $D$  such that  $D^2 = 0$  and no semisimple outer derivations, by Theorem 1 we see that  $L = L^2$ . Hence  $L$  is not solvable and  $R = [L, R]$ .

It is to be noted that there are some Lie algebras in the class  $\mathfrak{D}$  each of whose

outer derivations is a linear sum of the outer derivations, which are obtained as in the first part of the proof of Theorem 1, and of the inner derivations. The Lie algebras given in [1, (6), p. 126] and [2] are the examples of such Lie algebras. We omit the illustration.

**4. Lie algebras of type (T).** In this section we shall study certain types of Lie algebras in the class  $\mathfrak{D}$  which play a special rôle in the next section. We call a Lie algebra  $L$  over a field  $\Phi$  to be of type (T) provided  $L$  has a nonzero subspace  $S$  satisfying the following conditions:

- (1)  $L = S + L^2, S \cap L^2 = (0)$ .
- (2)  $[S, L^2] = (0)$ .
- (3)  $[S, S] = \Phi z$  with  $0 \neq z \in Z(L)$ .

(4) The pairing  $\theta$  which assigns to  $(x, y) \in S \times S$  the coefficient of  $z$  in  $[x, y]$  is a nondegenerate (alternate) form on  $S$ .

If a Lie algebra  $L$  of type (T) is solvable, then  $L^{(1)} = \Phi z + L^{(2)}$  and therefore  $L^{(2)} = L^{(3)} = \dots = (0)$ . Hence  $L^2 = Z(L) = \Phi z$ . We call this special type of nilpotent Lie algebras to be *pseudo-abelian*.

For a Lie algebra  $L$  of type (T),  $Z(L) = Z(L^2)$ . In fact, since  $\theta$  is nondegenerate, any nonzero element of  $S$  does not commute with some element of  $S$ . It follows that  $Z(L) \subset L^2$  and therefore  $Z(L) \subset Z(L^2)$ . The converse inclusion is evident.

**LEMMA 2.** *Let  $L$  be a Lie algebra in the class  $\mathfrak{D}$  such that  $Z(L) \subset L^2$  and  $Z(M) \not\subset L^2$  for every ideal  $M$  of codimension 1. Then for every ideal  $M$  of codimension 1 there exist an ideal  $M'$  of codimension 1 and the elements  $e$  and  $e'$  of  $L$  satisfying the following conditions:*

- (1)  $L = \Phi e + M = \Phi e' + M', e \in Z(M') \setminus L^2, e' \in Z(M) \setminus L^2$ ;
- (2)  $[e, e'] \neq 0, [e, L] = [e', L] = \Phi[e, e'] \subset Z(L)$ ;
- (3)  $M$  and  $M'$  have a common basis except  $e'$  and  $e$ .

**Proof.** Take an element  $e'$  in  $Z(M) \setminus L^2$  and choose a basis  $x_1, x_2, \dots, x_k$  of a subspace of  $M$  complementary to  $\Phi e' + L^2$ . Next take a subspace  $M'$  of  $L$  complementary to  $\Phi e'$  and containing  $x_1, x_2, \dots, x_k$  and  $L^2$ . Then  $M'$  is an ideal of  $L$  of codimension 1. Since  $Z(M') \not\subset L^2$ , choose an element  $e$  of  $Z(M') \setminus L^2$ . Then  $[e, e'] \neq 0$ . In fact, if  $[e, e'] = 0$ , then  $[e, \Phi e' + M'] = (0)$  and therefore  $e \in Z(L) \subset L^2$ , which contradicts the choice of  $e$ . Hence, together with the fact that  $e' \in Z(M)$ , we see that  $e \notin M$ , and therefore  $L = \Phi e + M$ . It follows that  $M' = \Phi e + \Phi x_1 + \dots + \Phi x_k + L^2$ . Thus (1), the first part of (2) and (3) are satisfied by  $M', e$  and  $e'$ . We now use (1) and (3) to infer  $[e, L] = [e, M] = \Phi[e, e']$  and similarly  $[e', L] = \Phi[e, e']$ . By using (1) we have

$$\begin{aligned} [[e, e'], L] &\subset [[e, e'], \Phi e] + [[e, e'], M] \\ &\subset [L^2, e] + [[e, M], e'] + [e, [e', M]] \\ &\subset [M', e] + [M, e'] + [e, M'] = (0). \end{aligned}$$

Hence  $[e, e'] \in Z(L)$ , completing the proof.

By making use of Lemma 2 we shall give the following characterization of Lie algebras of type (T).

**THEOREM 2.** *A Lie algebra  $L$  over a field  $\Phi$  is of type (T) if and only if  $L \neq L^2$ ,  $(0) \neq Z(L) \subset L^2$  and  $Z(M) \not\subset L^2$  for every ideal  $M$  of  $L$  of codimension 1.*

**Proof.** Assume that  $L$  is of type (T) and let  $S, \theta$  be as in the definition. Then  $L \neq L^2$  and  $(0) \neq Z(L) \subset L^2$ . Every ideal  $M$  of  $L$  of codimension 1 contains  $L^2$  and  $M \cap S$  is of codimension 1 in  $S$ . Hence the restriction of  $\theta$  to  $M \cap S$  has a radical  $x \neq 0$  in  $M \cap S$ . It follows that  $x \in S \cap Z(M)$ . Therefore  $Z(M) \not\subset L^2$ .

Conversely, assume that  $L \neq L^2$ ,  $(0) \neq Z(L) \subset L^2$  and  $Z(M) \not\subset L^2$  for every ideal  $M$  of  $L$  of codimension 1. Let  $M$  be a subspace of  $L$  of codimension 1 containing  $L^2$ . Then  $M$  is an ideal of  $L$ . Choose an ideal  $M'$  of codimension 1 and the elements  $e$  and  $e'$  of  $L$  as in Lemma 2. Then we can write  $M = \Phi e' + \Phi x_1 + \dots + \Phi x_k + L^2$  and  $M' = \Phi e + \Phi x_1 + \dots + \Phi x_k + L^2$ . If  $k \geq 1$ , put  $M_1 = \Phi e + \Phi e' + \Phi x_2 + \dots + \Phi x_k + L^2$ . Then  $M_1$  is an ideal of  $L$  of codimension 1 and  $Z(M_1) \subset \Phi x_2 + \dots + \Phi x_k + L^2$  since  $[e, e'] \neq 0$ . By our assumption  $Z(M_1) \not\subset L^2$  and therefore  $k \geq 2$ . Now, by using Lemma 2, we choose an ideal  $M'_1$  and the elements  $f$  and  $f'$  of  $L$  such that

- (1)  $L = \Phi f + M_1 = \Phi f' + M'_1, f \in Z(M'_1) \setminus L^2, f' \in Z(M_1) \setminus L^2$ ;
- (2)  $[f, f'] \neq 0, [f, L] = [f', L] = \Phi [f, f'] \subset Z(L)$ ;
- (3)  $M_1 = \Phi f' + \Phi e + \Phi e' + \Phi y_1 + \dots + \Phi y_{k-2} + L^2$  and  $M'_1 = \Phi f + \Phi e + \Phi e' + \Phi y_1 + \dots + \Phi y_{k-2} + L^2$ .

It is obvious that  $f \neq e, f \neq e', f' \neq e, f' \neq e', [f, e] = [f, e'] = [f', e] = [f', e'] = 0$ .

Continuing this procedure and changing the notations, we find that

$$L = \Phi e_1 + \Phi e_{1'} + \Phi e_2 + \Phi e_{2'} + \dots + \Phi e_n + \Phi e_{n'} + L^2,$$

$$0 \neq [e_i, e_{i'}] \in Z(L), [e_i, e_j] = [e_{i'}, e_{j'}] = [e_{i'}, e_j] = 0,$$

$$[e_i, L^2] = [e_{i'}, L^2] = (0) \quad \text{for } i, j = 1, 2, \dots, n \text{ and } i \neq j.$$

We now put  $z = [e_1, e_{1'}]$  and assert that  $[e_i, e_{i'}] = \alpha_i z, \alpha_i \neq 0 (i = 2, 3, \dots, n)$ . In fact, put  $[e_i, e_{i'}] = z_i$  and assume that  $z$  and  $z_i$  are linearly independent over  $\Phi$ . Put

$$M = \Phi(e_1 + e_i) + \Phi e_{1'} + \Phi e_{i'} + \sum_{j \neq 1, i} (\Phi e_j + \Phi e_{j'}) + L^2.$$

Then  $M$  is an ideal of  $L$  of codimension 1. Let  $x$  be an arbitrary element of  $Z(M)$ . Since  $x$  commutes with  $e_{1'}, e_j, e_{j'} (j \neq 1, i)$ ,  $x$  is expressed in the form  $x = \lambda e_{1'} + \mu e_{i'} + y$  with  $y \in L^2$ . It follows that  $[x, e_1 + e_i] = -\lambda z - \mu z_i = 0$  and therefore  $\lambda = \mu = 0$ . Consequently  $x = y \in L^2$ . Thus  $Z(M) \subset L^2$ , which contradicts our assumption. Therefore  $z_i = \alpha_i z, \alpha_i \neq 0$ , as was asserted. We now set  $S = \Phi e_1 + \Phi e_{1'} + \dots + \Phi e_n + \Phi e_{n'}$  and define a bilinear form  $\theta$  on  $S$  by  $[x, y] = \theta(x, y)z$ . Then the matrix of  $\theta$  with respect to the basis  $e_1, e_{1'}, \dots, e_n, e_{n'}$  has the determinant  $\alpha_2^2 \alpha_3^2 \dots \alpha_n^2 \neq 0$ . Hence  $\theta$  is nondegenerate. The other conditions are obviously satisfied and  $L$  is of type (T). Thus the proof is complete.

PROPOSITION 1. *Let  $L$  be a Lie algebra of type  $(T)$  over a field  $\Phi$ . Then*

$$Z(\mathfrak{D}(L)) = (0).$$

**Proof.** Suppose  $D \in Z(\mathfrak{D}(L))$ . Since  $L$  is of type  $(T)$ , take  $S, z$  and  $\theta$  as in the definition. Let  $x$  be any nonzero element of  $S$ . Since  $\theta$  is nondegenerate, there is a nonzero element  $y$  of  $S$  such that  $\theta(x, y) \neq 0$  and the set  $S_1$  of all  $u \in S$  such that  $\theta(x, u) = 0$  is a subspace of  $S$  of codimension 1. Put  $M = S_1 + L^2$ . Then  $L = \Phi y + M$  and  $x \in Z(M)$ . An endomorphism  $D'$  sending  $y$  to  $x$  and  $M$  into  $(0)$  is a derivation of  $L$ . From the facts that  $D \in \mathfrak{C}(L)$  and  $Z(L) \subset M$ , it follows that  $Dx = [D, D']y = 0$ . Therefore  $D = 0$ .

LEMMA 3. *Let  $L$  be a Lie algebra over a field  $\Phi$ . Under the same hypotheses as in Lemma 2,  $\dim[L, Z(M)] = 1$  and  $\dim Z(M) = \dim Z(L) + 1$  for every ideal  $M$  of  $L$  of codimension 1.*

**Proof.** Let  $M$  be an ideal of  $L$  of codimension 1. Choose the elements  $e$  and  $e'$  as in Lemma 2. Then, using Lemma 2, we have

$$\Phi[e, e'] \subset [L, Z(M)] = [e, Z(M)] \subset [e, L] = \Phi[e, e']$$

and therefore  $[L, Z(M)] = [e, Z(M)] = \Phi[e, e'] \neq (0)$ . Hence  $\dim[L, Z(M)] = 1$ . Thus the rank of  $\text{ad}_{Z(M)}e$  is 1. But the kernel of  $\text{ad}_{Z(M)}e$  is  $Z(L)$ . Hence

$$\dim Z(M) = \dim Z(L) + 1.$$

PROPOSITION 2. *A Lie algebra  $L$  over a field  $\Phi$  is one of type  $(T)$  such that  $\dim Z(L) = 1$  if and only if  $L \neq L^2$ ,  $(0) \neq Z(L) \subset L^2$  and  $[L, Z(M)] = Z(M) \cap L^2$  for every ideal  $M$  of  $L$  of codimension 1.*

**Proof.** Assume that  $L$  is a Lie algebra of type  $(T)$  such that  $\dim Z(L) = 1$ . Then  $L \neq L^2$  and  $(0) \neq Z(L) \subset L^2$ . For every ideal  $M$  of codimension 1,  $Z(M)$  is a characteristic ideal of  $M$  and therefore  $[L, Z(M)] \subset Z(M) \cap L^2 \subset Z(L^2) = Z(L)$ . Since  $\dim[L, Z(M)] = 1$  by Theorem 2 and Lemma 3 and since  $\dim Z(L) = 1$  by assumption, it follows that  $[L, Z(M)] = Z(M) \cap L^2$ .

Conversely, assume that  $L \neq L^2$ ,  $(0) \neq Z(L) \subset L^2$  and  $[L, Z(M)] = Z(M) \cap L^2$  for every ideal  $M$  of codimension 1. If  $Z(M) \subset L^2$  for some ideal  $M$  of codimension 1,  $[L, Z(M)] = Z(M)$  and this contradicts Lemma 1 since  $Z(L) \subset L^2 \subset M$ . Hence  $Z(M) \not\subset L^2$  for every ideal  $M$  of  $L$  of codimension 1. By Theorem 2,  $L$  is of type  $(T)$ . Now we have  $Z(L) = Z(L^2) \subset Z(M) \cap L^2 = [L, Z(M)]$ . Since  $Z(L) \neq (0)$  and since  $\dim[L, Z(M)] = 1$  by Lemma 3, it follows that  $\dim Z(L) = 1$ .

PROPOSITION 3. *Let  $L$  be a Lie algebra over a field  $\Phi$ . Then each of the following conditions is necessary and sufficient for  $L$  to be pseudo-abelian:*

- (1)  $Z(L) = L^2$  and  $\dim Z(L) = 1$ .
- (2)  $Z(L) = L^2$  and  $\dim Z(M) = 2 \dim Z(L)$  for every ideal  $M$  of  $L$  of codimension 1.
- (3)  $Z(L) = L^2$  and  $\dim (Z(M) \cap L^2) = 1$  for every ideal  $M$  of  $L$  of codimension 1.

**Proof.** Assume that  $L$  satisfies the conditions in (1). Let  $S$  be any subspace of  $L$  complementary to  $L^2$ . Then the bilinear form  $\theta$  on  $S$  defined by  $[x, y] = \theta(x, y)z$  with  $0 \neq z \in Z(L)$  is nondegenerate since  $Z(L) = L^2$ .  $S$  obviously satisfies the other conditions so that  $L$  is pseudo-abelian. The converse is evident.

If  $L$  is pseudo-abelian, by Lemma 3 we see that  $\dim Z(M) = \dim Z(L) + 1$  for every ideal  $M$  of codimension 1. Since  $\dim Z(L) = 1$ , it follows that  $\dim Z(M) = 2 \dim Z(L)$ . Conversely, assume that  $L$  satisfies the conditions in (2). Then  $Z(M) \not\subseteq L^2$  for every ideal  $M$  of  $L$  of codimension 1. By Theorem 2,  $L$  is of type (T). But  $L$  is solvable since  $Z(L) = L^2$ . Thus  $L$  is pseudo-abelian.

If  $Z(L) = L^2$ ,  $L$  is nilpotent. Every ideal  $M$  of  $L$  of codimension 1 contains  $L^2$  and therefore  $Z(L) \subset Z(M)$ . It follows that  $Z(L) = Z(M) \cap L^2$ . Hence the condition (3) is equivalent to the condition (1).

As an immediate consequence of Proposition 3 we have the following

**COROLLARY.** *Let  $L$  be a pseudo-abelian Lie algebra over a field  $\Phi$ . Then for every ideal  $M$  of codimension 1,  $\dim Z(M) = 2$  and  $Z(L) = Z(M) \cap L^2$ .*

Here we give an example of nonsolvable Lie algebras of type (T). Let  $L$  be the Lie algebra over a field of characteristic  $\neq 2$  described in terms of a basis  $e_1, e_2, \dots, e_8$  by the multiplication table:

$$\begin{aligned} [e_1, e_2] &= 2e_2, & [e_1, e_3] &= -2e_3, & [e_2, e_3] &= e_1, \\ [e_1, e_6] &= -e_6, & [e_1, e_7] &= e_7, & [e_2, e_6] &= -e_7, \\ [e_3, e_7] &= -e_6, & [e_4, e_5] &= e_8, & [e_6, e_7] &= e_8. \end{aligned}$$

In addition  $[e_i, e_j] = -[e_j, e_i]$  and, for  $i < j$ ,  $[e_i, e_j] = 0$  if it is not in the table above. Then  $L$  is obviously such a Lie algebra.

In connection with the statements of Proposition 3 and its corollary, one might expect that  $L$  is pseudo-abelian if  $Z(L) = L^2 = Z(M) \cap L^2$  and  $\dim Z(M) = 2$  for every ideal  $M$  of  $L$  of codimension 1. However this does not hold in general. We shall show this fact by the following example.

Let  $L$  be the Lie algebra over the field of real numbers described in terms of a basis  $e_1, e_2, \dots, e_6$  by the following multiplication table:

$$\begin{aligned} [e_1, e_2] &= e_5, & [e_1, e_3] &= e_6, & [e_1, e_4] &= e_6, \\ [e_3, e_4] &= e_5, & [e_2, e_3] &= e_6. \end{aligned}$$

In addition  $[e_i, e_j] = -[e_j, e_i]$  and, for  $i < j$ ,  $[e_i, e_j] = 0$  if it is not in the table above. For any ideal  $M$  of  $L$  of codimension 1, we have  $Z(M) = Z(L)$ . The original proof of this fact contained a long computation and the following short proof was suggested by the referee. Suppose that  $f \in Z(M)$ . Then  $f = \sum_{i=1}^6 \alpha_i e_i$ . Since  $\text{ad } f$  has rank  $\leq 1$ ,  $[f, e_2]$  and  $[f, e_4]$  are linearly dependent, from which it follows that  $\alpha_1^2 + \alpha_3^2 = 0$  and therefore  $\alpha_1 = \alpha_3 = 0$ . Similarly from the linear dependence of

$[f, e_1]$  and  $[f, e_3]$  we have  $\alpha_2 = \alpha_4 = 0$ . Hence  $f \in Z(L)$  and  $Z(M) = Z(L)$ . We see that  $L$  satisfies the conditions stated above, but  $L$  is not pseudo-abelian.

**5. Ideals of the derivation algebras containing outer derivations.** In this section, by making use of Theorem 2 in §4 we shall show some results generalizing and sharpening a result of Satô [5].

Through this section, we denote by  $\mathfrak{R}_0$  (resp.  $\mathfrak{C}_0$ ) the set of all the derivations of  $L$  which map  $L$  into  $L^2$  (resp.  $Z(L)$ ) and  $L^2$  (resp.  $Z(L)$ ) into  $(0)$ . Then from the fact that  $L^2$  and  $Z(L)$  are characteristic ideals of  $L$ , it follows that  $\mathfrak{R}_0$  and  $\mathfrak{C}_0$  are abelian ideals of  $\mathfrak{D}(L)$ .

**THEOREM 3.** *Let  $L$  be a nonabelian Lie algebra in the class  $\mathfrak{D}$ .*

(1) *If  $L$  has no nonzero abelian direct summands and if  $L$  is not of type  $(T)$ , then  $L$  has an outer derivation in  $\mathfrak{R}_0$ .*

(2) *Assume that  $L$  has a nonzero abelian direct summand.*

(a) *If  $Z(L)$  is not a direct summand of  $L$ , then  $L$  has an outer derivation in  $\mathfrak{R}_0 \cap \mathfrak{C}(L)$ .*

(b) *If  $Z(L)$  is a direct summand of  $L$  and  $L/Z(L)$  does not coincide with the derived algebra, then  $L$  has an outer derivation in  $\mathfrak{C}_0$ .*

(c) *If  $Z(L)$  is a direct summand of  $L$  and  $L/Z(L)$  coincides with the derived algebra, then  $L$  has an outer derivation in  $Z(\mathfrak{D}(L))$ .*

(3) *If  $L$  is of type  $(T)$  and  $L^{(1)} \neq L^{(2)}$ , then  $L$  has an outer derivation in the radical of  $\mathfrak{D}(L)$ .*

**Proof.** (1) Assume that  $L$  has no nonzero abelian direct summands and  $L$  is not of type  $(T)$ . Then  $Z(L) \subset L^2$ . Therefore by Theorem 2 there exists an ideal  $M$  of  $L$  of codimension 1 such that  $Z(M) \subset L^2$ . Every such  $M$  contains  $L^2$  and therefore  $Z(L)$ . By Lemma 1  $[L, Z(M)] \subset Z(M)$  but  $[L, Z(M)] \neq Z(M)$ . As in the first part of the proof of Theorem 1 we can show that every endomorphism  $D$  of  $L$  defined in such a way that  $DL \subset Z(M) \setminus [L, Z(M)]$  and  $DM = (0)$  is an outer derivation of  $L$ , which belongs to  $\mathfrak{R}_0$  since  $Z(M) \subset L^2 \subset M$ .

(2) Assume that  $L$  has a nonzero abelian direct summand. Then, as in the second part of the proof of Theorem 1, we see that  $L$  is the direct sum of a nonzero abelian ideal  $L_1$  and a nonzero ideal  $L_2$  such that  $Z(L_2) \subset L_2^2$ . If  $Z(L)$  is not a direct summand of  $L$ , then  $Z(L_2) \neq (0)$ . Every nonzero endomorphism  $D$  of  $L$  defined in such a way that  $DL_1 \subset Z(L_2)$  and  $DL_2 = (0)$  is an outer derivation of  $L$  belonging to  $\mathfrak{R}_0 \cap \mathfrak{C}(L)$ . If  $Z(L)$  is a direct summand and  $L/Z(L)$  does not coincide with its derived algebra, then  $Z(L) = L_1$  and  $L_2 \neq L_2^2$ . Every nonzero endomorphism  $D$  of  $L$  such that  $DL_2 \subset L_1$  and  $D(L_1 + L_2^2) = (0)$  is an outer derivation of  $L$  belonging to  $\mathfrak{C}_0$ . If  $Z(L)$  is a direct summand and  $L/Z(L)$  coincides with its derived algebra, then  $Z(L) = L_1$  and  $L_2 = L_2^2$ . The trivial extension of the identity endomorphism of  $L_1$  to  $L$  is an outer derivation belonging to  $Z(\mathfrak{D}(L))$ .

(3) Assume that  $L$  is a Lie algebra of type  $(T)$  such that  $L^{(1)} \neq L^{(2)}$ . Let  $S$  and  $z$  be as in the definition of Lie algebras of type  $(T)$ . Then  $L^{(1)} = \Phi z + L^{(2)}$  and  $z \notin L^{(2)}$ .

Put  $L_1 = S + \Phi Z$  and  $L_2 = L^{(2)}$ . Then  $L_1$  is pseudo-abelian and  $L$  is the direct sum of the ideals  $L_1$  and  $L_2$ . In the case where  $L_2 \neq (0)$ , define an endomorphism  $D_0$  of  $L$  in such a way that

$$\begin{aligned} D_0 x &= x \quad \text{for } x \in S, \\ D_0 z &= \begin{cases} 2z & \text{if the characteristic of } \Phi \text{ is } \neq 2, \\ 0 & \text{if the characteristic of } \Phi \text{ is } 2, \end{cases} \\ D_0 L_2 &= (0). \end{aligned}$$

Then  $D_0$  is a semisimple outer derivation of  $L$ . Putting  $\mathfrak{R}_0 = \Phi D_0 + \mathfrak{C}(L)$ , we assert that  $\mathfrak{R}_0$  is a solvable ideal of  $\mathfrak{D}(L)$ . In fact, since  $Z(L) \subset L^2$ ,  $\mathfrak{C}(L)$  is an abelian ideal of  $\mathfrak{D}(L)$ . Hence  $\mathfrak{R}_0$  is a solvable subalgebra of  $\mathfrak{D}(L)$ . Let  $D$  be an arbitrary derivation of  $L$ . For any  $x \in S$ , write  $Dx = y + \lambda z + u$  with  $y \in S$  and  $u \in L_2$ . Then  $u \in Z(L_2) \subset Z(L)$ . Therefore, if the characteristic of  $\Phi$  is  $\neq 2$  (resp. is 2), we have

$$[D, D_0]x = Dx - D_0 Dx = -\lambda z + u$$

(resp.  $\lambda z + u$ ). Hence  $[D, D_0]x \in Z(L)$ . Since  $Z(L)$  and  $L^{(2)}$  are characteristic ideals of  $L$ , we have  $[D, D_0]z \in Z(L)$  and  $[D, D_0]L_2 \subset D_0 DL_2 \subset D_0 L_2 = (0)$ . Therefore we conclude that  $[D, D_0] \in \mathfrak{C}(L)$ . Thus  $\mathfrak{R}_0$  is a solvable ideal of  $\mathfrak{D}(L)$ , as was asserted. In the case where  $L_2 = (0)$ , define  $D_0$  and  $\mathfrak{R}_0$  similarly. Then  $D_0$  is a semisimple outer derivation of  $L$  and  $\mathfrak{R}_0$  is a solvable ideal of  $\mathfrak{D}(L)$ . Thus in both cases we see that there is an outer semisimple derivation in the radical of  $\mathfrak{D}(L)$ . The proof is complete.

Here we shall show that there is an example of Lie algebras  $L$  satisfying the hypotheses in each case of Theorem 3 such that the indicated ideal of  $\mathfrak{D}(L)$  contains no ideals of  $\mathfrak{D}(L)$  different from itself and containing outer derivations.

Let  $L$  be the Lie algebra over a field  $\Phi$  of characteristic  $\neq 2$  described in terms of a basis  $e_1, e_2, e_3, e_4$  by the multiplication table

$$\begin{aligned} [e_1, e_2] &= e_2, & [e_1, e_3] &= -e_3, & [e_2, e_3] &= e_4, \\ [e_i, e_4] &= 0 & \text{for } i &= 1, 2, 3. \end{aligned}$$

Then  $L$  satisfies the hypotheses in the case (1) of Theorem 3. Let  $D$  be any derivation of  $L$  and put  $De_i = \sum_{j=1}^4 \lambda_{ij} e_j$  ( $i = 1, 2, 3, 4$ ). Then the matrix of  $D$  is

$$\begin{pmatrix} 0 & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ 0 & \lambda_{22} & 0 & -\lambda_{13} \\ 0 & 0 & \lambda_{33} & -\lambda_{12} \\ 0 & 0 & 0 & \lambda_{22} + \lambda_{33} \end{pmatrix}.$$

Denote by  $D_i$ , the derivation of  $L$  whose matrix is obtained from the above matrix by putting  $\lambda_{ij} = 1$  and  $\lambda_{kl} = 0$  for all  $(k, l) \neq (i, j)$ . Then  $\mathfrak{R}_0 = \Phi D_{14}$  where  $D_{14}$  is outer.

Let  $L$  be the Lie algebra over a field  $\Phi$  described in terms of a basis  $e_1, e_2, e_3, e_4$  by the multiplication table

$$[e_1, e_i] = 0 \quad \text{for } i = 2, 3, 4,$$

$$[e_2, e_3] = e_4, \quad [e_2, e_4] = [e_3, e_4] = 0.$$

Then  $L$  satisfies the hypotheses in the case (2)(a) of Theorem 3. Since  $L^2 \subset Z(L)$ ,  $\mathfrak{R}_0 \subset \mathfrak{C}(L)$  and therefore  $\mathfrak{R}_0 \cap \mathfrak{C}(L) = \mathfrak{R}_0$ . Let  $D$  be any derivation of  $L$ . Using the similar notations  $\lambda_{ij}$  and  $D_{ij}$  as above, the matrix of  $D$  is

$$\begin{pmatrix} \lambda_{11} & 0 & 0 & \lambda_{14} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} & \lambda_{24} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} & \lambda_{34} \\ 0 & 0 & 0 & \lambda_{22} + \lambda_{33} \end{pmatrix}.$$

Then  $\mathfrak{R}_0 = \Phi D_{14} + \Phi D_{24} + \Phi D_{34}$ , where  $D_{14}$  is outer and  $D_{24}, D_{34}$  are inner. Let  $\mathfrak{M}$  be an ideal of  $\mathfrak{D}(L)$  which is contained in  $\mathfrak{R}_0$  and contains outer derivations. Then  $\mathfrak{M}$  contains  $D_{14} + \lambda D_{24} + \mu D_{34}$ . Since  $[D_{14} + \lambda D_{24} + \mu D_{34}, D_{21}] = D_{24}$  and  $[D_{24}, D_{32}] = D_{34}$ ,  $\mathfrak{M}$  coincides with  $\mathfrak{R}_0$ .

Let  $L$  be the direct sum of the 1-dimensional Lie algebra and the 2-dimensional nonabelian solvable Lie algebra over a field  $\Phi$ . Then  $L$  satisfies the hypotheses in the case (2)(b) of Theorem 3 and  $\mathfrak{C}_0$  is 1-dimensional.

Let  $L$  be the direct sum of the 1-dimensional Lie algebra and a semisimple Lie algebra over a field of characteristic 0. Then  $L$  satisfies the hypotheses in the case (2)(c) of Theorem 3 and  $Z(\mathfrak{D}(L))$  is 1-dimensional.

Finally, let  $L$  be a pseudo-abelian Lie algebra over a field  $\Phi$  of characteristic  $\neq 2$ . Then  $L$  satisfies the hypotheses in the case (3) of Theorem 3. Let  $S, z$  and  $\theta$  be as in the definition of pseudo-abelian Lie algebras. Since  $\theta$  is nondegenerate,  $\mathfrak{C}(L) = \mathfrak{F}(L)$ . Let  $\mathfrak{S}$  be the set of all endomorphisms of  $S$  which are skew relative to  $\theta$ . Then  $\mathfrak{S}$  is a symplectic Lie algebra. Take the set of trivial extensions of elements of  $\mathfrak{S}$  to  $L$ , which we again denote by  $\mathfrak{S}$ . Then  $\mathfrak{S} \subset \mathfrak{D}(L)$ . Let  $D_0$  and  $\mathfrak{R}_0$  be as in the third part of the proof of Theorem 3. We assert that  $\mathfrak{D}(L) = \mathfrak{S} + \mathfrak{R}_0$ . In fact, any derivation  $D$  of  $L$  induces the endomorphism  $D'$  of  $L$  mapping  $S$  into  $Z(L)$  and  $Z(L)$  into  $(0)$ .  $D'$  is in  $\mathfrak{C}(L)$ . We denote by  $\lambda$  the coefficient of  $z$  in  $Dz$  and put  $D_1 = D - (\lambda/2)D_0 - D'$ . Then it is immediate that  $D_1 \in \mathfrak{S}$  and we have the assertion. Thus  $\mathfrak{R}_0$  is the radical of  $\mathfrak{D}(L)$ . Since  $[D', D_0] = D'$  for any  $D'$  in  $\mathfrak{F}(L)$ ,  $\mathfrak{R}_0$  contains no ideals of  $\mathfrak{D}(L)$  different from  $\mathfrak{R}_0$  and containing outer derivations.

**THEOREM 4.** *Let  $L$  be a solvable Lie algebra over a field  $\Phi$  such that  $Z(L) \neq (0)$ .*

(1) *If  $L$  is not abelian and not pseudo-abelian and if  $L$  has no nonzero abelian direct summands, then  $L$  has an outer derivation in  $\mathfrak{R}_0$ .*

(2) *Assume that  $L$  is not abelian and has a nonzero abelian direct summand. If  $Z(L)$  is not a direct summand of  $L$ , then  $L$  has an outer derivation in  $\mathfrak{R}_0 \cap \mathfrak{C}(L)$ . If  $Z(L)$  is a direct summand of  $L$ , then  $L$  has an outer derivation in  $\mathfrak{C}_0$ .*

(3) If  $L$  is abelian,  $L$  has an outer derivation in  $Z(\mathfrak{D}(L))$ . If  $L$  is pseudo-abelian,  $L$  has an outer derivation in the radical of  $\mathfrak{D}(L)$ .

**Proof.** Let  $L$  be a solvable Lie algebra over  $\Phi$  such that  $Z(L) \neq (0)$ . Then  $L \neq L^2$ . We do not have the case (c) in the statement (2) of Theorem 3. If  $L$  is of type (T), then  $L$  is pseudo-abelian and  $L^{(1)} \neq L^{(2)}$ . The statement is evident for the case where  $L$  is abelian. Thus the theorem follows from Theorem 3.

**COROLLARY 1.** Let  $L$  be a Lie algebra in the class  $\mathfrak{D}$ . If  $L$  is not of type (T),  $L$  has an outer derivation in an abelian ideal of  $\mathfrak{D}(L)$ .

**Proof.** This is immediate from Theorems 3 and 4.

**COROLLARY 2.** Every Lie algebra  $L$  such that  $L \neq L^{(1)}$ ,  $L^{(1)} \neq L^{(2)}$  and  $Z(L) \neq (0)$ , every solvable Lie algebra  $L$  such that  $Z(L) \neq (0)$ , and every nonzero nilpotent Lie algebra, over a field  $\Phi$ , have outer derivations in the radicals of their derivation algebras.

**Proof.** This is immediate from Theorems 3 and 4.

**COROLLARY 3.** Let  $L$  be a nilpotent Lie algebra over a field  $\Phi$ . Assume that  $L$  is neither abelian nor pseudo-abelian. Then  $L$  has an outer derivation in  $\mathfrak{R}_0$  and especially in  $\mathfrak{R}_0 \cap \mathfrak{C}(L)$  if  $L$  has a nonzero direct summand different from itself.

**Proof.** Owing to Theorem 4 it suffices to prove the statement in the case where  $L$  has a nonzero direct summand different from itself. In this case  $L$  has a nonabelian direct summand  $L_1$  which is different from  $L$ . If  $Z(L_1) \not\subset L_1^2$ , as in the second part of the proof of Theorem 1,  $L_1$  is the direct sum of a central ideal and a nonzero ideal  $L_0$  such that  $Z(L_0) \subset L_0^2$ . Thus we may assume that  $Z(L_1) \subset L_1^2$ . Denote by  $L_2$  the ideal of  $L$  complementary to  $L_1$ . Then  $L_2 \neq (0)$ . Define a nonzero endomorphism  $D$  of  $L$  in such a way that  $DL_1 = (0)$ ,  $DL_2 \subset Z(L_1)$  and  $DL_2^2 = (0)$ . Then  $D$  is an outer derivation belonging to  $\mathfrak{R}_0 \cap \mathfrak{C}(L)$ .

**6. Nonsolvable Lie algebras which have semisimple outer derivations.** Besides the class  $\mathfrak{D}$  of Lie algebras considered in §§ 3, 4 and 5, there is another class of Lie algebras which have outer derivations. For example, let  $L_1$  be a simple Lie algebra and  $M$  be a faithful irreducible  $L_1$ -module. Regarding  $M$  as an abelian Lie algebra, we put  $L = L_1 + M$  (semidirect sum). Then the endomorphism of  $L$  acting as identity on  $M$  and zero on  $L_1$  is a semisimple derivation of  $L$  which is obviously not inner.

In the rest of this section, we assume that  $L$  is a Lie algebra over a field of characteristic 0. For a maximal semisimple subalgebra  $S$  of  $L$ , we denote by  $\mathfrak{A}(S)$  the set of all derivations of  $L$  which map  $S$  into  $(0)$ . Then we have

**LEMMA 4.** Let  $R$  be the radical of  $L$  and let  $L = S + R$  be a Levi decomposition of  $L$ . Then among maximal toroidal subalgebras of the radical of  $\mathfrak{D}(R)$  there exists one which can be imbedded in  $\mathfrak{A}(S)$ .

**Proof.** Let  $\mathfrak{S}$  be a maximal semisimple subalgebra of  $\mathfrak{D}(R)$  containing  $\text{ad}_R S$ . Since  $\mathfrak{D}(R)$  is algebraic, there exists a maximal toroidal subalgebra  $\mathfrak{T}$  of the radical

of  $\mathfrak{D}(R)$  such that  $[\mathfrak{S}, \mathfrak{X}] = (0)$ . Hence  $[\text{ad}_R S, \mathfrak{X}] = (0)$ . For any  $D$  in  $\mathfrak{X}$  we have  $D[s, r] = [s, Dr]$  for any  $s \in S$  and any  $r \in R$ . Therefore  $D$  can be extended to be a derivation of  $L$  by putting  $Ds = (0)$ . Thus  $\mathfrak{X}$  can be imbedded in  $\mathfrak{A}(S)$ .

**THEOREM 5.** *Let  $L$  be a nonsolvable Lie algebra over a field of characteristic 0. Let  $R$  be the radical of  $L$ . Then, if  $R$  has a semisimple outer derivation in the radical of  $\mathfrak{D}(R)$ ,  $L$  has a semisimple outer derivation. Especially, in the case where  $R$  is nilpotent, if  $R$  has a semisimple derivation in the radical of  $\mathfrak{D}(R)$ , then  $L$  has a semisimple outer derivation.*

**Proof.** Let  $L = S + R$  be a Levi decomposition of  $L$ . Let  $\mathfrak{R}$  be the radical of  $\mathfrak{D}(R)$ . By Lemma 4 there exists a maximal toroidal subalgebra  $\mathfrak{X}$  of  $\mathfrak{R}$  which can be imbedded in  $\mathfrak{A}(S)$ . Assume that  $D'$  is a semisimple outer derivation belonging to  $\mathfrak{R}$ . Then there exists an element  $E$  of  $\mathfrak{R}^2$  such that  $D'_1 = \sigma D' \in \mathfrak{X}$  with  $\sigma = \exp(\text{ad}_{\mathfrak{R}} E)$ . Let  $D_1$  be the derivation in  $\mathfrak{A}(S)$  which is a trivial extension of  $D'_1$ . Then  $D_1$  is a semisimple derivation of  $L$ . Furthermore  $D_1$  is not inner. In fact, if  $D_1$  is inner,  $D_1 = \text{ad}(s+r)$  with  $s \in S$  and  $r \in R$ . It follows that  $D_1 S = [s+r, S] = (0)$  and therefore  $[s, S] = [r, S] \subset S \cap R = (0)$ . Hence  $s = 0$  and  $D_1 = \text{ad } r$ . It follows that  $D'_1 = \text{ad}_R r$  and therefore  $D' = \sigma^{-1} D'_1 = \text{ad}_R u$  with  $u = (\exp(-E))r$ , which contradicts our assumption. Thus  $D_1$  is a semisimple outer derivation of  $L$  and the first statement is proved. In the case where  $R$  is nilpotent, every semisimple derivation of  $R$  is outer. Hence the second statement is immediate from the first one.

**COROLLARY.** *Let  $L$  be a Lie algebra over a field of characteristic 0 whose radical  $R$  is nilpotent. If  $R$  has a derivation whose trace is  $\neq 0$ , then  $L$  has a semisimple outer derivation. In particular, this is the case if  $R$  is quasi-cyclic.*

**Proof.** Assume that  $L$  has no semisimple outer derivations. In the case where  $L$  is not solvable, by Theorem 5 we see that  $R$  has no semisimple derivations in the radical  $\mathfrak{R}$  of  $\mathfrak{D}(R)$ . Let  $\mathfrak{D}(R) = \mathfrak{S} + \mathfrak{R}$  be a Levi decomposition of  $\mathfrak{D}(R)$ . Then  $\mathfrak{R}$  is splittable and therefore consists of nilpotent elements. Since  $\mathfrak{S} = [\mathfrak{S}, \mathfrak{S}]$ , the trace of every derivation of  $R$  is 0. In the case where  $L$  is solvable,  $L = R$  and therefore  $R$  has no semisimple derivations. Since  $\mathfrak{D}(R)$  is splittable,  $\mathfrak{D}(R)$  consists of only nilpotent derivations. Hence the trace of every derivation of  $R$  is 0. Thus the first statement is proved.

If  $R$  is quasi-cyclic, there is a subspace  $U$  of  $R$  such that  $R = \sum_i U^i$  with  $U^i \cap U^j = (0)$  for  $i \neq j$ , where  $U^1 = U$  and  $U^i = [U, U^{i-1}]$  for  $i \geq 2$ . Define the endomorphism  $D$  of  $R$  in such a way that  $Du = iu$  for  $u \in U^i$  ( $i = 1, 2, \dots$ ). Then  $D$  is a derivation of  $R$  whose trace is  $\neq 0$ .

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