

# A WEAK CONTAINMENT THEOREM FOR GROUPS WITH A QUOTIENT $R$ -GROUP

BY  
LAWRENCE BAGGETT

**Introduction.** In this paper we investigate the following problem: Suppose  $G$  is a locally compact group and let  $N$  be a closed normal subgroup of  $G$ . What hypotheses are needed on  $G$  and  $N$  in order that each element  $W$  of the dual space  $\hat{G}$  of  $G$  should be weakly contained in some induced representation of the form  $U^\chi$ , where  $\chi$  is an element of the dual space  $\hat{N}$  of  $N$ ? This is a generalization of the classical question: What conditions must we put on a locally compact group  $G$  in order that each of its irreducible unitary representations should be weakly contained in the regular representation of  $G$ ? What we show here is that the generalized question may be reduced, in the presence of some quite general assumptions, to the classical one.

Our presentation leans heavily on the analysis of  $\hat{G}$  in terms of  $\hat{N}$  à la Mackey, and we therefore do not claim the simplest possible proof. However, consideration of this question seems to indicate a necessity for using some kind of detailed analysis of representations.

In §1 we give a sketch of the particular version of Mackey's theory which we will use. We are then able to state the "quite general assumptions" mentioned above. In §2 we prove the equivalence of two representations. This equivalence is not needed in the proofs of the main theorems of this paper, but it is of interest in itself and will be referred to in a subsequent paper. In §3 the idea of "weak containment" for cocycle representations is presented, and in §4 we give the main theorems.

We introduce here the following conventions in notation.

i. If  $X$  is a locally compact topological space, then  $L(X)$  denotes the linear space of all continuous complex valued functions with compact support on  $X$ .

ii. A topological group is called "separable" if, as a topological space, it satisfies the second axiom of countability.

iii. We write  $e$  for the identity of all groups. Whenever any confusion could arise, we will be precise.

iv. If  $f$  is a function on a space  $X$ , and if  $Z$  is a subset of  $X$ , we write  $f|_Z$  for the restriction of the function  $f$  to the subset  $Z$ .

v. If  $G$  is a locally compact group,  $K$  is a closed subgroup of  $G$ , and  $S$  is a unitary representation of  $K$ , we write  $U^S$ , or  ${}_G U^S$ , for the representation of  $G$  induced from  $S$ .

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- vi. If  $G$  is a locally compact group and if  $T$  and  $S$  are unitary representations of  $G$ , we write  $T \otimes S$  for the tensor product representation.
- vii. If  $E$  is a normed linear space, we denote the norm in  $E$  by  $\| \cdot \|_E$ . If  $H$  is a Hilbert space, we denote the inner product in  $H$  by  $( \cdot , \cdot )_H$ .

### 1. Preliminaries.

1.1-A. DEFINITION. Let  $G$  be a locally compact group. By the dual space  $\hat{G}$  of  $G$  we mean the set of all equivalence classes of irreducible unitary representations of  $G$  equipped with the hull-kernel topology. (See [2].)

For basic definitions and propositions about "weak containment" for unitary representations and for the elements of the dual spaces of groups, also see [2]. For general facts about representations, i.e., equivalence, irreducibility, etc., see [11].

1.1-B. Suppose  $G$  is a separable locally compact group. A *cocycle* on  $G$  is to mean a Borel mapping  $a$  of  $G \times G$  into the set of complex numbers of absolute value one such that, for all  $x, y$ , and  $z$  in  $G$ ,

- i.  $a(x, e) = a(e, x) = 1$ .
- ii.  $a(xy, z)a(x, y) = a(x, yz)a(y, z)$ .

A *cocycle representation* of  $G$  is a mapping  $x \rightarrow T_x$  of  $G$  into the set of all unitary operators on some separable Hilbert space  $H(T)$  such that:

- i. For each pair of vectors  $(p, q)$  of  $H(T)$ , the mapping  $x \rightarrow ([T_x(p)], q)_{H(T)}$  is a Borel mapping on  $G$ .
- ii. For each pair of elements  $(x, y)$  of  $G$ , there exists a complex number  $a(x, y)$  such that  $T_x T_y = a(x, y) T_{xy}$ .

REMARK. The mapping  $a$  of condition ii above is a cocycle and is called the "cocycle of the (cocycle) representation."  $T$  is also called an " $a$ -representation."

If  $a$  is a cocycle on  $G$ , define the "regular  $a$ -representation"  $R^a$  of  $G$  as follows:  $H(R^a) = L^2(G)$ . If  $f$  is in  $L^2(G)$  and  $x$  and  $y$  are  $G$ , put

$$[R_y^a(f)](x) = f(y^{-1}x)a(y, y^{-1}x).$$

Let  $T$  be a cocycle representation of  $G$ . A *fundamental function* on  $G$  is a mapping  $x \rightarrow ([T_x(q)], q)_{H(T)}$ , where  $q$  is a vector in  $H(T)$ . Such a fundamental function is said to be *associated* with  $T$ .

REMARK. Fundamental functions resemble "functions of positive type." We have not used that terminology because fundamental functions need not be continuous while functions of positive type always are continuous. See [13].

1.1-C. Now suppose  $G$  is an arbitrary locally compact group and let  $N$  be a closed normal subgroup of  $G$ . By a *cross section* of  $G/N$  into  $G$  we mean a mapping  $p$  of  $G/N$  into  $G$  which satisfies:

- i. If  $y$  is in  $G/N$ , then  $p(y)$  is contained in the coset of  $N$  which is determined by  $y$ .
- ii.  $p$  effects a Borel isomorphism between  $G/N$  and  $p(G/N)$ .
- iii. If  $C$  is a compact subset of  $G/N$ , then  $p(C)$  has compact closure in  $G$ .

**PROPOSITION.** *Let  $G$  and  $N$  be as above. If  $G$  is separable, then a cross section of  $G/N$  into  $G$  always exists.*

For the proof, see [10].

1.1-D. Again let  $G$  be an arbitrary locally compact group and  $N$  a closed normal subgroup of  $G$ . Then  $G$  acts to the left (via inner automorphisms) as a group of transformations on the dual space  $\hat{N}$  of  $N$ . If  $\chi$  is an element of  $\hat{N}$ , define the *stability subgroup*  $G_\chi$  of  $G$  for  $\chi$  as the set of all elements  $x$  of  $G$  such that  $x \cdot \chi$  equals  $\chi$ . Then  $G_\chi$  is a subgroup of  $G$  which contains  $N$ .

**DEFINITION.** Let  $G$  and  $N$  be as above and assume that  $G$  is separable. We say that  $N$  is *regularly imbedded* in  $G$  if, for each element  $\chi$  of  $\hat{N}$ , the natural mapping of  $G/(G_\chi)$  onto the orbit of  $\hat{N}$  to which  $\chi$  belongs is a homeomorphism.

**REMARK.** This definition is equivalent to Mackey's. See [10].

**DEFINITION.** An orbit  $\theta$  of  $\hat{N}$  is called *accommodating* if, for each element  $\chi$  of  $\theta$ , there exists a unitary extension of  $\chi$  to  $G_\chi$ , i.e., there exists a unitary representation  $\chi'$  of  $G_\chi$  such that  $\chi'|_N$  is equivalent to  $\chi$ .

1.1-E. We are now ready to state the two inequivalent sets of hypotheses on  $G$  and  $N$  with which we will deal simultaneously throughout the sequel. The value of these hypotheses is at least partially indicated in Propositions 1 and 3 of 1.1-F below.

1.  $N$  is abelian and  $G/N$  is compact.
2.  $G$  is separable,  $N$  is of type I, and  $N$  is regularly imbedded in  $G$ .

**LEMMA 1.** *Let  $(G, N)$  satisfy 1 or 2 above. Then each stability subgroup  $G_\chi$ , for  $\chi$  in  $\hat{N}$ , is closed.*

**Proof.** See [7] and [9].

**LEMMA 2.** *Suppose  $G$  and  $N$  satisfy either 1 or 2 above. Then, to each irreducible unitary representation  $T$  of  $G$ , there corresponds (in a canonical way) a unique orbit of  $\hat{N}$ . This orbit is said to be associated with the representation  $T$ .*

**Proof.** See [10].

**PROPOSITION.** *Let  $(G, N)$  satisfy 1 or 2 above. Let  $T$  be an irreducible representation of  $G$  and let  $\chi$  be an element of the orbit of  $\hat{N}$  with which  $T$  is associated. Then there exists an irreducible representation  $S$  of  $G_\chi$  such that:*

- i.  $S|_N$  is a multiple of  $\chi$ .
- ii.  $U^S$  is equivalent to  $T$ .

For the proof in case 2, see [10]. This proof goes through in case 1 as well.

1.1-F. **PROPOSITION 1.** *Assume  $G$  and  $N$  satisfy 1 of E above and suppose further that each orbit of  $\hat{N}$  is accommodating. Let  $T$  be an irreducible unitary representation of  $G$  and let  $\theta$  be the orbit of  $\hat{N}$  with which  $T$  is associated. Then, if  $\chi$  is an element of  $\theta$ ,  $T$  is equivalent to  $U^{\chi' \otimes S \cdot \pi}$ , where  $\chi'$  is a unitary extension of  $\chi$  to its stability subgroup  $G_\chi$ , and where  $S$  is an irreducible unitary representation of  $G_\chi/N$ .*

For the proof when  $G$  is separable, see [10]. Under the present hypotheses, Mackey's proof in [11] can be extended to cover the case when  $G$  is not separable.

**PROPOSITION 2.** *Assume  $G$  and  $N$  satisfy 2 of E above. Let  $\chi$  be an element of  $\hat{N}$ . Then, there exist a cocycle  $a$  of  $G_x/N$  and a cocycle representation  $\chi'$  of  $G_x$  which satisfy:*

- i.  $\chi'|_N$  is equivalent to  $\chi$ .
- ii. The cocycle  $a'$  for  $\chi'$  is given by:

$$a'(x, y) = a(\pi(x), \pi(y)).$$

For the proof, see [12].

**PROPOSITION 3.** *Assume  $G$  and  $N$  satisfy 2 of E above. Let  $T$  be an irreducible unitary representation of  $G$ . Then, if  $\chi$  is an element of the orbit of  $\hat{N}$  with which  $T$  is associated,  $T$  is equivalent to  $U^x \otimes S^{\cdot \pi}$ , where  $\chi'$  is the cocycle representation of  $G_x$  guaranteed by Proposition 2 above, and where  $S$  is an irreducible cocycle representation of  $G_x/N$  whose cocycle is  $\bar{a}$ . ("a" is the cocycle of Proposition 2.)*

For the proof, see [12].

1.2. Assume  $G$  and  $N$  satisfy one of two sets of hypotheses of 1.1-E. If  $(G, N)$  satisfies 2, then, by 1.1-C, there exists a cross section  $p$  of  $G/N$  into  $G$ . From here to the end of §2, we assume the following: If  $(G, N)$  satisfies 1 of 1.1-E but not 2, then there exists a continuous cross section  $p$  of  $G/N$  into  $G$ . Let  $K$  equal  $G/N$ . Fix an element  $\chi$  of  $\hat{N}$ , and call the space  $H(\chi)$  of  $\chi$  simply  $H$ . It follows from our assumptions that  $H$  is separable.

1.2-A. In view of the normality of  $N$ , the separability of  $H$ , and the presence of the cross section  $p$ , we may state the definition of the induced representation  $U^x$  as follows.

**DEFINITION.** The space  $H(U^x)$  consists of all functions  $f$  on  $G$  into  $H$  which satisfy:

- i. For each vector  $q$  of  $H$ , the mapping  $x \rightarrow (f(x), q)_H$  is locally measurable on  $G$ .
- ii. If  $x$  is in  $G$  and  $n$  is in  $N$ , then  $f(xn) = \chi_{(n^{-1})}[f(x)]$ , or, if  $k$  is in  $K$  and  $n$  is in  $N$ , then  $f[p(k)n] = \chi_{(n^{-1})}[f[p(k)]]$ .
- iii.  $\int_K \|f[p(k)]\|_H^2 dk$  is finite.

The inner product in  $H(U^x)$  is given by:

$$(f, g)_{H(U^x)} = \int_K ([f[p(k)]], [g[p(k)]])_H dk.$$

This definition is essentially the one given in [10].

Finally, the action of  $U^x$  is given as follows. If  $x$  and  $y$  are in  $G$  and  $f$  is in  $H(U^x)$ , put  $[U_y^x(f)](x) = f(y^{-1}x)$ .

1.2-B. We now transfer the action of  $U^x$  to another Hilbert space, namely the space  $L^2(K, H)$ , i.e., the space of all functions  $h$  on  $K$  into  $H$  which satisfy:

i. The function  $k \rightarrow (h(k), q)_H$  is a locally measurable function on  $K$  for each vector  $q$  in  $H$ .

ii.  $\int_K \|h(k)\|_H^2 dk$  is finite.

Let  $X$  be the dense subspace of  $H(U^x)$  consisting of continuous functions on  $G$  into  $H$ . (See [1].) Define  $\theta$  to be the following mapping of  $X$  into the set of all  $H$ -valued functions on  $K$ .

If  $f$  is in  $X$  and  $k$  is in  $K$ , put

$$[\theta(f)](k) = f[p(k)].$$

Now the function  $\theta(f)$  is clearly a Borel mapping of  $K$  into  $H$  and it satisfies the measurability condition of i above. Also,  $\theta$  is clearly linear on  $X$ .

1.2-C. PROPOSITION.  $\theta$  is an isometry of  $X$  into  $L^2(K, H)$ .

The proof is routine.

1.2-D. PROPOSITION.  $\theta$  maps  $H(U^x)$  onto all of  $L^2(K, H)$ .

**Proof.** Of course the statement implies that  $\theta$  is defined on all of  $H(U^x)$  which, as yet, we have not observed. However, since  $\theta$  is an isometry, we may extend  $\theta$  to the closure of its domain,  $X$ , which equals all of  $H(U^x)$ .

Now let  $g$  be in  $L^2(K, H)$ . Define  $q(g)$  to be the following proposed element of  $H(U^x)$ .

If  $k$  is in  $K$ , and  $n$  is in  $N$ ,

$$[q(g)][p(k)n] = \chi_{(n^{-1})}(g(k)).$$

We may establish that  $q(g)$  is in  $H(U^x)$ , and clearly  $\theta(q(g)) = g$ .

1.2-E. It remains to transfer the action of  $U^x$  to its new home  $L^2(K, H)$ . For the purposes of this computation, let  $T$  denote the representation  $\theta \cdot U^x \cdot \theta^{-1}$ .

Now if  $k$  is in  $K$ ,  $n$  is in  $N$ ,  $l$  is in  $K$ ,  $f$  is in  $H(U^x)$ , and  $h$  equals  $\theta(f)$ ,

$$\begin{aligned} [T_{p(k)n}(h)](l) &= [T_{p(k)n}(\theta(f))](l) \\ &= [\theta[U_{p(k)n}^x(f)]](l) \\ &= [U_{p(k)n}^x(f)][p(l)] \\ &= f[n^{-1}p(k)^{-1}p(l)] \\ &= f[p(k^{-1}l)p(k^{-1}l)^{-1}n^{-1}p(k)^{-1}p(l)] \\ &= \chi_{[p(l)^{-1}p(k)n p(k^{-1}l)]}[f[p(k^{-1}l)]] \\ &= \chi_{[p(l)^{-1}p(k)n p(k^{-1}l)]}[[\theta(f)](k^{-1}l)] \\ &= \chi_{[p(l)^{-1}p(k)n p(k^{-1}l)]}(h(k^{-1}l)). \end{aligned}$$

REMARK. These equalities depend on properties of the cross section, namely that

$$p(k)p(l)[p(kl)^{-1}] \text{ is in } N.$$

REMARK. We dispense with the notation  $T$  and simply think of  $U^x$  as acting in the space  $L^2(K, H)$ .

2. **The representation  ${}_J U^\chi$ .** In this section, we show that  $U^\chi$  can be related to the regular  $a$ -representation of some quotient group.

Assume that  $G$  and  $N$  satisfy hypotheses 1 or 2 of 1.1-E, and fix an element  $\chi$  of  $\hat{N}$ . Since, by [1], we know that the representation  $U^\chi$  is equivalent to the representation  $U^{J, U^\chi}$ , where  $J$  is any closed subgroup of  $G$  which contains  $N$ , we investigate the representation  ${}_J U^\chi$  for clues to the status of the representations  $U^\chi$  with respect to the irreducible representations of  $G$ .

2.1-A. In particular, we consider the situation when  $J$  is the stability subgroup  $G_\chi$  for  $\chi$ , i.e.,  $J$  is the set of all  $x$  in  $G$  such that  $x\chi$  is equivalent to  $\chi$ .

Let the quotient group  $J/N$  be called  $M$ , so that we know, by 1.2, that  ${}_J U^\chi$  acts in the Hilbert space  $L^2(M, H)$ , where  $H$  denotes  $H(\chi)$ .

$\pi$  of course denotes the natural mapping of  $G$  onto  $G/N$ , which we call  $K$ . We also write  $\pi$  for the restriction of this natural mapping to  $J$ .

2.1-B. Now if  $G$  and  $N$  satisfy hypotheses 2 of 1.1-E, then Proposition 2 of 1.1-F asserts the existence of a cocycle  $a$  on  $M$  and a cocycle representation  $\chi'$  of  $J$  such that:

- i.  $\chi'|_N$  is equivalent to  $\chi$ .
- ii. The cocycle  $a'$  of the cocycle representation  $\chi'$  is given as follows:

If  $x$  and  $y$  are in  $J$ ,  $a'(x, y) = a[\pi(x), \pi(y)]$ .

If  $G$  and  $N$  satisfy the first set of hypotheses of 1.1-E, but not the second set, then we make the additional assumption that the orbit of  $\hat{N}$  to which  $\chi$  belongs is accommodating. Then, of course, there exists a unitary representation  $\chi'$  of  $J$  such that  $\chi'|_N$  is equivalent to  $\chi$ .

2.1-C. Throughout the remainder of this section  $\chi'$  and  $a$  will have the following meanings:

1. If  $G$  and  $N$  satisfy our second set of hypotheses, then  $\chi'$  is to be the cocycle representation of  $J$  guaranteed by Proposition 2 of 1.1-F, and  $a$  is to be the cocycle on  $M$  which is guaranteed by that same proposition.

2. If  $G$  and  $N$  satisfy the first set of hypotheses, but not the second set, then  $\chi'$  is to mean the unitary representation of  $J$  which we have hypothesized in B above, and  $a$  is to mean the complex-valued function on  $M \times M$  which is identically one.

2.1-D. Now let  $R$  be the regular  $\bar{a}$ -representation of  $M$ . (If  $G$  and  $N$  satisfy 2 above,  $R$  is simply the left regular representation of  $M$ .)

Denote by  $R'$  the representation of  $J$  lifted from  $R$ , that is, the space of  $R'$  is the space of  $R$ , which is  $L^2(M)$ , and, if  $x$  is in  $J$ ,

$$R'_x = R_{\pi(x)}.$$

(Of course, if  $G$  and  $N$  satisfy 1 of 2.1-C, then  $R'$  is a cocycle representation of  $J$ . Referring to  $R'$  as a "representation" is somewhat imprecise but nevertheless clear from the context.)

2.2. We now give the main theorem of the present section. This theorem shows that the representation  $U^\chi$  is related to a regular representation of a quotient group,

and this fact is a hint to the relationship between the question posed in the introduction and the classical problem mentioned there.

**THEOREM.** *The two representations  ${}_jU^x$  and  $\chi' \otimes R'$  are equivalent.*

**Proof.** 2.2-A. We shall define an operator on the Hilbert space  $H \otimes L^2(M)$ , the space of  $\chi' \otimes R'$ , to the Hilbert space  $L^2(M, H)$ , the space of  ${}_jU^x$ . We begin by defining an operator  $\theta$  on the spanning set of vectors of  $H \otimes L^2(M)$  of the form  $x \otimes f$ , where  $x$  is in  $H$  and  $f$  is in  $L^2(M)$ .

Thus, if  $x$  is in  $H$ ,  $f$  is in  $L^2(M)$ , and  $l$  is in  $M$ , we put

$$[\theta(x \otimes f)](l) = \bar{a}(l^{-1}, l)f(1)\chi'_{(p(l)^{-1})(x)}.$$

$\theta$  is clearly linear in  $x$  and  $f$  and hence can be extended to the linear span of such vectors. Thus  $\theta$  is defined on a dense subspace of  $H \otimes L^2(M)$ .

2.2-B. **LEMMA.**  *$\theta$  is an isometry of a dense subspace of  $H \otimes L^2(M)$  into  $L^2(M, H)$ .*

**Proof.** Let  $X$  be the dense subspace of  $H \otimes L^2(M)$  on which  $\theta$  is defined, and let  $z$  be in  $X$ , i.e.,

$$z = \sum_{i=1}^n (x^i \otimes f^i),$$

where each  $x^i$  is in  $H$  and each  $f^i$  is in  $L^2(M)$ .

We need to show that,

$$(\theta(z), \theta(z))_{L^2(M, H)} = (z, z)_{H \otimes L^2(M)},$$

i.e., we need to show that

$$\sum_{i,j=1}^n (\theta(x^i \otimes f^i), \theta(x^j \otimes f^j))_{L^2(M, H)} = \sum_{i,j=1}^n ((x^i \otimes f^i), (x^j \otimes f^j))_{H \otimes L^2(M)}.$$

This will follow from the following:

Let  $x$  and  $y$  be in  $H$  and let  $f$  and  $g$  be in  $L^2(M)$ . Then

$$\begin{aligned} &([\theta(x \otimes f)], [\theta(y \otimes g)])_{L^2(M, H)} \\ &= \int_M ([\theta(x \otimes f)](l), [\theta(y \otimes g)](l))_H dl \\ &= \int_M ([\bar{a}(l^{-1}, l)f(1)\chi'_{(p(l)^{-1})(x)}], [\bar{a}(l^{-1}, l)g(1)\chi'_{(p(l)^{-1})(y)}])_H dl \\ &= \int_M f(l)\overline{g(l)}(x, y)_H dl \\ &= (f, g)_{L^2(M)}(x, y)_H \\ &= ((x \otimes f), (y \otimes g))_{(H \otimes L^2(M))}. \end{aligned}$$

**REMARK.** The mapping  $\theta(x \otimes f)$  is clearly a Borel mapping of  $M$  into  $H$ , and hence the above calculations are admissible.

2.2-C. LEMMA. *The mapping  $\theta$  is onto a dense subspace of  $L^2(M, H)$ .*

**Proof.** Let  $g$  be an element of  $L^2(M, H)$  such that  $([\theta(x \otimes f)], g)_{L^2(M, H)} = 0$  for all  $x$  in  $H$  and all  $f$  in  $L^2(M)$ . This implies that, for all  $f$  in  $L^2(M)$ , we have:

$$\begin{aligned} 0 &= \int_M \bar{a}(l^{-1}, l) f(l) ([\chi'_{p(l)^{-1}}(x)], g(l))_H dl \\ &= \int_M \bar{a}(l^{-1}, l) f(l) (x, [\chi'_{p(l)}(g(l))])_H dl. \end{aligned}$$

But this implies that, for all  $x$  in  $H$ , the function  $l \rightarrow (x, [\chi'_{p(l)}(g(l))])_H$  is zero for almost all  $l$  in  $M$ . Since  $H$  is separable (1.2), this implies that  $\chi'_{p(l)}(g(l))$  is zero for almost all  $l$  in  $M$ , i.e., since  $\chi'_{p(l)}$  is a unitary operator,  $g(l)$  is zero for almost all  $l$  in  $M$ . Therefore  $g$  is the zero element of  $L^2(M, H)$ . This proves the lemma.

Now B and C show that  $\theta$  can be extended to all of  $H \otimes L^2(M)$  and that  $\theta$  is an isometry of  $H \otimes L^2(M)$  onto  $L^2(M, H)$ .

2.2-D. LEMMA. *The isometry  $\theta$  is intertwining for  $\chi' \otimes R'$  and  ${}_jU^x$ . (We write, for the purposes of this lemma,  $U^x$  for  ${}_jU^x$ .)*

**Proof.** We show the intertwining equality for spanning vectors of  $H \otimes L^2(M)$  of the form  $x \otimes f$ . This will be sufficient since these vectors span a dense subspace of  $H \otimes L^2(M)$ .

Thus, if  $x$  is in  $H$ ,  $f$  is in  $L^2(M)$ ,  $k$  and  $l$  are in  $M$ , and  $n$  is in  $N$ , we have:

$$\begin{aligned} &[U^x_{p(k)n}[\theta(x \otimes f)]](l) \\ &= \chi_{[p(l)^{-1}p(k)n p(k^{-1}l)]} [[\theta(x \otimes f)](k^{-1}l)] \\ &= \bar{a}(l^{-1}k, k^{-1}l) f(k^{-1}l) \chi_{[p(l)^{-1}p(k)n p(k^{-1}l)]} [\chi'_{p(k^{-1}l)^{-1}}(x)] \\ &= \bar{a}(l^{-1}k, k^{-1}l) f(k^{-1}l) \chi'_{[p(l)^{-1}p(k)n p(k^{-1}l)]} [\chi'_{p(k^{-1}l)^{-1}}(x)] \\ &= \bar{a}(l^{-1}k, k^{-1}l) f(k^{-1}l) a'([p(l)^{-1}p(k)n p(k^{-1}l)], (p(k^{-1}l)^{-1})) \chi'_{[p(l)^{-1}p(k)n]}(x) \\ &= \bar{a}(l^{-1}k, k^{-1}l) f(k^{-1}l) a(e, l^{-1}k) \chi'_{[p(l)^{-1}p(k)n]}(x) \\ &= \bar{a}(l^{-1}k, k^{-1}l) f(k^{-1}l) \chi'_{[p(l)^{-1}p(k)n]}(x) \\ &= \bar{a}(l^{-1}k, k^{-1}l) f(k^{-1}l) \bar{a}'([p(l)^{-1}], p(k)n) \chi'_{[p(l)^{-1}]} [\chi'_{p(k)n}(x)] \\ &= \bar{a}(l^{-1}k, k^{-1}l) \bar{a}(l^{-1}, k) f(k^{-1}l) \chi'_{[p(l)^{-1}]} [\chi'_{p(k)n}(x)] \\ &= \bar{a}(l^{-1}, l) \bar{a}(k, k^{-1}l) f(k^{-1}l) \chi'_{[p(l)^{-1}]} [\chi'_{p(k)n}(x)] \\ &= \bar{a}(l^{-1}, l) [R'_k(f)](l) \chi'_{[p(l)^{-1}]} [\chi'_{p(k)n}(x)] \\ &= \bar{a}(l^{-1}, l) [R'_{p(k)n}(f)](l) \chi'_{[p(l)^{-1}]} [\chi'_{p(k)n}(x)] \\ &= [\theta[(\chi'_{p(k)n}(x)) \otimes (R'_{p(k)n}(f))]](l) \\ &= [\theta[(\chi' \otimes R')_{p(k)n}(x \otimes f)]](l). \end{aligned}$$

Q.E.D.

Now, by B, C, and D the proof of the theorem is complete.

### 3. Weak containment for cocycle representations.

3.1. Suppose  $G$  is a separable locally compact group and that  $a$  is a fixed cocycle on  $G$ . In this section we extend the idea of weak containment to the case of  $a$ -representations on  $G$ .

3.1-A. We define a group  $G_a$  as follows:

The underlying Borel space for  $G_a$  is the Borel space  $C \times G$ , where  $C$  is the multiplicative group of complex numbers of absolute value one. The underlying topological space for  $G_a$  need not be the topological product space  $C \times G$ .

Multiplication in  $G_a$  is defined as follows:

If  $(s, x)$  and  $(t, y)$  are in  $G_a$ , then

$$(s, x)(t, y) = (a(x, y)st, xy).$$

The properties of the cocycle  $a$  make this definition of multiplication an associative binary operation.

**THEOREM.** *There exists a unique locally compact topology for  $G_a$  such that the Borel structure generated by this topology is the Borel structure of the Borel space  $C \times G$ , and such that, with the operation defined above,  $G_a$  is a locally compact separable topological group. Further, the subgroup of  $G_a$  consisting of all pairs  $(s, e)$ , where  $s$  is in  $C$ , is a central closed normal subgroup of  $G_a$  which can be homeomorphically identified in the natural way with the group  $C$ . Also,  $G_a/C$  is homeomorphic to  $G$ .*

For the proof, see [12].

**REMARK 1.** If  $(s, x)$  is an element of  $G_a$ , then

$$(s, x)^{-1} = (\bar{a}(x, x^{-1})s^{-1}, x^{-1}).$$

**REMARK 2.** The mapping  $p$  on  $G$  which sends an element  $x$  of  $G$  to the point  $(e, x)$  of  $G_a$  is a cross section of  $G$  into  $G_a$ .

**REMARK 3.** The left Haar measure on  $G_a$  can be taken as the product measure of the left Haar measures on  $C$  and  $G$ .

3.1-B. Let  $Z$  denote the group of integers, i.e., the character group of  $C$ . If  $J$  is an integer, then the character of  $C$  corresponding to  $J$  is the character which sends an element  $s$  of  $C$  to the complex number  $s^J$ . Denote by  $\delta$  the character corresponding to the integer 1.

**PROPOSITION.** *There is a one-to-one correspondence between the set of all  $a$ -representations of the separable locally compact group  $G$  and the set of all unitary representations of  $G_a$  whose restrictions to the normal subgroup  $C$  are multiples of the character  $\delta$ . Further, this correspondence preserves the spaces of the representations, stable subspaces, and equivalence. In fact, the correspondence, in one direction, is as follows: If  $T$  is a unitary representation of  $G_a$  whose restriction to  $C$  is a multiple of  $\delta$ , then let  $T'$  be the  $a$ -representation of  $G$  given by  $T'_x = T_{(e, x)}$ .*

For the proof, see [12].

3.1-C. Let  $R$  be the left regular representation of  $G_a$  acting in  $L^2(G_a)$ . Since  $C$  is compact,  $R|_C$  is completely reducible, and hence we may write  $L^2(G_a)$  as the direct sum  $\sum_{j \in Z} (H_j)$ , where  $Z$  is the group of integers, i.e., the dual space of  $C$ , and where each subspace  $H_j$  is the  $j$ -subspace. (See [11].) Observe that the  $\delta$ -subspace is the 1-subspace.

Now if  $f$  is an element of  $L^2(G_a)$ ,

$$f(sr, x) = [R_{(s^{-1})}(f)](r, x)$$

for almost all pairs  $(r, x)$  in  $G_a$ . Thus:

**PROPOSITION 1.** *Let  $f$  be in  $L^2(G_a)$ . Then  $f$  is in the subspace  $H_j$  if and only if for almost all pairs  $(r, x)$  in  $G_a$ ,  $f(sr, x) = s^{-j}f(r, x)$ .*

We obtain from the above proposition the following: If  $(t, y)$  is in  $G_a$ ,  $s$  is in  $C$ , and  $f$  is in  $H_j$ , then, for almost all pairs  $(r, x)$  in  $G_a$ ,

$$\begin{aligned} [R_{(t, y)}(f)](rs, x) &= f[(t, y)^{-1}(rs, x)] \\ &= f[\bar{a}(y, y^{-1}x)t^{-1}rs, y^{-1}x] \\ &= [\bar{a}(y, y^{-1}x)t^{-1}s]^{-j}f(r, y^{-1}x) \\ &= s^{-j}f[\bar{a}(y, y^{-1}x)t^{-1}r, y^{-1}x] \\ &= s^{-j}f[(t, y)^{-1}(r, x)] \\ &= s^{-j}[R_{(t, y)}(f)](r, x). \end{aligned}$$

Thus each subspace  $H_j$  is stable under the action of  $R$ .

**PROPOSITION 2.** *Each subspace  $H_j$  has a dense subspace consisting of continuous functions.*

**Proof.** The proof follows from the following three facts.

- i.  $H_j$  is stable under  $R$ .
- ii. The convolution of two functions in  $L^2(G_a)$  is a continuous function on  $G_a$ .
- iii. The set  $[L(G_a) * L^2(G_a)]$  is dense in  $L^2(G_a)$ .

Define  $R^1$  to be the restriction of  $R$  to the 1-subspace  $H_1$  of  $L^2(G_a)$ . Then, by C above,  $R^1$  is an  $a$ -representation of  $G$ .

Denote by  $R^a$  the regular  $a$ -representation of  $G$ .

3.1-D. **THEOREM.** *The two  $a$ -representations  $R^1$  and  $R^a$  are equivalent.*

**Proof.** We wish to define a linear isometry of  $H_1$  onto  $L^2(G)$  which intertwines the two representations. Let  $H'$  be the dense subspace of  $H_1$  consisting of continuous functions. We begin by defining an operator  $\theta$  on  $H'$  as follows:

If  $f$  is in  $H'$  and  $x$  is in  $G$ ,

$$[\theta(f)](x) = f(e, x).$$

We show first that  $\theta$  is an isometry of  $H'$  into  $L^2(G)$ . Let  $f$  be in  $H'$ . Then,

$$\begin{aligned} \int_G |[\theta(f)](x)|^2 dx &= \int_G |f(e, x)|^2 dx \\ &= \int_C \int_G |s^{-1}|^2 |f(e, x)|^2 dx ds \\ &= \int_C \int_G |f(s, x)|^2 dx ds \\ &= \int_{(G_a)} |f(s, x)|^2 dx ds. \end{aligned} \quad \text{Q.E.D.}$$

Now let  $g'$  be an element of  $L(G)$ . Define  $g$  to be the function on  $G_a$  given by: If  $(s, x)$  is in  $G_a$ ,

$$g(s, x) = s^{-1}g'(x).$$

Then, by Proposition 1 of 3.1-C,  $g$  is in  $H_1$ . Let  $[f_n]$  be a sequence of continuous members of  $H_1$  which converges to  $g$  in the  $L^2$  norm. Then,

$$\begin{aligned} 0 &= \lim_n \int_C \int_G |g(s, x) - f_n(s, x)|^2 dx ds \\ &= \lim_n \int_C \int_G |s^{-1}g'(x) - s^{-1}f_n(e, x)|^2 dx ds \\ &= \lim_n \int_G |g'(x) - [\theta(f_n)](x)|^2 dx. \end{aligned}$$

Hence,  $g'$  is in the closure of the range of  $\theta$ . Since  $\theta$  is defined on a dense subspace of  $H_1$ , and since  $\theta$  is an isometry, we see that we may extend  $\theta$  to an isometry of  $H_1$  onto all of  $L^2(G)$ .

We show now that  $\theta$  is the desired intertwining operator. If  $f$  is an element of  $H'$ , and  $x$  and  $y$  are elements of  $G$ ,

$$\begin{aligned} [\theta[R_y^1(f)]](x) &= [R_y^1(f)](e, x) \\ &= [R_{(e, y)}^1(f)](e, x) \\ &= f[(e, y)^{-1}(e, x)] \\ &= f[\bar{a}(y, y^{-1}x), y^{-1}x] \\ &= a(y, y^{-1}x)f(e, y^{-1}x) \\ &= a(y, y^{-1}x)[\theta(f)](y^{-1}x) \\ &= [R_y^a[\theta(f)]](x). \end{aligned} \quad \text{Q.E.D.}$$

This completes the proof of the theorem.

3.2. We now define the notion of weak containment for  $a$ -representations of  $G$ . We preserve the notation of the subsection just preceding.

3.2-A. Let  $T'$  be an  $a$ -representation of  $G$  and let  $S'$  be a collection of  $a$ -representations of  $G$ . We say that  $T'$  is *weakly contained* in the collection  $S'$  if the unitary

representation  $T$  of  $G_a$ , corresponding to  $T'$  as in the above, is weakly contained in the collection  $S$  of unitary representations of  $G_a$  corresponding to the collection  $S'$  of  $a$ -representations of  $G$  as in the above.

**LEMMA.** *Let  $\pi$  denote the natural mapping of  $G_a$  onto  $G$ . Then a subset  $K$  of  $G$  is compact if and only if  $\pi^{-1}(K)$  is a compact subset of  $G_a$ .*

The proof follows from the fact that  $C$  is compact.

Making use of the lemma above, the following proposition is immediate.

**PROPOSITION.** *Let  $T'$  be an  $a$ -representation of  $G$  and let  $S'$  be a collection of  $a$ -representations of  $G$ . Then  $T'$  is weakly contained in  $S'$  if and only if, for every fundamental function  $f$  on  $G$  associated with  $T'$ , every compact subset  $K$  of  $G$ , and every positive number  $\epsilon$ , there exists a function  $h$  on  $G$  which satisfies:*

- i.  $h$  is a finite sum of fundamental functions associated with  $S'$ .
- ii.  $|f(x) - h(x)| < \epsilon$  for all  $x$  in  $K$ .

3.2-B. Here are three propositions, all of which are simple consequences of the proposition in A above.

**PROPOSITION 1.** *Suppose  $T$  and  $S$  are both  $a$ -representations of  $G$  such that  $T$  is weakly contained in  $S$ . Let  $H$  be a closed subgroup of  $G$ . Then the  $a$ -representation  $T|_H$  of  $H$  is weakly contained in the  $a$ -representation  $S|_H$ . (See [2].)*

**PROPOSITION 2.** *Suppose  $S$  and  $T$  are  $a$ -representations of  $G$  such that  $S$  is weakly contained in  $T$ . Let  $V$  be a  $b$ -representation of  $G$ . Suppose further that all the cocycle representations  $S, T, S \otimes V$ , and  $T \otimes V$  are irreducible. Then  $S \otimes V$  is weakly contained in  $T \otimes V$ . (See [4].)*

**PROPOSITION 3.** *Suppose  $H$  is a closed normal subgroup of the separable group  $G$ . Let  $\pi$  be the natural mapping of  $G$  onto  $G/H$ . Let  $T$  and  $S$  be  $a$ -representations of  $G/H$  such that  $T$  is weakly contained in  $S$ . Then the  $a'$ -representation  $T \cdot \pi$  of  $G$  is weakly contained in  $S \cdot \pi$ , where  $a'(x, y) = a(\pi(x), \pi(y))$ .*

(The proof is routine.)

3.3. In this subsection, we show that if  $G$  has the property that its regular representation weakly contains all irreducible unitary representations, then the regular  $a$ -representation, where  $a$  is a fixed cocycle on the group, weakly contains all irreducible  $a$ -representations.

3.3-A. By [2], we know that if  $T$  is a representation of  $G_a$  which is weakly contained in the representation  $S$  of  $G_a$ , then the representation  $T|_C$  is weakly contained in the representation  $S|_C$ .

**LEMMA.** *Let  $T$  be an irreducible representation of  $G_a$  such that  $T|_C$  is a multiple of the character  $\delta$ . Let  $S$  be  $R|_{(H^\perp)}$ . Then  $T$  is not weakly contained in  $S$ .*

**Proof.** If  $T$  were weakly contained in  $S$ , then  $T|_C$  would be weakly contained in  $S|_C$ . This would imply that the function of positive type,  $\delta$ , associated with  $T|_C$

would be approximable by functions of positive type  $\lambda$  associated with  $S|_C$ . This implies that the character  $\delta$  is in the closure of the set of all characters  $\lambda$  contained in  $S|_C$ , and since the dual group  $Z$  of  $C$  is discrete, this implies that  $\delta$  is actually contained in  $S|_C$ . But this is a contradiction. Q.E.D.

3.3-B. DEFINITION. A locally compact group  $G$  is called an  $R$ -group if its left regular representation weakly contains all irreducible unitary representations.

Note that compact groups and abelian groups are  $R$ -groups.

3.3-C. THEOREM. *If  $G$  is a separable locally compact  $R$ -group, and if  $a$  is a cocycle on  $G$ , then the regular  $a$ -representation  $R^a$  weakly contains all irreducible  $a$ -representations of  $G$ .*

**Proof.** Let  $T'$  be an irreducible  $a$ -representation of  $G$ , and let  $T$  be the unitary representation of  $G_a$  corresponding to  $T'$ , i.e.,  $T_{(s, x)} = sT'_x$ .

By 3.1-D we know that  $R^a$  is equivalent to  $R^1$ , and we need to show that  $T'$  is weakly contained in  $R^1$ . But by definition this means we must show that  $T$  is weakly contained in  $R^1$ .

Now, by [8],  $G_a$  is an  $R$ -group because it is an extension of the  $R$ -group  $C$  by the  $R$ -group  $G$ . Therefore,  $T$  is weakly contained in  $R$ . By 3.1-B,  $T|_C$  is a multiple of the character  $\delta$ , and therefore, by the lemma in A above,  $T$  is not weakly contained in  $R|_{(H_1^+)}$ . Hence  $T$  must be weakly contained in  $R|_{(H_1)}$  which equals  $R^1$ . Q.E.D.

3.3-D. Again by [8], every closed subgroup  $H$  of an  $R$ -group is an  $R$ -group. Hence we have the following:

**THEOREM.** *Let  $G$  be a separable locally compact  $R$ -group. Then, for every closed subgroup  $H$  and every cocycle  $a$  on  $H$ , the regular  $a$ -representation of  $H$  weakly contains all irreducible  $a$ -representations of  $H$ .*

#### 4. A weak containment theorem.

4.1. LEMMA. *Let  $T$  be a unitary representation of the locally compact group  $G$  and let  $S$  be a unitary representation of a closed subgroup  $K$  of  $G$ . Then  $T \otimes U^S$  is equivalent to  $U^{(T|_K \otimes S)}$ .*

For the proof in the separable case, see Lemma 4.2 of [3]. We sketch a proof in general.

To show the equivalence we must produce a linear isometry from the space  $H(T \otimes U^S)$  onto the space  $H(U^{(T|_K \otimes S)})$  which intertwines the two representations. We begin then by defining an operator  $\theta$  on the spanning subset of  $H(T) \otimes H(U^S)$  consisting of vectors  $x \otimes f$ , where  $x$  is in  $H(T)$  and  $f$  is in  $H(U^S)$ . The value  $\theta(x \otimes f)$  must be a vector in  $H(U^{(T|_K \otimes S)})$ , i.e., a function on  $G$  into  $H(T) \otimes H(S)$ . Thus, if  $x$  is in  $H(T)$ ,  $f$  is in  $H(U^S)$ , and  $y$  is in  $G$ , put:

$$[\theta(x \otimes f)](y) = [T_{(y^{-1})}(x)] \otimes f(y).$$

Now  $\theta$  can be shown to be the required intertwining operator.

4.1-A. THEOREM. Let  $(G, N)$  satisfy hypotheses 2 of 1.1-E and assume that  $G/N$  is an  $R$ -group. Let  $W$  be an element of  $\hat{G}$  and let  $\chi$  be an element of the orbit of  $\hat{N}$  with which  $W$  is associated. Then  $W$  is weakly contained in  $U^\chi$ .

**Proof.** By Proposition 1.1-E, choose an irreducible representation  $T$  of  $G_\chi$  whose restriction to  $N$  is a multiple of  $\chi$  and such that  $U^T$  is equivalent to  $W$ . Let  $I$  be the trivial representation of  $G_\chi$  and  $I'$  be the trivial representation of  $N$ . Denote by  $\pi$  the natural mapping of  $G_\chi$  onto  $G_\chi/N$ , and denote by  $R$  the regular representation of  $G_\chi/N$ .

Now it is easy to see that  $U^{I'}$  is equivalent to  $R \cdot \pi$ , and hence, since  $G_\chi/N$  is an  $R$ -group,  $I$  is weakly contained in  $U^{I'}$ .

Now  $T$  is equivalent to  $T \otimes I$ , which is weakly contained in  $T \otimes U^{I'}$ , which, by the lemma above, is equivalent to  $U^{T|_N \otimes I'}$ , which is equivalent to  $nU^{\chi \otimes I'}$ , which is equivalent to  $nU^\chi$ .

But  $T$  is irreducible, and hence  $T$  is weakly contained in  $U^\chi$ , whence  $W$  is weakly contained in  $U^\chi$ . Q.E.D.

4.1-B. THEOREM. Let  $(G, N)$  satisfy 1 of 1.1-E. Suppose  $W$  is an element of  $\hat{G}$ , and suppose  $\chi$  is an element of the orbit of  $\hat{N}$  with which  $W$  is associated. Then  $W$  is contained as a direct summand in  $U^\chi$ .

**Proof.** We may merely rewrite the proof of the theorem in A, replacing "weakly contained in" by "contained as a direct summand in."

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