1. Introduction. Let \( \mathcal{D} \) be the class of bounded homogeneous star-shaped domains \( D \subset \text{space } C^n \) of \( n \) complex variables \( z = (z^1, \ldots, z^n) \). A domain \( D \) is homogeneous if any point of \( D \) can be transformed into any other by a holomorphic automorphism; \( D \) is star-shaped with respect to a point \( z_0 \in D \) if \( z \in D \) implies that \( r(z - z_0) \in D \) for \( 0 < r \leq 1 \). A bounded domain \( D \) possesses the Bergman metric, which is invariant under biholomorphic mappings. Let \( \mathcal{K} \) be a class of Kähler manifolds \( \Delta \) such that the components of its Ricci curvature tensor satisfy certain boundedness conditions (see formulas (2.9)). We consider biholomorphic mappings \( w = w(z) \) of \( D \in \mathcal{D} \) into \( \Delta \in \mathcal{K} \), that is, \( w = (w^1, \ldots, w^n) \), where \( w \) is local coordinate on \( \Delta \), and \( w' = w'(z^1, \ldots, z^n) \), are holomorphic functions on \( D \) with Jacobian determinant

\[
J_w(z) = \frac{\partial(w)}{\partial(z)} \neq 0.
\]

In §2 we generalize the Ahlfors version of the Schwarz-Pick lemma in \( C^1 \) to invariant volume in bounded homogeneous domains \( D \in \mathcal{D} \) in \( C^n \). This theorem states that if \( w = w(z) \) is a holomorphic mapping of the disk \( |z| < 1 \) into a Riemann surface \( \mathcal{W} \) and if the metric \( d\sigma = \lambda |dw| \), \( \lambda > 0 \), of \( \mathcal{W} \) has a negative curvature \( \leq -4 \) everywhere on \( \mathcal{W} \), then

\[
\lambda |dw/dz| \leq 1/(1 - |z|^2)
\]

for \( |z| < 1 \) [1]. Theorem 1 gives the invariant form of this generalization (with respect to biholomorphic mappings) and Theorem 2 an inequality which generalizes an inequality obtained by Dinghas when \( D \) is the unit hypersphere [4]. The proof of these theorems uses the method of Ahlfors in [1] and depends on properties of certain relative invariants of \( D \), in particular, the fact that the Bergman kernel function of a bounded homogeneous domain is infinite everywhere on the boundary (Lemma 1). In §3 various applications and extensions of the ideas in §2 are given in Theorems 3-6 and corollary. In §4 the results of §2 are applied to a study of the relative invariants of the classical Cartan domains \( R_j \) (\( j = 1, \ldots, IV \)), in particular, the invariant \( I_j \) (see (2.4)) is calculated. This procedure leads to a solution of a certain nonhomogeneous partial differential equation formed from the Hessian determinant of a holomorphic function on \( R_j \).

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2. Generalization of Schwarz-Pick lemma. 1. Let \( D \) be a bounded homogeneous star-shaped domain of class \( \mathcal{D} \). Its Bergman metric is given by

\[
\begin{align*}
\frac{ds_D^2}{dz} = T_{a\bar{a}} \, dz^a \, d\bar{z}^\bar{a}
\end{align*}
\]

(the summation convention is used), where

\[
T_{a\bar{a}} = T_{a\bar{a}}(z, \bar{z}) = \frac{\partial^a \log K_D(z, \bar{z})}{\partial z^a \partial \bar{\bar{z}}^\bar{a}},
\]

(2)

\[
T_D = T_D(z, \bar{z}) = \det(T_{a\bar{a}}),
\]

\( K_D(z, \bar{z}) \) the Bergman kernel function of \( D \) \( [2] \). The Bergman metric is a Kähler metric on \( D \) which is invariant under biholomorphic mappings of \( D \). The domains \( D \) in \( \mathcal{D} \) have the additional properties:

(i) The image domains \( D_r \) of \( D \) under the similarity transformation

\[
w = r(z - z_0), \quad 0 < r \leq 1,
\]

are such that \( D_{r_1} \subset D_{r_2} \) if \( r_1 \leq r_2 \). Also \( D = \bigcup_{j=1}^{\infty} D_{r_j} \), where \( r_j, \, 0 < r_j < 1 \), is an increasing sequence with limit 1. These facts follow since \( D \) is star-shaped. (Without loss of generality we may take \( z_0 = 0 \) in (3).)

(ii) The kernel function \( K_D(z, \bar{z}) \) becomes infinite on the boundary \( \partial D \) of \( D \) (This means that the set \( \{ z : z \in D \text{ and } K_D(z, \bar{z}) < M \} \) is relatively compact on \( D \).) This result is included in Lemma 1.

**Definition.** A real-valued function \( R_D(z, \bar{z}) \) on \( D \) is a relative invariant of \( D \) if under any biholomorphic mapping \( w : D \to D^* \)

\[
R_D(w(z), \bar{w}) = R_D(z, \bar{z}).
\]

The functions \( K_D(z, \bar{z}) \) and \( T_D(z, \bar{z}) \) are relative invariants of \( D \) \( [2] \) and consequently the function

\[
I_D(z, \bar{z}) = \frac{K_D(z, \bar{z})}{T_D(z, \bar{z})}
\]

is invariant under biholomorphic mappings:

\[
I_D(z, \bar{z}) = I_D(w(z), \bar{w}).
\]

It is clear that if \( J_D \) is another invariant of \( D \), then \( J_D = kI_D \) for some constant \( k \). Therefore, an invariant on a homogeneous domain is uniquely determined up to a constant multiple.

**Lemma 1.** Any relative invariant \( R_D(z, \bar{z}) \) of a bounded homogeneous domain \( D \) becomes infinite on \( \partial D \).

**Proof.** Let \( \Gamma \) be the group of holomorphic automorphisms of \( D \). Since the set of elements of \( \Gamma \) is uniformly bounded it forms a normal family. Let \( a \) be an
arbitrary point and $a_0$ a fixed point of $D$. Since $D$ is homogeneous there is an automorphism $t = t_1(z)$ which maps $a_0$ into $a$. Since $R_p(a, \bar{a})$ is a relative invariant of $D$

$$R_D(a, \bar{a}) = R_D(a_0, \bar{a}_0) |J_{a_0}(a_0)|^{-2}.$$  

Let $b \in \partial D$. It follows from a well-known theorem of H. Cartan [3] that

$$\lim_{a_1 \to b} J_{a_0}(z) = 0.$$  

The lemma follows from (6) and (7).

Let $\mathcal{K}$ be the class of Kähler manifolds $\Delta$ with metric given by

$$\begin{align*}
\sigma_{\Delta}^2 &= g_{a\bar{b}}(w, \bar{w}) \, dw^a \, d\bar{w}^b, \\
g_{\Delta} &= g_{\Delta}(w, \bar{w}) = \det(g_{a\bar{b}}),
\end{align*}$$

where $w$ is a local coordinate of a point on $\Delta$. We also assume

\begin{align*}
-\frac{\partial^2}{\partial w^a \partial \bar{w}^b} g_{\Delta} &\geq 0, \\
\det(-r_{a\bar{b}}) &\geq g_{\Delta},
\end{align*}

where

$$r_{a\bar{b}} = -\frac{\partial^2 \log g_{\Delta}}{\partial w^a \partial \bar{w}^b}$$

are the components of the Ricci curvature tensor of the metric (8) [8, 126]. Since $I_D = K_D/T_D$ is constant for a homogeneous domain, it follows that the components of the Ricci curvature tensor of the metric (1) have the form

$$-\frac{\partial^2 \log T_D}{\partial z^a \partial \bar{z}^b} = -T_{a\bar{b}}$$

so that (9) is satisfied. Hence $\mathcal{D}$ is a subclass of $\mathcal{K}$.

2. Theorem 1. If a bounded homogeneous domain $D$ of class $\mathcal{D}$ can be mapped biholomorphically by $w = w(z)$ into a Kähler manifold $\Delta \in \mathcal{K}$, then

$$g_{\Delta}(w, \bar{w}) |J_w(z)|^2 \leq T_D(z, \bar{z})$$

on $D$. Equality holds if the mapping is onto and the Kähler metric of $\Delta$ equals the Bergman metric of $\Delta$.

Proof. Let $z \in D$. By (i) there exists an $r < 1$, such that $z \in Dr$. Now $Dr$ is a homogeneous domain: $z_1, z_2 \in Dr$ implies $z_1/r, z_2/r \in D$ by the similarity transformation $s$ given by (3) with $z_0 = 0$; thus there is an automorphism $t$ of $D$ which takes $z_1/r$ into $z_2/r$ and $s^{-1}ts$ is an automorphism of $D$, and takes $z_1$ into $z_2$. Let $\zeta = \zeta(z)$ be an automorphism of $D$, which takes an arbitrary point $z$ of $Dr$ into 0. From the definition of relative invariant

$$I_{Dr} = \frac{K_{Dr}(0, 0)}{T_{Dr}(0, 0)} = \frac{K_{Dr}(z, \bar{z})}{T_{Dr}(z, \bar{z})} = I_{Dr}(z, \bar{z})$$
so that the invariant $I_{D_1}(z, \bar{z})$ is a constant on $D_1$, and by (5)

(12) \[ I_{D_1} = I_D. \]

Let

$$G_{a\bar{b}} \ d\bar{z}^a \ dz^\beta$$

be the hermitian form on $D$ corresponding to the metric (8) on $\Delta$ under the inverse mapping $z = z(w)$ of $w: D \to \Delta$ [5, 79]. Then

$$G_{a\bar{b}}(z, \bar{z}) = g_{a\bar{b}}(w, \bar{w}) \frac{\partial w^a}{\partial z^\alpha} \frac{\partial \bar{w}^\beta}{\partial \bar{z}^\beta},$$

and

(13) \[ G_D(z, \bar{z}) = g_\Delta(w, \bar{w}) |J_w(z)|^2 > 0. \]

Let

$$R_{a\bar{b}}(z, \bar{z}) = -\frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta} \log G_D(z, \bar{z}).$$

Then

$$r_{a\bar{b}}(w, \bar{w}) \ \frac{\partial w^a}{\partial z^\alpha} \frac{\partial \bar{w}^\beta}{\partial \bar{z}^\beta} = R_a(z, \bar{z})$$

and

$$\det(-r_{a\bar{b}}(w, \bar{w})) = \det(-R_{a\bar{b}}(z, \bar{z})) |J_w(z)|^{-2}.$$

Thus from hypothesis (9b)

(14) \[ \det(-R_{a\bar{b}}(z, \bar{z})) \geq g_\Delta(w, \bar{w}) |J_w(z)|^2 = G_D(z, \bar{z}). \]

Following the proof in Dinghas and Ahlfors [4, 11] let

$$U = \log \frac{G_D(z, \bar{z})}{T_D(0, 0)}, \quad V = \log \frac{K_{D_1}(z, \bar{z})}{K_D(0, 0)},$$

(15) \[ \Psi = U - V, \]

and set $E = \{ z \in D: U > V \} \ (E \text{ open})$. Since under the transformation (3) $\partial D \to \partial D_1$, from Lemma 1 $K_{D_1}(z, \bar{z})$ becomes infinite on $\partial D_1$. Thus since $U$ is continuous on $D_1 \subset D$ and $V$ on $D_1$, $E \subset D$. Let $O$ be any component of $E$. Then $\bar{O} \subset \bar{E} \subset D$, so that $\bar{O}$ is compact. Thus the continuous function $\Psi$ takes its maximum at a point $z_0 \in \bar{O}$ but $\Psi(z_0) = \max \Psi > 0$ so that $z_0 \in E$. Since $\Psi$ has a maximum on $E$,

$$\frac{\partial^2 \Psi}{\partial z^\alpha \partial \bar{z}^\beta} u^\alpha \bar{u}^\beta \leq 0$$

at $z_0$ for any vector $(u^\alpha)$ or by (15)

(16) \[ \frac{\partial^2 U}{\partial z^\alpha \partial \bar{z}^\beta} u^\alpha \bar{u}^\beta \leq \frac{\partial^2 V}{\partial z^\alpha \partial \bar{z}^\beta} u^\alpha \bar{u}^\beta. \]
From (15), (13) and (9a) and (c)

\[
\frac{\partial^2 U}{\partial z^a \partial \bar{z}^b} u^a \bar{u}^b = -r \mu \bar{u}^a \bar{u}^b \geq 0,
\]

where \( (\mu e) = (\partial w^a/\partial z^a) u^a \). From the definition of \( V \) the matrix \( A = (\partial^2 V/\partial z^a \partial \bar{z}^b) \) is positive definite. Hence by a classical theorem on the simultaneous reduction of a pair of hermitian quadratic forms there exists a nonsingular matrix \( T \) such that

\[
A = TT', \quad B = T \Lambda T', \quad \Lambda = [\lambda_1, \ldots, \lambda_n]
\]

[7, 191], \( B = (\partial^2 U/\partial z^a \partial \bar{z}^b) \), and from (16) and (17)

\[
0 \leq \lambda_i \xi_i^T \leq \sum_{i=1}^n \xi_i^T
\]

where \( \xi = uT \). By taking \( \xi \) successively equal to \((1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\), we get \( 0 \leq \lambda_i \leq 1 \) \((i = 1, \ldots, n)\). Hence at \( z_0 \) since \( \det TT' > 0 \)

\[
0 \leq \det \left( \frac{\partial^2 U}{\partial z^a \partial \bar{z}^b} \right) \leq \det \left( \frac{\partial^2 V}{\partial z^a \partial \bar{z}^b} \right).
\]

But from (15) and (14)

\[
\det \left( \frac{\partial^2 U}{\partial z^a \partial \bar{z}^b} \right) \geq G_D(z, \bar{z})
\]

while from (15), (2), (11) and (12)

\[
\det \left( \frac{\partial^2 V}{\partial z^a \partial \bar{z}^b} \right) = T_{D_\epsilon}(z, \bar{z}) = K_{D_\epsilon}(z, \bar{z}) \frac{T_D(0, 0)}{K_D(0, 0)}
\]

so that at \( z_0 \)

\[
G_D(z, \bar{z}) \leq \frac{K_{D_\epsilon}(z, \bar{z})}{K_D(0, 0)},
\]

that is, \( U \leq V \), which is a contradiction. Thus \( O \) and \( E \) are empty so that \( U \leq V \) on \( D_r \), and relation (18) holds on \( D_r \). Since \( K_{D_\epsilon} \) is a relative invariant,

\[
\frac{G_D(z, \bar{z})}{T_D(0, 0)} \leq r^{-2n} \frac{K_D(z, \bar{z})}{K_D(0, 0)}
\]

and letting \( r \to 1 \) from the continuity of \( K_D(z, \bar{z}) \) on \( D_r \) follows

\[
\frac{G_D(z, \bar{z})}{T_D(0, 0)} \leq \frac{K_D(z, \bar{z})}{K_D(0, 0)}
\]

which gives (10) by (13), (4) and the fact that \( I_D(0, 0) = I_D(z, \bar{z}) \).
To prove the last part of the theorem we note that $d\sigma_{\Delta} = ds_{\Delta}$ and $g_{\Delta} = T_{\Delta}$ and since $T_{\Delta}$ is a relative invariant equality follows in (10).

**Remark.** The inequality of Theorem 1 is invariant under biholomorphic mappings in the following sense. Let $D \in \mathscr{D} \rightarrow D^*$ and $\Delta \in \mathcal{K} \rightarrow \Delta^*$ under the biholomorphic maps $z^* = z^*(z)$ and $w^* = w^*(w)$ respectively. Suppose there exists a biholomorphic mapping $w^* = f(z^*)$ of $D^*$ into $\Delta^*$. Then

\begin{equation}
\tag{19}
g_{\Delta}(w^*, \bar{w}^*) |J_f(z^*)|^2 \leq T_{D^*}(z^*, \bar{z}^*)
\end{equation}

for $z^* \in D^*$.

**Proof.** Set $h = w^{-1} f z^*$. Then $w = h(z)$ is a biholomorphic mapping of $D$ into $\Delta$ and by Theorem 1

\[ g_{\Delta}(w, \bar{w}) |J_h(z)|^2 \leq T_D(z, \bar{z}). \]

Since

\[ T_D(z, \bar{z}) = T_{D^*}(z^*, \bar{z}^*) |J_{z^*}(z)|^2 \quad \text{and} \quad g_{\Delta}(w, \bar{w}) = g_{\Delta^*}(w^*, \bar{w}^*) |J_{w^*}(w)|^2, \]

and from the definition of $h$

\[ |J_h(z)|^2 = |J_{f(z^*)}|^2 |J_{z^*}(z)|^2 |J_{w^*}(w)|^{-2}, \]

from which (19) follows.

The following form of Theorem 1 is not invariant under biholomorphic mappings. It may be proved by replacing $U$ in (15) by $U = \log G_D(z, \bar{z})$.

**Theorem 2.** If there exists a biholomorphic mapping $w = w(z)$ of a domain $D \in \mathscr{D}$ into a Kähler manifold $\Delta \in \mathcal{K}$ with condition (9b) replaced by

\[ \det(-r_{ad}) \geq T_D(0, 0)g_{\Delta}(w, \bar{w}), \]

then on $D$

\[ g_{\Delta}(w, \bar{w}) |J_w(z)|^2 \leq T_D(z, \bar{z})/T_D(0, 0). \]

**Remarks.**

1. The result obtained by Dinghas in [4] is a special case of Theorem 2 when $D$ is the unit hypersphere.

2. A disadvantage of Theorem 2 is the fact that the original domain $D$ furnished with the Bergman metric cannot belong to our admissible class of Kähler manifolds $\Delta$.

3. **Further extensions and applications of Theorem 1.**

1. **Area theorem for manifolds of $n$ real dimensions.** We also may derive an inequality for manifolds of $n$ (real) dimensions. Let $M \subset D$ be a continuously differentiable manifold of dimension $n$ parametrized by $z^i = z^i(u)$, $u$ a point of the $n$-cube $I^n$. The $n$-dimensional noneuclidean analytic volume of $M$ is

\[ dV_M(z) = T_D^{1/2}(z, \bar{z}) |\partial(z^1, \ldots, z^n)/\partial(u^1, \ldots, u^n)| \, du^u, \]

and is invariant under biholomorphic mappings of $D$ [5, 330]. Then
**Theorem 3.** Let $M$ be a continuously differentiable manifold of real dimension $n$ in $D$ and $w(M)$ its image under a biholomorphic mapping of $D \in \mathfrak{D}$ into $\Delta \in \mathfrak{K}$. Then $V_\Delta(w(M)) \leq V_\phi(M)$ and equality holds if the mapping is onto and $d\sigma_\Delta = ds_D$.

We note that if we take $\Delta = D$ and $d\sigma_\Delta = ds_D$, we obtain a generalization of the Schwarz-Pick lemma to this real $n$-dimensional noneuclidean volume (see also Theorem 25.1 in [5]).

Theorem 3 is invariant under biholomorphic mappings of $D$ and $\Delta$ in the sense described in §2.2 since all quantities involved are invariant under such biholomorphic mappings.

2. Properties of certain domain functions. The following theorems give some useful inequalities connecting the relative invariants of a domain $D$ and $I_D(z, \bar{z})$.

The first theorem follows easily from Theorem 1 for homogeneous star-shaped domains but also holds for any bounded domain.

**Theorem 4.** Let $D$ be any bounded domain and $w = w(z)$ a biholomorphic mapping of $D$ into $D$. The invariant $I_D(z, \bar{z}) \leq 1$ on $D$ if and only if $K_D(w, \bar{w}) |J_w(z)|^2 \leq T_D(z, \bar{z})$.

**Proof.** Suppose that $I_D(z, \bar{z}) \leq 1$ on $D$ and assume that the conclusion does not hold, that is, there is a point $z_0 \in D$ such that

$$K_D(w_0, \bar{w}_0) |J_w(z_0)|^2 > T_D(z_0, \bar{z}_0) \quad (w_0 = w(z_0)).$$

Now $D^* = w(D) \subset D$ and hence $K_D^*(w_0, \bar{w}_0) \geq K_D(w_0, \bar{w}_0)$ [2, 45]. Therefore

$$K_D^*(w_0, \bar{w}_0) |J_w(z_0)|^2 = K_D(z_0, \bar{z}_0) > T_D(z_0, \bar{z}_0) \quad \text{or} \quad I_D(z_0, \bar{z}_0) > 1$$

which is a contradiction. Since the Jacobian of the identity mapping is 1, the converse of the theorem is trivial.

A useful application of Theorem 4 is

**Theorem 5.** Let $D$ be a bounded complete circular domain in $C^n$ with center at the origin and $w = w(z)$ a biholomorphic mapping of $D$ into $D$. If $I_D \leq 1$ on $D$, then $|J_w(z)|^2 \leq \omega(D) T_D(z, \bar{z})$. Also $\omega(w(G)) \leq \omega(D) V_D(G)$ ($\omega$ euclidean volume) for any measurable $G \subset D$, where $V_D(G) = \int_G T_D(z, \bar{z}) \, d\omega_z$.

**Proof.** Since $D$ is a bounded complete circular domain, $K_D(z, \bar{z})$ attains its minimum at $z = 0$ [6, 79] and the minimum value is $1/\omega(D)$. Then the conclusions of the theorem follow from Theorem 4.

Since $V_D(G) = V_D(w(G))$ we have

**Corollary.** Under the hypotheses of Theorem 5

$$\frac{\omega(w(G))}{V_D(w(G))} \leq \omega(D)$$

for any measurable set $G \subset D$ with nonzero measure. In particular if $w = w(z)$ is the identity mapping, then

$$\omega(G)/V_D(G) \leq \omega(D).$$
We remark that $I_D < 1$ for the classical Cartan domains. (See §4.) In fact we do not know examples of domains for which $I_D(\mathbf{z}_0, \overline{\mathbf{z}}_0) = 1$ at some point $\mathbf{z}_0 \in D$.

Finally if we apply Theorem 1 under the identity mapping $w = z$, we get

**Theorem 6.** Let $\Delta$ be a homogeneous bounded domain in $\mathbb{C}^n$ and $D$ a subdomain which is equivalent to a domain in $\mathcal{D}$. Then for $z \in D$

\[ T_\Delta(z, \bar{z}) \leq T_D(z, \bar{z}). \]

4. **Relative invariants on the classical Cartan domains.**

1. The theorems in §2 give interesting results for the classical Cartan domains. These domains along with two special domains are the 4 types of bounded irreducible symmetric domains in $\mathbb{C}^n$, into which all bounded symmetric domains in $\mathbb{C}^n$ can be mapped biholomorphically. Let $z$ be a matrix of complex elements, $z'$ its transpose, $z^*$ its conjugate transpose, and $I$ the identity matrix. The first 3 types are represented by

\[ R_j = \{ z : I - zz^* > 0 \} \]

($j=1, II, III$) where $z$ is a matrix of type $(n, m)$ on $R_I$, $z$ is a symmetric matrix of order $n$ on $R_{II}$ and a skew-symmetric matrix of order $n$ on $R_{III}$, and $" > 0"$ means that the quadratic form is positive definite. The fourth type $R_{IV}$ is the set of $n$ dimensional vectors such that

\[ |zz'| < 1, 1 - 2\overline{z}z' + |z|^2 > 0. \]

These domains belong to class $\mathcal{D}$. The Bergman kernel function of these domains is known [6] so that to get inequality (2.10) it is sufficient to find $T_j = T_{R_j}(0, 0)$ and use formula (2.11). For the first 3 types

\[ K_I(z, z^*) = \frac{1}{\omega_I \det^p(I - zz^*)}, \]

$p = m + n$ for $R_I$, $n + 1$ for $R_{II}$ and $n - 1$ for $R_{III}$ and $\omega_I$ is the euclidean volume of $R_I$. For $R_{IV}$

\[ K_{IV}(z, \bar{z}) = \frac{1}{\omega_{IV}(1 + |z|^2 - 2z\bar{z})^n}. \]

In case I since \( \log K_I(z, z^*) = -(m+n) \log Q_I - Q_I \), where $Q_I = \det(I - zz^*)$ we need the value of $\frac{\partial^2 Q_I}{\partial z\bar{z}}$ at $z = 0$. To evaluate this use the expansion for the characteristic equation of $zz^*$ with $\lambda = 1$:

\[ \det(\lambda I - zz^*) = \lambda^n - \sigma \lambda^{n-1} + t_{n-2} \lambda^{n-2} - \cdots \pm \det zz^*, \]

where $t_i$ is the sum of the principal $i$-rowed minors of $zz^*$, $\sigma$ being the trace. Since $t_i$ is a homogeneous polynomial of degree $2i$, only the second derivatives of $\sigma$
contribute nonzero terms at \( z=0 \). Also all the first derivatives of \( t_i \) and \( \sigma \) are 0 at \( z=0 \). Now

\[
\frac{\partial}{\partial z^a \overline{\partial z}^b} \log K_1(z, z^*) = -(m+n)Q_1^{-1} \frac{\partial}{\partial z^a \overline{\partial z}^b} Q_1 
\]

\[
\frac{\partial^2}{\partial z^a \overline{\partial z}^b \partial z^c \overline{\partial z}^d} \log K_1(z, z^*) = (m+n)Q_1^{-2} \frac{\partial}{\partial z^a \overline{\partial z}^b} Q_1 \frac{\partial}{\partial z^c \overline{\partial z}^d} Q_1 - (m+n)Q_1^{-1} \frac{\partial^2}{\partial z^a \overline{\partial z}^b} Q_1 \frac{\partial}{\partial z^c \overline{\partial z}^d} Q_1.
\]

These remarks and formulas apply also to cases II and III.

Since \( \sigma = \sum_{i, k} z^i \overline{z}^k \), at \( z=0 \)

\[
\frac{\partial^2 \sigma}{\partial z^a \overline{\partial z}^b} = \delta_{ik} \delta_{jk},
\]

\[
\frac{\partial^2 \log K_1(z, z^*)}{\partial z^a \overline{\partial z}^b} = m + n,
\]

and all other derivatives are zero so that

\[
T_1(0, 0) = \det[(m+n)I] = (m+n)^m,
\]

\[
I_1 = \frac{1}{\omega_1(m+n)^m},
\]

and from (2.4)

\[
T_1(z, z^*) = \frac{(m+n)^m}{\det^{m+n}(I-zz^*)}.
\]

The trace of \( zz^* \) for a symmetric matrix is

\[
\sigma = \sum_i z^i \overline{z}^i + 2 \sum_{i \neq k} z^i \overline{z}^k
\]

so that at \( z=0 \)

\[
\frac{\partial^2 \sigma}{\partial z^i \overline{\partial z}^j} = 1, \quad \frac{\partial^2 \sigma}{\partial z^j \overline{\partial z}^k} = 2 \quad (j \neq k)
\]

and all other derivatives of \( \sigma \) are zero. Thus

\[
T_{II}(0, 0) = (n+1)^{n((n+1)/2)} 2^{n(n-1)/2},
\]

\[
I_{II} = \frac{1}{\omega_{II} 2^{n(n-1)/2}(n+1)^{n(n+1)/2}}
\]

and

\[
T_{II}(z, z^*) = \frac{2^{n(n-1)/2}(n+1)^{n(n+1)/2}}{\det^{n+1}(I-zz^*)}.
\]
For a skew symmetric matrix \( o = 2 \sum_{i<k} z^{ik} z^{jk} \), where the matrix \( z \) has only \( n(n-1)/2 \) distinct nonzero elements, so that we get

\[
T_{III}(0, 0) = [2(n-1)]^{n(n-1)/2}, \quad I_{III} = \frac{1}{\omega_{III}[2(n-1)]^{n(n-1)/2}}
\]

(3)

\[
T_{III}(z, \bar{z}) = \frac{[2(n-1)]^{n(n-1)/2}}{\det^{n-1}(I+z \bar{z})}.
\]

For case IV setting \( \Delta = 1 + |zz'|^2 - 2zz' \)

\[
\frac{\partial}{\partial z^a} \Delta = 2z^a[(\bar{z})^2 + \cdots + (\bar{z}^n)^2] - 2z^a,
\]

which is 0 at \( z = 0 \) and similarly for \( \partial \Delta/\partial \bar{z}^a \) and

\[
\frac{\partial^2 \Delta}{\partial z^a \partial \bar{z}^\beta} = 4z^a \bar{z}^\beta - 2\delta_{a\beta}
\]

so that at \( z = 0 \)

\[
\frac{\partial^2 \log K_{IV}(z, \bar{z})}{\partial z^a \partial \bar{z}^\beta} = -n(-2\delta_{a\beta})
\]

and

\[
T_{IV}(0, 0) = (2n)^n, \quad I_{IV} = \frac{1}{\omega_{IV}(2n)^n}.
\]

(4)

\[
T_{IV}(z, \bar{z}) = \frac{(2n)^n}{(1+|zz'|^2 - 2\bar{z}z')^n}.
\]

Formulas (1), (2) and (3) give the interesting result

**Theorem 7.** The function \( \log \det^{-1}(I-zz^*) \) satisfies the partial differential equation

\[
\frac{\partial^2 V}{\partial z^a \partial \bar{z}^\beta} = ae^{2V}
\]

on \( R_j \) \((j=I, II, III)\) where \( a = 1, b = m + n \) for an \((n, m)\) matrix, \( a = 2^{n(n-1)/2} \) for a symmetric or skew-symmetric matrix and \( b = n + 1 \) for a symmetric and \( n - 1 \) for a skew-symmetric matrix. For case IV \(-\log(1+|zz'|^2 - 2\bar{z}z')\) satisfies the partial differential equation

\[
\frac{\partial^2 V}{\partial z^a \partial \bar{z}^\beta} = 2^n e^{2V}
\]

on \( R_{IV} \).

This theorem corresponds to Lemma 4 of Dinghas [4] for the function

\( -\log(1-z^t \bar{z}) \).

Using the values for the euclidean volume given in [6] and induction on \( n \) we find that the invariant \( I_j < 1 \) \((j=I, II, III, IV)\). Thus Theorem 5 holds for the
Cartan domains and from this result, Theorem 4.2.1 of [6] and the expressions (1)–(4) for $T_j(z, z^*)$ we obtain an interesting distortion theorem on the Jacobian of an interior mapping of a Cartan domain:

**Theorem 8.** Let $w = w_j(z)$ be a biholomorphic mapping of the Cartan domain $R_j$ into itself. Then

$$|J_{w_j}(z)|^2 \leq \frac{|J_j(z, \bar{z})|^2}{I_j}$$

for $z \in R_j$, where $I_j$ is the invariant of $R_j$ and $J_j(z, \bar{z})$ the Jacobian of the holomorphic automorphism of $R_j$ which maps $z$ into the origin ($j = I, II, III, IV$).

**References**