

TWO THEOREMS ON HYPERHYPERSIMPLE SETS

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There are two main results. First, there is a hyperhypersimple set which is not quasimaximal. Second, there is an r -maximal set which is not hyperhypersimple. These results answer questions raised by Young [10, pp. 75 and 81] and McLaughlin [3, p. 87]. McLaughlin [4] reports a weaker, "non-co-r.e." version of our second result, due to D. A. Martin. He also improves Martin's result, though the "co-r.e." versions of both would be equivalent to our second result. Lachlan [2] has obtained different proofs of the two main results, as discussed in the last section.

The proof of the first result involves a construction closely akin to the maximal set construction as handled by Yates [8]. The priority system applicable to maximal set constructions was introduced by Friedberg [1], and is essentially unaltered. The proof of the second result requires that the maximal set priorities be altered in an essential way, and thus represents more of a departure.

1. **Preliminaries.** We are concerned only with recursively enumerable (r.e.) subsets of the nonnegative integers. Definitions of the special classes of r.e. sets which we deal with may be found in [3], [5], and [6]. By an observation of Yates [8, p. 344] it is clear that any quasimaximal set is hyperhypersimple. The examples of hyperhypersimple sets appearing in the literature, as in [8], are all quasimaximal. Thus our first construction provides a new variety of hyperhypersimple set. The only examples of r -maximal sets (r.e. sets maximal with respect to the recursive sets) previously known are just maximal sets. Every maximal set is certainly r -maximal. The converse of this triviality is disposed of by our second construction.

The set of all nonnegative integers is denoted by N , and the empty set by \emptyset . All of the sets under consideration will be subsets of N , all functions of one variable will be subsets of $N \times N$, etc. All quantification is to be taken over N or some indicated subset of N . A relation is called recursive (in the exhibited variables) if its representing function is recursive (in the corresponding variables), and a set or sequence of sets is called recursive if its membership relation is recursive. A sequence of sets $S(i)$ is called r.e. just if there is some recursive sequence $S(i, s)$ such that $S(i) = \bigcup S(i, s)$ ($s \geq 0$) for all i .

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Let $W(i, s)$ be a recursive sequence such that if $W(i) = \bigcup W(i, s) (s \geq 0)$ for all i , then $\{W(i) \mid i \geq 0\}$ is the set of all r.e. sets. It is further required that

- (1) $(y)(i)(s)(y \in W(i, s) \Rightarrow y \leq s)$ and
- (2) $(i)(s)(W(i, s) \subseteq W(i, s+1))$.

Such a sequence is obtained by taking $W(i, s)$ to be the set of integers placed in the i th r.e. set by the s th stage of some standard enumeration of the r.e. sets.

Also we require a specific pairing function p , defined by $p(x, y) = \frac{1}{2}(x+y) \cdot (x+y+1) + x$, along with the inverses r and c . These satisfy the relation $p(r(x), c(x)) = x$ for all x . The functions p , r , and c are all recursive, and p maps $N \times N$ isomorphically onto N . The following three facts are assumed without mention in our proofs. First, for every x the set $r^{-1}(x)$ is infinite. Next, $p(x, y) \geq x$ for all x and y . Finally, $p(x, y) > p(0, x+y)$ for all y and all $x > 0$.

2. THEOREM 1. *There is a hyperhypersimple set which is not quasimaximal.*

A recursive sequence of sets $H(i, s)$ is defined, along with a recursive predicate $P(k, s, x)$ and a recursive function $x(k, s)$, all simultaneously by induction on s . For a fixed s the definition of $P(k, s, x)$ and $x(k, s)$ proceeds by simultaneous induction on k for all x , with the help of an auxiliary recursive function.

Let $E(k, s, y) = \sum 2^{k-j} (j \leq k \ \& \ y \in W(j, s))$ for all k, s , and y . It may be noted that $E(k, s, y)$ is the k -state of y at stage s in Friedberg's terminology [1]. An immediate result of the definition is

- (3) $(j)(k)(x)(y)(s) ((j \geq k \ \& \ E(k, s, x) > E(k, s, y)) \Rightarrow E(j, s, x) > E(j, s, y))$.

In view of (2) it also follows that

- (4) $(x)(k)(s)(t)(t \geq s \Rightarrow E(k, t, x) \geq E(k, s, x))$.

At stage $s=0$, set $H(i, 0) = \emptyset$ for all i .

For any stage $s \geq 0$, and any k and x , we set

- (5) $P(k, s, x) \equiv x \notin H(r(k), s) \ \& \ (j)(j < k \Rightarrow x(j, s) < x)$, and

- (6) $x(k, s) = \mu x(P(k, s, x) \ \& \ (y)(P(k, s, y) \Rightarrow E(k, s, y) \leq E(k, s, x)))$.

Finally, for any stage $s+1 > 0$, and any i , let

$$H(i, s+1) = \{x \mid x \leq s+1 \ \& \ (k)(x = x(k, s) \Rightarrow r(k) < i)\}.$$

These definitions result in recursive sets, functions, and predicates, for in the definition of $x(k, s)$ the least number operator is always well defined, and the quantifier is implicitly bounded by s , due to (1). We conclude by setting

$$H(i) = \bigcup H(i, s) \quad (s \geq 0).$$

The construction may be viewed as a series of attempts at specifying the sets $\bar{H}(i)$. The approximation at stage s to the set $\bar{H}(i)$ consists of the numbers $x(k, s)$ for

all k such that $r(k) \geq i$. The choice of $x(k, s)$ is limited by the "availability predicate" P . This is because $P(k, s, x(k, s))$ must always be true, by (6). The first half of (5) insures that the sets $H(i)$ are nested, and the second half makes $H(i+1) - H(i)$ infinite for all i . Modulo the restriction imposed by P , the choice of $x(k, s)$ is determined by the function E . That is, $x(k, s)$ is chosen so as to maximize the value of $E(k, s, x(k, s))$. The function $E(k, s, x)$ is simply a convenient way of putting the highest priority on the membership of x in $W(0)$, the next highest priority on membership in $W(1)$, and so on down to $W(k)$. This choice of priorities gives the r.e. sequence $H(0), H(1), \dots$ properties strongly resembling maximality, as will be seen in Lemma 3.

Certain simple properties follow at once from the definitions. First, for any s and i , $x(i, s) < x(i+1, s)$, so that the function $\lambda i x(i, s)$ is always 1-1. Next, it is evident that for every s and i , $H(i, s) \subseteq H(i+1, s)$, directly from the definition. Further, for every s and i , $H(i, s) \subseteq H(i, s+1)$. For suppose $x \in H(i, s)$, and $x = x(k, s)$. Then $P(k, s, x)$ must be true, so that $x \notin H(r(k), s)$. Since $x \in H(i, s)$ and $H(i, s) \subseteq H(i+n, s)$ for all n , $i > r(k)$. If $x = x(j, s)$ then $j = k$, and so of course $i > r(j)$. Since $x \leq s$ by virtue of $x \in H(i, s)$, it follows that $x \in H(i, s+1)$. From these two facts we see that $H(i) \subseteq H(i+1)$ for every i .

LEMMA 1. For each j , $\lim_s x(j, s)$ exists.

Proof. We proceed by induction on j . Suppose, then, that s_0 is such that for all $s \geq s_0$ and all $k < j$, we have $x(k, s) = x(k, s_0)$. The definition of $H(r(j), s+1)$ requires that $x(j, s) \notin H(r(j), s+1)$. By (6), $P(j, s, x(j, s))$ is true. If $s \geq s_0$, the right half of (5) remains unchanged as it applies to the definition of $P(j, s+1, x(j, s))$, and so $P(j, s+1, x(j, s))$ is true. By (4) and (6) it follows that

$$E(j, s+1, x(j, s+1)) \geq E(j, s, x(j, s)),$$

for all $s \geq s_0$. As $\lambda s \lambda x E(j, s, x)$ can take on at most 2^{j+1} different values, there is a stage $s_1 \geq s_0$ such that if $s \geq s_1$, then $E(j, s, x(j, s)) = E(j, s_1, x(j, s_1))$. But for all $s \geq s_1$ it is now easy to see from (6) that $x(j, s+1) \leq x(j, s)$, and so $\lim_s x(j, s)$ does exist.

Let $x(j) = \lim_s x(j, s)$ for all j . Since $x(0, s) < x(1, s) < \dots$ for all s , then in the limit $x(0) < x(1) < \dots$.

LEMMA 2. $N = \bigcup H(i)$ ($i \geq 0$), and for all i $H(i+1) - H(i) = \{x(j) \mid r(j) = i\}$.

Proof. First, let y be such that $y \notin \{x(j) \mid j \geq 0\}$. Then we claim that $y \in H(0)$. By Lemma 1 we can find s_0 such that for any $s \geq s_0$ and $k \leq y$, $x(k, s+1) = x(k, s)$. It is readily seen that $y \leq x(y, s)$ for every s , and of course $x(y, s) < x(y+1, s) < \dots$. Thus for any $s \geq s_0$, $y \notin \{x(j, s) \mid j \geq 0\}$. So if $s_1 = \max\{s_0, y\}$, it is clear that $y \in H(0, s_1)$.

Second, consider $y = x(j)$ for some j . As seen in the proof of Lemma 1, $x(j, s) \notin H(r(j), s+1)$ for all s . Thus in the limit $x(j) \notin H(r(j))$. By Lemma 1, let s_0 be

such that for all $s \geq s_0$, $x(j, s) = x(j)$. Take $s_1 = \max \{x(j), s_0\}$. Then it is clear that $x(j) \in H(r(j) + 1, s_1)$, so that $x(j) \in H(r(j) + 1) - H(r(j))$. Recalling that $H(k) \subseteq H(k + 1)$ for all k , the lemma now follows easily.

LEMMA 3. *For every e there is a number $i(e)$ such that for all $k, j \geq i(e)$, $(r(k) = r(j)) \Rightarrow (x(k) \in W(e) \Leftrightarrow x(j) \in W(e))$, and either (1) for all $i \geq i(e)$, $H(i + 1) - H(i) \subseteq W(e)$, or (2) for all $i \geq i(e)$, $(H(i + 1) - H(i)) \cap W(e) = \emptyset$.*

Proof. Induct on e , supposing the lemma to hold for all $f < e$. Take $j_0 = \max \{e\} \cup \{i(f) \mid f < e\}$. Suppose the lemma to be false for e . Then there is a pair j_1, j_2 such that $j_0 < j_1 < j_2$, $r(j_1) \leq r(j_2)$, $x(j_1) \notin W(e)$, and $x(j_2) \in W(e)$. For if not, recalling that $r(j) \leq j$ for all j , the construction of $i(e)$ would be routine. Let s_0 be such that if $s \geq s_0$, $j \leq j_2$, and $f \leq e$, then $x(j, s) = x(j)$ and $x(j) \in W(f, s) \Leftrightarrow x(j) \in W(f)$. From the definitions of j_0 and s_0 , we see that for any $s \geq s_0$ and $f < e$, $E(f, s, x(j_1)) = E(f, s, x(j_2))$. Thus since $x(j_1) \notin W(e)$ and $x(j_2) \in W(e)$, we have that $E(e, s, x(j_1)) < E(e, s, x(j_2))$. By (3), $E(j_1, s, x(j_1)) < E(j_1, s, x(j_2))$, or what is the same thing for $s \geq s_0$, $E(j_1, s + 1, x(j_1, s)) < E(j_1, s + 1, x(j_2, s))$. We have $H(r(j_1)) \subseteq H(r(j_2))$ as $r(j_1) \leq r(j_2)$, and since $j_1 < j_2$ it follows that $P(j_2, s, y) \Rightarrow P(j_1, s, y)$ for any y . In particular $P(j_2, s + 1, x(j_2, s + 1))$ must hold, and $x(j_2, s) = x(j_2, s + 1)$ since $s \geq s_0$, so that $P(j_1, s + 1, x(j_2, s))$ is true. But the latter along with $E(j_1, s + 1, x(j_1, s)) < E(j_1, s + 1, x(j_2, s))$ implies that $x(j_1, s) \neq x(j_1, s + 1)$, where $s \geq s_0$, a contradiction of the definition of s_0 . Hence the lemma must hold for e .

We have already remarked that for all i , $H(i) \subseteq H(i + 1)$. By Lemma 2 and the remark preceding it, $H(i + 1) - H(i)$ is infinite for all i . Thus $H(i)$ cannot be quasi-maximal for any i .

It remains to see that $H(0)$ is hyperhypersimple. First, we claim that $H(i + 1) - H(i)$ is always cohesive. This is immediate from Lemma 2 and the first part of Lemma 3. Now, suppose that $H(0)$ were not hyperhypersimple. Then there would be an r.e. sequence $\{S(i) \mid i \geq 0\}$ of pairwise disjoint r.e. sets such that $S(i) \cap \overline{H}(0)$ is infinite for all i . Let $K(e) = \{j \mid (H(j + 1) - H(j)) \cap S(e) \text{ is infinite}\}$, for all e . We show that for any e , $K(e) \neq \emptyset$. Let $S(e) = W(k)$ and let $i(k)$ be as in Lemma 3. Then using Lemma 2 we see that $S(e) \subseteq \overline{H}(i(k))$ or else $S(e) \cap \overline{H}(i(k)) = \emptyset$. In the former case $K(e)$ is infinite. In the latter case, $S(e) \subseteq H(i(k))$, and so $(H(i(k)) - H(0)) \cap S(e)$ is infinite. Since $H(i(k)) - H(0)$ is the finite disjoint union of $H(i(k)) - H(i(k) - 1), \dots, H(1) - H(0)$, it follows that for some $j < i(k)$, $(H(j + 1) - H(j)) \cap S(e)$ is infinite. So $j \in K(e)$, and again $K(e) \neq \emptyset$. Now, if $i \neq j$ then $K(i) \cap K(j) = \emptyset$, because each set $H(e + 1) - H(e)$ is cohesive, and $S(i) \cap S(j) = \emptyset$. Consider $S(E) = \bigcup S(2i)$ ($i \geq 0$), $S(Q) = \bigcup S(2i + 1)$ ($i \geq 0$), $K(E) = \bigcup K(2i)$ ($i \geq 0$), and $K(Q) = \bigcup K(2i + 1)$ ($i \geq 0$). $S(E)$ and $S(Q)$ are disjoint r.e. sets, and $K(E)$ and $K(Q)$ are infinite and disjoint. If $k \in K(E)$, then $(H(k + 1) - H(k)) \cap S(E)$ is clearly infinite, and similarly for Q in place of E . Suppose that $S(E) = W(e)$. Then Lemma 3 fails at e , for by our construction neither of the alternatives (1) or (2) can be true for any value of $i(e)$. By contradiction $H(0)$ must be hyperhypersimple, and so is $H(i)$ for all i .

3. THEOREM 2. *There is an r -maximal set which is not hyperhypersimple.*

Sequences of sets $H(s)$ and $R(i, s)$, functions $x(j, s)$ and $E(j, s, y)$, and a predicate $P(j, s, x)$ are all defined simultaneously by induction on s . For each s , the definitions of $x(j, s)$, $E(j, s, y)$, and $P(j, s, x)$ proceed by simultaneous induction on j , for all x and y at each step. The definition of $E(j, s, y)$ depends on whether or not $r(j)=0$.

At stage $s=0$ set $H(0)=\emptyset$ and $R(i, 0)=\emptyset$ for all $i \geq 0$.

At any stage $s \geq 0$, set

$$(7) \quad E(p(0, d), s, y) = \sum 2^{d-k}(k \leq d \ \& \ y \in W(k, s))$$

for every d and y . Next, for all n and i such that $n+i=d$ and $n > 0$, set

$$(8) \quad E(p(n, i), s, y) = \sum 2^{d-k}(k \leq d \ \& \ y \in W(k, s) \ \& \ x(p(0, d), s) \in W(k, s)).$$

For all x and k let

$$(9) \quad P(k, s, x) \equiv (j)(j < k \Rightarrow x > x(j, s)) \ \& \ (x \notin R(0, s) \vee x \in R(r(k), s) - H(s)),$$

and for all k let

$$(10) \quad x(k, s) = \mu x(P(k, s, x) \ \& \ (y)(P(k, s, y) \Rightarrow E(k, s, x) \geq E(k, s, y))).$$

Finally, at stage $s+1 > 0$, let

$$\begin{aligned} H(s+1) &= \{y \mid y \leq s+1 \ \& \ (j)(y \neq x(j, s))\}, \\ R(0, s+1) &= \{y \mid y \leq s+1 \ \& \ (j)(y = x(j, s) \Rightarrow r(j) \neq 0)\}, \ \text{and} \\ R(i, s+1) &= \{y \mid y \leq s+1 \ \& \ (j)(y = x(j, s) \Rightarrow r(j) = i)\} \end{aligned}$$

for all $i > 0$.

The quantifiers employed in these definitions are implicitly bounded by s or k , and the least number operator is easily seen to be well defined. Thus all our functions, sequences of sets, and predicates are recursive. In conclusion, we set $H = \bigcup H(s)$ ($s \geq 0$) and $R(i) = \bigcup R(i, s)$ ($s \geq 0$) for all i . These sets are then r.e., and of course the sequence $\{R(i) \mid i > 0\}$ is r.e. also.

As in the construction of Theorem 1, the predicate P defined in (9) imposes certain restrictions on the choice of $x(k, s)$, since by (10) $P(k, s, x(k, s))$ must hold. Beyond this the choice of $x(k, s)$ is dictated by the priority scheme embodied in the definition of $E(k, s, x)$. If $r(k)=0$, (7) applies and the priority scheme is unchanged from Theorem 1. The numbers $x(k, s)$ for $r(k)=0$ constitute the s th approximation to $\bar{R}(0)$, and so $R(0)$ turns out to be maximal. But if $r(k) > 0$, (8) applies and the highest priority goes to membership in $W(0, s)$ only if $x(p(0, n+j), s) \in W(0, s)$ (where $k=p(j, n)$). The same restriction may nullify the value of membership in $W(1), \dots$, down to $W(n+j)$. This has the effect of rejecting, eventually, every set $W(k)$ such that $\bar{R}(0) \cap W(k)$ is finite. The numbers $x(k, s)$ for $r(k)=i > 0$ represent the s th approximation to the difference $R(i) - H$. The sets

$R(i) - H$ are not cohesive; instead, the use of the restricted priority scheme yields that if $\bar{R}(0) \cap W(k)$ is infinite, then $\bar{H} - W(k)$ is finite, as seen in Lemma 5.

Directly from the definitions we see that $R(0, s) = \bigcup R(i, s)$ ($i > 0$) for every s , and $H(s) = R(i, s) \cap R(j, s)$ for every s, i , and j such that $i > 0, j > 0$, and $i \neq j$. Also for every $s, x(0, s) < x(1, s) < \dots$. We claim that $R(i, s) \subseteq R(i, s+1)$ for every s and every $i > 0$. Suppose that $x \in R(i, s)$ with $i > 0$. Then $x \leq s$, and $P(k, s, x)$ is true only if $r(k) = i$. Consequently we cannot have $x = x(k, s)$ unless $r(k) = i$, and so by definition $x \in R(i, s+1)$. In addition, this yields that $H(s) \subseteq H(s+1)$ and $R(0, s) \subseteq R(0, s+1)$ for all s . Going to the limit in s we now see that $R(0) = \bigcup R(i)$ ($i > 0$) and $H = R(i) \cap R(j)$ for all $0 < i \neq j > 0$.

LEMMA 4. $\lim_s x(j, s)$ exists for all j .

Proof. Induct on j , assuming the lemma to hold for all $k < j$. Suppose, first of all, that $j = p(n, i)$ for some i and some $n > 0$. Then $p(n, i) > p(0, n+i)$. By the induction hypothesis there is a stage s_0 such that for all $s \geq s_0$ and $k < j, x(k, s) = x(k, s_0)$, and for all $e \leq j, x(k, s) \in W(e, s) \Leftrightarrow x(k, s_0) \in W(e, s_0)$. In particular let $I = \{m \mid m \leq n+i \ \& \ x(0, n+i) \in W(m)\}$. Then for $s \geq s_0$ (8) becomes

$$E(j, s, x) = \sum 2^{n+i-m} (m \in I \ \& \ x \in W(m, s)),$$

and hence we have the analogue of (4) for our particular j and s_0 ,

$$(x)(t)(s)(t \geq s \geq s_0 \Rightarrow E(j, t, x) \geq E(j, s, x)).$$

It is easy to verify now, as in Lemma 1, that for $s \geq s_0, P(j, s+1, x(j, s))$ is always true. Thus we see that $E(j, s+1, x(j, s+1)) \geq E(j, s, x(j, s))$. Of course for all s and $x, E(j, s, x) < 2^{j+1}$, and so $E(j, s, x(j, s))$ goes to a limit, say by stage $s_1 > s_0$. Then for all $s \geq s_1, x(j, s) \geq x(j, s+1)$, and $\lim_s x(j, s)$ exists. Suppose, finally, that $j = p(0, d)$ for some d . Then (4) holds without alteration, and the proof proceeds as before.

Let $x(j) = \lim_s x(j, s)$ for all j . It is now easy to see that $\bar{R}(0) = \{x(p(0, n)) \mid n \geq 0\}, R(i) - H = \{x(p(i, n)) \mid n \geq 0\}$ for all $i > 0$, and $\bar{H} = \{x(k) \mid k \geq 0\}$. We omit the proof because of its similarity to that of Lemma 2. These sets are all infinite since $x(0) < x(1) < \dots$, which is the limit of a previously noted set of inequalities.

Further, $\bar{R}(0)$ is cohesive. We omit the proof of this because it would resemble the proof of Lemma 3 (as it bears on the first part of the lemma).

LEMMA 5. For any e such that $W(e) \cap \bar{R}(0)$ is infinite, there is a number $k(e)$ such that for all $k \geq k(e), x(k) \in W(e)$.

Proof. For any e , let $d(e) \geq e$ be such that either (1) for all $d \geq d(e), x(p(0, d)) \in W(e)$ or (2) for all $d \geq d(e), x(p(0, d)) \notin W(e)$. This is possible because $R(0)$ is maximal. Suppose the lemma to be true for all $f < e$. Let $h(e)$ be such that $p(0, h(e)) \geq k(f)$ if $f < e$ and $W(f) \cap \bar{R}(0)$ is infinite, and $p(0, h(e)) \geq p(0, d(f))$ for all $f \leq e$. Then we

claim that we can set $k(e) = p(0, h(e))$ if $W(e) \cap \bar{R}(0)$ is infinite. For in this case consider any $h \geq p(0, h(e))$.

If $h = p(0, d)$ for some d , then $h \geq p(0, h(e)) \geq p(0, d(e))$, and $d \geq d(e)$. Thus by definition of $d(e)$, since $W(e) \cap \bar{R}(0)$ is infinite, $x(h) = x(p(0, d)) \in W(e)$.

Suppose that for some d , $p(0, d) < h < p(0, d+1)$. Then $p(0, d) \geq p(0, h(e))$, so that $d \geq d(f)$ for all $f \leq e$. Let s_0 be a stage such that for all $s \geq s_0$, all $f \leq e$, and all $j \leq p(0, d+1)$ we have $x(j, s) = x(j)$ and $x(j) \in W(f, s) \Leftrightarrow x(j) \in W(f)$. So for all $f < e$, $x(p(0, d+1), s) \in W(f, s) \Leftrightarrow x(p(0, d), s) \in W(f, s)$. So if $x(h) = x(h, s) \notin W(e)$, then $E(h, s, x(h, s)) < E(h, s, x(p(0, d+1), s))$, directly from (8) and the fact that $h \geq p(0, d)$ where $d \geq e$. But this is impossible, for $P(p(0, d+1), s, x(p(0, d+1), s))$ certainly holds, and hence $P(h, s, x(p(0, d+1), s))$, contradicting the definition of $x(h, s)$. Thus $x(h) \in W(e)$. Since we assumed only that $h \geq p(0, h(e))$, the lemma is proved.

It is clear from Lemma 5 that H is r -maximal. For, suppose S is a recursive set. Then either $S \cap \bar{R}(0)$ or $\bar{S} \cap \bar{R}(0)$ is infinite; say, without loss of generality, that the latter is the case. Then \bar{S} is r.e., so $\bar{S} = W(e)$ for some e . By Lemma 5, $\bar{H} - W(e)$ is finite, and so $\bar{H} \cap S$ is finite.

Moreover, H cannot be hyperhypersimple. For we can obviously construct an r.e. sequence $\{S(i) \mid i > 0\}$ of pairwise disjoint r.e. sets such that for all $i > 0$, $S(i) \subseteq R(i)$, and $\bigcup S(i) (i > 0) = \bigcup R(i) (i > 0) = R(0)$. Since $R(i) \cap R(j) = H$ for all $0 < i \neq j > 0$ and $R(i) - H$ is infinite for every $i > 0$, this implies that $S(i) \cap \bar{H}$ is infinite for every $i > 0$.

4. Related results. It has recently been shown by Lachlan [2, Theorem 2] that if A and B are r.e. sets and $A - B$ is hyperhyperimmune, then $A - B$ is co-r.e. In particular, the differences $H(i+1) - H(i)$ from the proof of Theorem 1 are co-maximal, and so $H(0)$ is contained in each of infinitely many maximal sets, no two of which have a finite difference. In contrast to this, Lachlan [2, §3] constructs a hyperhypersimple set which is not contained in any maximal set. This provides an alternative proof of Theorem 1, since any quasimaximal set must be contained in a maximal set.

In addition, Lachlan [2, Theorem 7] shows that any maximal set contains a nonmaximal r -maximal set, and the theorem rephrased above shows that no such set is hyperhypersimple. In this indirect way Lachlan provides a stronger version of our Theorem 2.

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