REAL CHARACTERS AND THE RADICAL OF AN ABELIAN GROUP

BY

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1. **Introduction.** A real character of a topological abelian group G is a continuous homomorphism of G into the additive group R of real numbers. In this note, we shall show that the radical of the group G, as defined by Wright [18], is exactly the set of elements annihilated by every real character of the group. Thus a group G has sufficiently many real characters to separate points if and only if G is radical free. For connected groups, this result was implicit in [18]. In 1948, Mackey characterized locally compact abelian groups with sufficiently many real characters [13]; these are the groups of the form $R^n \oplus D$, where R^n is an n-dimensional real vector space and D is a discrete torsion-free group. In 1957, Wright [18] identified the same family as the radical-free locally compact abelian groups.

This characterization of the radical is applied to settle, negatively, a problem raised in [18]: is the radical of a group itself a radical group? An example recently published by N. T. Peck [16] shows, when coupled with this characterization, that this need not be the case. (The result for connected groups, already more or less well known, together with Peck's example, would have sufficed.)

A second application corrects an error in [19], where a sufficient condition that a group be representable as a subgroup of a locally convex space is mistakenly asserted to be both necessary and sufficient.

The techniques are quite straightforward. The notion of a prime semigroup in G plays the central role, and minimal primes are crucial, since they give rise to real characters of G. The deepest result invoked is the Hahn Embedding Theorem for totally ordered abelian groups. We conclude with some comments on the categorical setting of the radical, in the category of all topological abelian groups.

2. **Prime semigroups.** In this section, we deal with purely algebraic considerations involving semigroups in a group. We shall recapitulate, and reorganize somewhat, the material in §§2 and 3 of [18]. In that note, maximality played the central role. Here we shall investigate the intimate relation between maximality and minimality, as given in Theorem 2.12. We shall omit most proofs.

Throughout this section, the word "group" means an abelian group, written additively. In all subsequent sections, we shall consider the word "group" to mean a topological abelian group, written additively. We assume that addition is

Presented to the Society, November 25, 1966; received by the editors September 8, 1966.

(1) This research was supported by National Science Foundation Grants GP-1791 and GP-3917 respectively.

jointly continuous, but we do not assume Hausdorff separation unless explicitly indicated.

2.1. DEFINITION. A nonempty set S in a group G is called a semigroup in G if $(x+y) \in S$ whenever $x, y \in S$. A subset A of G is said to be 0-proper if $0 \notin A$. For any nonempty set A in G we define the sets:

(i) $s(A) = \{x \in G : x+A \subset A\}$;

(ii) $b(A) = s(A) \cap s(-A)$;

(iii) $c(A) = G \sim (A \cup -A)$.

In this definition, $G \sim (A \cup -A)$ denotes the complement of $A \cup -A$ in G; in general, $A \sim B$ denotes the relative complement of B in A.

2.2. PROPOSITION. *Let A be a nonempty subset of a group G. Then*

(i) *A is a semigroup in G if and only if $A \subset s(A)$;*

(ii) $-s(A) = s(-A)$;

(iii) *$s(A)$ is a semigroup in G;*

(iv) *$b(A)$ is a group in G;*

(v) $b(A) = \{x \in G : x+A = A\}$.

2.3. PROPOSITION. *If S is a semigroup in a group G, then the following statements are equivalent:*

(i) *S is 0-proper;*

(ii) $s(S) \cap -S = \varnothing$;

(iii) $b(S) \subset c(S)$;

(iv) $S \cap -S = \varnothing$.

2.4. DEFINITION. A subset A of a group G is a prime set if its complement $G \sim A$ is a semigroup in G.

The complement of any semigroup, and hence of any group, is a prime set; in particular, the complement A of a subgroup is a symmetric prime set ($A = -A$). On the other hand, no subgroup H is a prime set; for if $x \notin H$, then $-x \notin H$, while $0 = x - x \in H$. The group G is itself not a prime set, since its complement is empty. The empty set is a prime set.

Recall that a subset A of G is called antisymmetric if $A \cap -A = \varnothing$. Every antisymmetric set is 0-proper, and a semigroup in G is antisymmetric if and only if it is 0-proper, by Proposition 2.3.

2.5. PROPOSITION. *Let A be a nonempty set in the group G. Then the following are equivalent:*

(i) *A is a prime 0-proper semigroup in G;*

(ii) *A is an antisymmetric prime set in G.*

2.6. PROPOSITION. *Let S be a 0-proper semigroup in G. The following are equivalent:*

(i) *S is prime;*

(ii) *$c(S)$ is a group;*

(iii) $c(S) \subset b(S)$;

(iv) $c(S) = b(S)$.

2.7. DEFINITION. A semigroup S in group G is said to be generating if $G = S - S = \{x - y : x, y \in S\}$.

2.8. PROPOSITION. *If S is a prime semigroup in G, then S is generating.*

2.9. PROPOSITION. *Let S be a prime 0-proper semigroup in G. Then*:

(i) $s(S) = S \cup b(S)$;

(ii) *$s(S)$ is a prime semigroup*;

(iii) $s(s(S)) = s(S)$.

2.10. DEFINITION. Let S be a semigroup in G. (i) We call S a maximal semigroup if $S \neq G$, if S is generating, and if the only semigroups containing S are S and G. (ii) We call S a minimal prime semigroup if S is prime and if S properly contains no prime semigroup. (iii) We call S a maximal prime semigroup if it is a prime semigroup and is not contained in any other prime semigroup.

We require a maximal semigroup to be generating because a semigroup which is maximal in the usual sense is either a group or is maximal in the sense of Definition 2.10; we wish to rule out an accidental encounter with a group.

2.11. PROPOSITION. *A minimal prime semigroup S in G is 0-proper.*

Proof. If S is not 0-proper, then $T = G \sim S$ is 0-proper and is a prime semigroup. Then $T \cap -T = \varnothing$, so $-T \subset S$. Since S is minimal, $-T = S$, and then, since $0 \notin -T$, we have $0 \notin S$, a contradiction.

2.12. THEOREM. *Let S be a semigroup in G. The following are equivalent*:

(i) *S is a maximal semigroup in G*;

(ii) *S is a maximal prime semigroup in G*;

(iii) *$S = s(T)$, where T is some minimal prime semigroup in G.*

Proof. Suppose that S is a maximal semigroup. We show first that $G = S \cup -S$, or, equivalently, that $G \sim S \subset -S$. Let $x \notin S$, and set $W = \bigcup_{i=0}^{\infty} (jx + S)$. Then W is a semigroup containing S. Since S is maximal, $0 \in S$, and hence $x \in W$. Thus W properly contains S, and, by the maximality of S, we must have $W = G$. Then there exists a least integer $k \geqq 0$ so that $(k+1)x \in -S$. Set $V = \bigcup_{j=0}^{k} (jx - S)$. Then V is a semigroup containing $-S$. Since $-S$ is maximal, either $V = -S$ or $V = G$. If $V = G$, there is an integer j with $0 \leqq j \leqq k$ such that $-x \in jx - S$. Then

$$-x \in (k - j - 1)x + S,$$

so that $k - j - 1 \geqq k$, and thus $-1 \geqq j \geqq 0$, a contradiction. Hence $V = -S$. Hence $x \in -S$, so that $G \sim S \subset -S$.

Set $X = G \sim S$. Then $X \cap -X = \varnothing$, so X is antisymmetric. Since S is a semigroup, then X is a prime set. Thus X is a prime semigroup by Proposition 2.5. Then $G \sim X = S$ is prime. Hence (i) implies (ii).

Suppose S is a maximal prime semigroup. Then $T = G \sim S$ is clearly a minimal prime semigroup. By Propositions 2.6 and 2.9, $S = G \sim T = s(-T)$. Hence (ii) implies (iii).

Finally assume that $S=s(T)$, where T is a minimal prime semigroup. By Propositions 2.8 and 2.9, S is generating. Let M be a semigroup such that $S=s(T)\subset M$. If $T\sim(-M)=\varnothing$, then $(G\sim T)\cup -M=G$, so that $s(-T)\cup -M=G$. Since $s(-T)\subset -M$, then $-M=G$ and hence $M=G$. If $T\sim(-M)\neq\varnothing$, then it is an antisymmetric set, since $T\cap -T=\varnothing$, by Propositions 2.3 and 2.11. The complement of $T\sim(-M)$ is $-M$, which is a semigroup, and hence $T\sim(-M)$ is a prime 0-proper open semigroup, by Proposition 2.5. Since T is a minimal prime semigroup, $T=T\sim(-M)$. Thus $-M\subset G\sim T=s(-T)\subset -M$, and hence $M=s(T)$. Hence $s(T)$ is a maximal semigroup.

2.13. COROLLARY. *Let S be a prime 0-proper semigroup in G. Then S is a minimal prime semigroup if and only if $s(S)$ is a maximal semigroup in G.*

There is some intersection between the results of Hayes [9] and the results of this section. Hayes is largely concerned with discrete groups, and with the problem of extending real characters. Maximal semigroups play a major role in Hayes work, as indeed they did in [18]. Further details on the extension problem may be found in Giles [6], and in the definitive theorem of Dixmier [4] for the locally compact case.

3. **Open semigroups.** In this section we introduce topological considerations. It will be seen that open prime 0-proper semigroups play an important role in the structure. In this section we characterize, quite sharply, the minimal elements in this class.

If A is a subset of G, then A^* will denote the topological closure of A, and A° denotes the interior of A. Recall that a set A is called regularly open if $A=A^{*\circ}$, and that a semigroup S in G is called angular if $0\in S^*$. This term is due to Hille and Zorn, [10] and [11], who also established the following proposition.

3.1. PROPOSITION. *Let A be a nonempty subset of G. Then*

(i) *$s(A^*)$ is a closed set;*

(ii) *if A is open, then $s(A)\subset s(A^*)\subset s(A^{*\circ})$;*

(iii) *if A is regularly open, then $s(A)=s(A^*)$;*

(iv) *if A is a regularly open semigroup, then $A^*+A\subset A$;*

(v) *if A is an open angular semigroup, then A is a regularly open semigroup.*

3.2. PROPOSITION. *Let S be an open prime 0-proper semigroup in G. Then*

(i) *S is angular if and only if S is not closed;*

(ii) *S is regularly open;*

(iii) *both $s(S)$ and $b(S)=c(S)$ are closed.*

Proof. (i) If S is closed, S is not angular since S is 0-proper. Conversely, suppose S is not angular. Then $0\notin S^*$, and hence $0\notin(-S)^*$. Thus $0\notin b(S)^\circ$, and hence $b(S)$ is an open subgroup of G. It follows easily that S is closed. Assertion (ii) now follows from Proposition 3.1, and (iii) follows from Propositions 2.6 and 2.9.

3.3. PROPOSITION. *Let S be an open prime 0-proper semigroup in G. Then S is angular if and only if $s(S)=S^*$.*

3.4. DEFINITION. A maximal open 0-proper semigroup in G is an open 0-proper semigroup which is not properly contained in any open 0-proper semigroup in G.

3.5. PROPOSITION. *Every maximal open 0-proper semigroup is a prime semigroup. Every open 0-proper semigroup is contained in a maximal open 0-proper semigroup.*

Proof. The first assertion follows from [18, Theorem 3.3], and the second follows from Zorn's lemma.

3.6. LEMMA. *Let S be an open prime 0-proper semigroup in G, and let $x \in S$. Then the set $T=\bigcap_{n=1}^{\infty}(nx+S)$ is either empty or is an open prime 0-proper semigroup in G contained in S.*

Proof. We have $T \subset S$ in any case, and if $T \neq \varnothing$, it is clearly a 0-proper semigroup. If $a \notin T$ and $b \notin T$, then $a \notin nx+S$ and $b \notin mx+S$ for some m, n. Then $a-nx$ and $b-mx$ are not in S. Since S is prime, $a+b-(n+m)x \notin S$, and hence $a+b \notin T$. Thus T is prime.

Let $z \in T$; since $z=(z-x)+x$, and since $x \in S$, we have $z \in (z-x)+S$. For any $y \in S$, $(z-x)+y-nx=z-(n+1)x+y \in S$ for all $n \geqq 1$, since $z-(n+1)x \in S$ for all $n \geqq 1$. Hence $(z-x)+S \subset nx+S$ for all $n \geqq 1$, and hence $z \in (z-x)+S \subset T$. Since $(z-x)+S$ is open, T is open.

3.7. THEOREM. *Let S be a semigroup in G. Then the following are equivalent:*

(i) *S is an open 0-proper semigroup which is a minimal prime semigroup;*

(ii) *S is an open 0-proper semigroup which is prime and is minimal in the family of open prime semigroups.*

Proof. Clearly (i) implies (ii). Assume (ii), and let $x \in S$. If the set

$$T = \bigcup_{n=1}^{\infty} (nx+S)$$

is not empty, then $T=S$, by the previous lemma. This implies that $x+S=S$, and hence that $x \in b(S)$, by Proposition 2.2. But $S \cap b(S)=\varnothing$, by Proposition 2.6. Hence the set T is empty. Since $G \sim S=s(-S)$, we must therefore have

$$G = G \sim T = \bigcup_{n=1}^{\infty} (nx+s(-S)).$$

In other words, if $x \notin s(-S)$, then $G=\bigcup_{n=1}^{\infty}(nx+s(-S))$. This asserts that $s(-S)$ is a maximal semigroup. By Corollary 2.13, S is a minimal prime semigroup.

Theorem 3.7 shows that there is no ambiguity attached to the word "minimal" in the expression "minimal open prime 0-proper semigroup". In [19], a maximal open 0-proper semigroup M is called "hypermaximal" if $s(M)$ is a maximal semigroup. This now appears to be an unfortunate term, since Theorem 3.7 and

Corollary 2.13 show that a maximal open 0-proper semigroup is hypermaximal if and only if it is a minimal prime semigroup.

4. **Primal, planar, and residual subgroups.** The concept of a residual subgroup was introduced in [18]. The definition was based on a property of *maximal* open 0-proper semigroups which can now be seen, from §2 of this note, to be characteristic of open *prime* 0-proper semigroups. This fact will lead to the characterization of the radical in §6. It will be convenient to have some new terminology.

4.1. DEFINITION. Let G be a topological abelian group, and let H be a subgroup of G. Then we say

(i) H is a primal subgroup if $H=b(S)$ for some open prime 0-proper semigroup S in G;

(ii) H is a planar subgroup if $H=b(S)$ for some minimal open prime 0-proper semigroup S in G;

(iii) H is a residual subgroup if $H=b(S)$ for some maximal open 0-proper semigroup S in G.

We note that primal, planar, and residual subgroups are all closed subgroups. From Proposition 3.5 we draw an immediate conclusion.

4.2. PROPOSITION. *Every residual subgroup is a primal subgroup, and every primal subgroup contains a residual subgroup.*

4.3. DEFINITION. The radical of a topological abelian group G is the intersection of all residual subgroups of G.

This is the definition given in [18]. From Proposition 4.2, we have at once the following characterization of the radical.

4.4. PROPOSITION. *The radical of G is the intersection of all the primal subgroups of G.*

In §6, we prove the much deeper fact that the radical of G is the intersection of all the planar subgroups of G.

5. **Totally ordered groups.** In this section, we show that primal subgroups correspond to continuous homomorphisms of G onto totally ordered groups endowed with the interval topology. The planar subgroups then correspond to the real characters of the group G.

If G is a totally ordered abelian group, and if $S=\{x \in G : x>0\}$, then it is easy to see that S is a 0-proper semigroup in G such that $b(S)=\{0\}$, and, in particular, S is prime. Conversely, if S is a 0-proper semigroup in G with $b(S)=\{0\}$, then S is prime by Proposition 2.6. We may then define $x \leqq y$ in G to mean $y-x \in S \cup \{0\}$; this makes G a totally ordered abelian group. These simple observations provide the basis for this section. In fact, we shall treat a totally ordered abelian group as a pair (G, S) where S is a prime 0-proper semigroup such that $b(S)=\{0\}$ in the abelian group G. Minimal prime semigroups then have an important significance.

5.1. PROPOSITION. *A totally ordered abelian group (G, S) is archimedean if and only if S is a minimal prime semigroup in G.*

Proof. From Corollary 2.13, it follows that S is a minimal prime semigroup if and only if $s(-S)$ is a maximal semigroup. But $s(-S)$ is a maximal semigroup if and only if $y \in \bigcup_{n=1}^{\infty} (nx+s(-S))$ whenever $x, y \in G \sim s(-S) = S$. This, in turn, holds if and only if, for each $x, y \in S$, there is an integer $n>0$ such that $y-nx \in s(-S) = -S \cup \{0\}$. This is exactly the assertion that if $x>0$ and $y>0$, there is an integer $n>0$ such that $y \leqq nx$.

If (G, S) is any totally ordered abelian group, then G is a topological group in its interval topology, which is the topology generated by the collection of open intervals $(a, b) = \{x \in G : a<x<b\}$, where $a, b \in G$, $a<b$. In the sequel, when we speak of a totally ordered abelian group, we shall always assume that the group is endowed with its interval topology. This usage makes a map from an abelian topological group G to a totally ordered abelian group G continuous if it is continuous with respect to the interval topology on H and the given topology on G.

It may be noted that if (G, S) is a totally ordered abelian group, then the semigroup S is open in the interval topology.

5.2. DEFINITION. Let G be a topological abelian group. Then we say

(i) G is a topologically ordered group if there is a totally ordered abelian group H and a homeomorphic isomorphism of G onto H;

(ii) G is a continuously ordered group if there is a totally ordered abelian group H and a continuous isomorphism of G onto H.

In either case, if the group H is an archimedean ordered group, we prefix the adjective "archimedean."

5.3. PROPOSITION. *In a topological abelian group G, the following are equivalent:*

(i) *G is continuously ordered;*

(ii) *$\{0\}$ is a primal subgroup of G;*

(iii) *$\{0\}$ is a residual subgroup of G.*

Proof. Let $f: G \to H$ be a continuous isomorphism of G onto an ordered group H. Let $S = \{x \in H : x>0\}$, so that S is an open prime 0-proper semigroup in H with $b(S) = \{0\}$. Then $S_1 = f^{-1}S$ is an open prime 0-proper semigroup in G with $b(S_1) = \{0\}$. Thus (i) implies (ii).

If $\{0\}$ is primal, then $\{0\}$ is residual by Proposition 4.2, and hence (ii) implies (iii).

Finally, suppose $\{0\}$ is residual, so that $\{0\} = b(M)$, where M is a maximal 0-proper open semigroup in G. By Proposition 3.5, (G, M) is a totally ordered group. Since M is open, the interval $(a, b) = (a+M) \cap (b-M)$ is open in G, so that the identity map of G onto (G, M) is continuous.

In what follows, R will always denote the additive group of real numbers, endowed with the usual order, and consequently with the interval topology. If R_1 is a subgroup of R, then R_1 inherits a total order from R, and consequently R_1 has its own interval topology. This topology coincides with the subspace topology

which R_1 inherits from R. This follows from the fact that R_1 is either closed in R or is dense in R. (This comment does not extend to subgroups of arbitrary totally ordered groups, a fact of some concern in the next section.)

5.4. PROPOSITION. *Let G be an abelian topological group. The following are equivalent*:

(i) *G is archimedean continuously ordered*;

(ii) *there is a continuous isomorphism of G into R*;

(iii) *$\{0\}$ is a planar subgroup of G.*

Proof. Let $f: G \to H$ be a continuous isomorphism of G onto an archimedean totally ordered group H. By Hölder's Theorem [1, p. 222] there is an order-preserving group isomorphism $g: H \to R$, of G onto a subgroup R_1 of R. Since g is order preserving, it is a homeomorphism. Then $g \circ f$ furnishes the desired mapping into R. Hence (i) implies (ii).

To show that (ii) implies (iii), let $f: G \to R$, with image R_1. Let

$$S = f^{-1}\{x \in R_1 : x > 0\}.$$

Then S is an open 0-proper semigroup in G, with $b(S) = \{0\}$, so that S is prime. Since f is one-one, S is a minimal prime semigroup, because $\{x \in R_1 : x > 0\}$ is a minimal prime semigroup in R_1. By Theorem 3.7, $\{0\}$ is a planar subgroup of G.

Finally, let $\{0\}$ be a planar subgroup in G, with $\{0\} = b(S)$. Then by Proposition 3.1 and Theorem 3.7, (G, S) is an archimedean group, and the identity map of G onto (G, S) is continuous. Hence (iii) implies (i).

5.5. LEMMA. *Let S be an open prime 0-proper semigroup in G, and let $\pi: G \to G/b(S)$ be the canonical homomorphism. Then $(G/b(S), \pi(S))$ is a totally ordered group, and $\pi: G \to (G/b(S), \pi(S))$ is continuous.*

Proof. We first show that $\pi(S)$ is a prime 0-proper semigroup in $G/b(S)$ such that $b(\pi/(S)) = 0$. Since $S \cap b(S) = \varnothing$, we have that $\pi(S)$ is a 0-proper semigroup. Since $G = S \cup b(S) \cup -S$ is a disjoint union, and since $b(S) = c(S)$, then $c(\pi(S)) = \{0\}$, and, by Propositions 2.3 and 2.6, we have $b(\pi(S)) = 0$, so $\pi(S)$ is prime. Hence $(G/b(S), \pi(S))$ is totally ordered. Next, note that $\pi(S)$ is open in the quotient topology on $G/b(S)$. Hence open intervals in $(G/b(S), \pi(S))$ are open in the quotient topology. Thus π is continuous as a map from G to $(G/b(S), \pi(S))$.

5.6. PROPOSITION. *A closed subgroup H of an abelian topological group G is primal if and only if H is the kernel of a continuous homomorphism of G onto a totally ordered abelian group.*

Proof. If $f: G \to (L, P)$ is a continuous homomorphism of G onto the totally ordered group (L, P), then $S = f^{-1}P$ is an open prime 0-proper semigroup in G such that $b(S)$ is the kernel of f. The converse follows at once from Lemma 5.5.

A real character of a topological abelian group G is a continuous homomorphism of G into R.

5.7. PROPOSITION. *The following are equivalent for a closed subgroup H of G:*

(i) *H is a planar subgroup of G;*

(ii) *H is the kernel of a continuous homomorphism of G onto an archimedean totally ordered group;*

(iii) *H is the kernel of a real character of G.*

Proof. In view of Proposition 5.1 and Lemma 5.5, it suffices, in the notation of 5.5, to show that S is a minimal open, prime 0-proper semigroup in S if and only if $\pi(S)$ is a minimal open prime semigroup in $G/b(S)$. If $x \in S$ and if $\pi(x)=\pi(y)$, then $y \in x+b(S) \subset S$. Thus $S=\pi^{-1}\pi S$. If $T \subset \pi(S)$, then $\pi^{-1}T \subset \pi^{-1}\pi S=S$. If S is minimal, and if T is a prime semigroup, then $\pi^{-1}T=S$, and hence $T=\pi S$. Thus $\pi(S)$ is minimal. Conversely if $\pi(S)$ is minimal, then $s(\pi(S))$ is a maximal semigroup in $G/b(S)$, by Corollary 2.13. If W is a semigroup in G such that $W \supset s(S)$, then $W \supset b(S)$, and hence $W=\pi^{-1}\pi W$. Then

$$\pi(W) \supset \pi(s(S)) = \pi(b(S) \cup S) = \pi(b(S)) \cup \pi(S) = s(\pi(S)),$$

so that either $\pi(W)=\pi(s(S))$ or $\pi(W)=G/b(S)$. Then either

$$W = \pi^{-1}\pi(W) = \pi^{-1}\pi(s(S)) = s(S) \quad \text{or} \quad W = \pi^{-1}\pi W = \pi^{-1}(G/b(S)) = G.$$

Hence $s(S)$ is maximal, so that S is minimal, by Corollary 2.13. This establishes the equivalence of (i) and (ii). The equivalence of (ii) and (iii) is immediate from Proposition 5.4.

The equivalence of (i) and (iii) shows that planar subgroups in G are analogues (and generalizations) of hyperplanes in linear topological spaces, and furnishes an a posteriori justification for the term "planar subgroup."

6. **Characterization of the radical.** It is shown in [18] that the radical of a real linear topological space consists of all those elements annihilated by every continuous linear functional. In this section we show that the radical of any topological abelian group is the set of elements annihilated by every real character of the group. The basic tool for this proof is the Hahn Embedding Theorem [8] for abelian totally ordered groups. Before applying this theorem, we handle some technicalities.

If (G, S) is a totally ordered abelian group, and if H is a subgroup of G, then $(H, H \cap S)$ is a totally ordered group. The subspace topology on H is always finer than the interval topology on $(H, H \cap S)$, but it may actually be strictly finer. For instance, consider the lexicographically ordered plane G (with the x-axis dominating), and let H be the x-axis. We have remarked earlier that if G is the group R, then the two topologies on the subgroup H agree. We shall have need of the following instance in which the two topologies coincide.

6.1. LEMMA. *Let H be a subgroup of a totally ordered abelian group (G, S), and suppose that for each $a \in S$, there is an integer $n>0$ such that $na \in H \cap S$. Then the interval topology and the subspace topology on H coincide.*

Proof. It is sufficient to show that, for any $a \in S$, there exists an element $b \in H \cap S$ such that $\{x \in H : -b < x < b\} \subset (-a, a)$. If there is an element $b \in (0, a) \cap H$, this is obvious. Otherwise, we have $(0, a) \cap H = \varnothing$, and consequently $(-a, a) \cap H = \{0\}$. Let n be the least positive integer such that $na \in H \cap S$. If $(-na, na) \cap H = \{0\}$, then the interval topology on H is discrete, and the two topologies clearly coincide. So we may suppose that there is a nonzero positive element z of H with $0 < z < na$. Since $z \in H$, then $z \neq ka$ for $0 \leqq k \leqq n-1$, and hence $z \in \bigcup_{j=0}^{n-1} (ja, (j+1)a)$. Since $(0, a) \cap H = \varnothing$, there is a least positive integer m such that $(0, ma) \cap H = \varnothing$ and $(ma, (m+1)a) \cap H \neq \varnothing$. Let $x \in (ma, (m+1)a) \cap H$. If there were an element $y \in (ma, x) \cap H$, then $y - x$ would be in $(0, a) \cap H$. Hence $(0, x) \cap H = \varnothing$, so that $(-x, x) \cap H = \{0\}$. Thus the interval topology on H is discrete, and again the topologies coincide. This completes the proof.

We apply this result in the following setting. Let G be a totally ordered abelian group, and let D denote the divisible hull of G. Then [7, pp. 65–66] for each nonzero element a of D, there is an integer n such that $na \in G$. Hence we may extend the total order of G to a total order on D by decreeing that $a \in D$ is positive if na is a positive element of G for some integer $n > 0$. It follows from Lemma 6.1 that the inclusion map $i: G \to D$ is continuous when both G and D carry their interval topologies.

We now turn to the application of Hahn's theorem. Let Γ be a totally ordered set, and suppose that, for each $\gamma \in \Gamma$, there is given a nonzero subgroup R_γ of the real group R. We denote the direct product of this indexed family by $\prod_{\gamma \in \Gamma} R_\gamma$, and we denote the direct sum of the indexed family by $\sum_{\gamma \in \Gamma} R_\gamma$. We regard $\sum_{\gamma \in \Gamma} R_\gamma$ as a subgroup of $\prod_{\gamma \in \Gamma} R_\gamma$. We shall denote by $W(\Gamma, (R_\gamma)_{\gamma \in \Gamma})$, or more simply by $W(\Gamma)$, the subset of $\prod_{\gamma \in \Gamma} R_\gamma$ given by

$$W(\Gamma) = \{(r_\gamma)_{\gamma \in \Gamma} : \{\alpha \in \Gamma : r_\alpha \neq 0\} \text{ is well-ordered.}\}$$

Then $W(\Gamma)$ is a subgroup of $\prod_{\gamma \in \Gamma} R_\gamma$ and contains the subgroup $\sum_{\gamma \in \Gamma} R_\gamma$. We introduce a total order on $W(\Gamma)$ by declaring that an element $(r_\gamma)_{\gamma \in \Gamma} \in W(\Gamma)$ is strictly positive if $r_\alpha > 0$, where α is the least element of the set $\{\beta \in \Gamma : r_\beta \neq 0\}$. With this lexicographic order, $W(\Gamma)$ is called a Hahn group.

Let G be any subgroup of $W(\Gamma)$ which contains the subgroup $\sum_{\gamma \in \Gamma} R_\gamma$. For $\beta \in \Gamma$, we define the mapping $p_\beta: G \to R_\beta$ by setting $p_\beta((r_\gamma)_{\gamma \in \Gamma}) = r_\beta$. There p_β is a homomorphism of G into R_β, for each $\beta \in \Gamma$. We show that p_β is continuous when G and R_β have their interval topologies. To this end, let $b \in R_\beta$ be positive. It suffices to show that there exists a positive element $b_1 \in G$ such that

$$p_\beta(\{g \in G : -b_1 < g < b_1\}) \subset (-b, b).$$

There are two cases. First, suppose β is not the last element of Γ. Let $\alpha \in \Gamma$ be such that $\alpha > \beta$, and let a be a positive element of R_α. Let $j_\alpha: R_\alpha \to W(\Gamma)$ be the injection given by $(j_\alpha(x))_\gamma = 0$ if $\alpha \neq \gamma$, $(j_\alpha(x))_\alpha = x$, for $x \in R_\alpha$. Then $j_\alpha(a) \in G$, since G contains $\sum_{\gamma \in \Gamma} R_\gamma$, and the set $N = \{g \in G : -j_\alpha(a) < g < j_\alpha(a)\}$ is such that $N) \subset (-b, b)$ since $p_\beta(p_\beta(g) = 0$ for all $g \in N$. In the second case, where β is the last

element of Γ (if such a last element exists), we let $N=\{g \in G : -j_\beta(b)<g<j_\beta(b)\}$, where $j_\beta: R_\beta \to W(\Gamma)$ is the injection as before. Then $p_\beta(N)\subset(-b, b)$.

We remark, and shall use the fact, that the projections p_β may be regarded as continuous homomorphisms of G into R with images R_β. This is because the interval topology on R_β and the subspace topology on R_β coincide, as remarked in §5.

6.2. THEOREM. *Let G be an abelian topological group, and let f be a continuous homomorphism of G onto a totally ordered group (H, S). Then there exists a family of real characters $\{g_\alpha : \alpha \in \Gamma\}$ of G such that* $\ker(f)=\bigcap\{\ker(g_\alpha) : \alpha \in \Gamma\}$.

Proof. Here, $\ker(f)$ denotes the kernel of f. Let D be the divisible hull of H, and let D be endowed with its interval topology. Let $i: H \to D$ be the inclusion mapping, so that i is continuous, by the earlier remarks. By the Hahn Embedding Theorem (see [2, Corollary 8.1], for example) there is a family $\{R_\alpha : \alpha \in \Gamma\}$ of nonzero subgroups of R, a subgroup H_1 of $W(\Gamma, (R_\alpha)_{\alpha\in\Gamma})$ containing $\sum_{\alpha\in\Gamma} R_\alpha$, and an order-isomorphism φ of D onto H_1. Since φ is order preserving, $\varphi: D \to H_1$ is continuous when H and H_1 have their interval topologies. Set $g_\alpha=p_\alpha \circ \varphi \circ i \circ f$. Then $g_\alpha: G \to R_\alpha$ is continuous, and we may regard $g_\alpha: G \to R_\alpha$ as a map into R. For $x \in G$, $g_\alpha(x)=0$ for all $\alpha \in \Gamma$ if and only if $\varphi \circ i \circ f(x)=0$, and this holds if and only if $f(x)=0$, since φ is an isomorphism. This proves the theorem.

6.3. COROLLARY. *If G is a topological abelian group, and if H is a primal subgroup of G, then H is the intersection of all the planar subgroups containing H.*

Proof. This follows at once from the above theorem and from Propositions 5.6 and 5.7.

6.4. THEOREM. *The radical of a topological abelian group G is the intersection of all the planar subgroups of G. Equivalently, the radical of G is the set of all those elements of G which are annihilated by every real character of G.*

Proof. This is immediate from Proposition 4.4 and the above corollary.

6.5. COROLLARY. *A topological abelian group G has zero radical if and only if, for each $x\neq 0$ in G, there is a real character f of G such that $f(x)\neq 0$.*

The above corollary may be regarded as known in the locally compact case, by combining results of Mackey [13] and Wright [18], as indicated in the introduction.

7. **Final comments.** If G is a topological abelian group, the set G^* of all real characters can be made into a linear topological space, under the usual pointwise operations and with the usual weak * topology. The conjugate space G^{**} of G^*, again in the weak * topology, is a locally convex linear space, and there is a natural continuous homomorphism of G into G^{**} with kernel equal the radical of G. It follows that a topological abelian group G has a continuous isomorphism into a locally convex linear space if and only if G has zero radical. This observation corrects an error in Theorem D of [19].

The homomorphism described above can be arranged to be a coreflection of the category of abelian topological groups into a subcategory (of "weakly reflexive" spaces) of the category of locally convex spaces. (We use the terminology of Mitchell [15] for categories, noting that it sometimes conflicts with other sources [14], [5].)

The definition of the radical enables one to define a *radical group* as a group which is equal to its own radical. It is observed in [8] that every topological abelian group G contains a maximal radical subgroup M, which is closed and which is contained in the radical T. The question was raised in [18] as to whether these two groups M and T always coincide. A recent example in linear topological spaces, due to Peck [16], provides an instance in which they do not coincide. The example furnishes a Hausdorff linear topological space E having a one-dimensional subspace F which is precisely the set annihilated by every continuous functional as E, and is thus the radical of E. However, F is trivially a group with zero radical.

The radical clearly contains every element which generates a precompact subgroup of G, and in the locally compact case, this set is the radical [18]. However, by combining Peck's example with a radical linear topological space [3], [12], [17], such as L_p, where $0<p<1$, then one can easily obtain an example in which $0\subset M\subset T\subset G$, with all inclusions proper, in which there are no elements with precompact closure.

Both M and T behave as radicals should in the category $\mathfrak{G}$ of all topological abelian groups. If $\mathscr{F}$ denotes the subcategory of $\mathfrak{G}$ composed of groups with zero radical, and if $\mathscr{R}$ denotes the subcategory of radical groups, then $\mathscr{F}$ and $\mathscr{R}$ have both products and coproducts, $\mathscr{F}$ is closed under subobjects and $\mathscr{R}$ is closed under quotients. However, since $T\neq M$ in general, one cannot always construct an exact sequence $0\to R\to G\to F\to 0$, with $R\in\mathscr{R}$ and $F\in\mathscr{F}$. Hence, although the radical furnishes a natural generalization of the torsion subgroup, it does not furnish a natural extension of torsion theory, except in special cases, such as locally compact groups, locally convex spaces, or discrete groups.

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