A STUDY OF METRIC-DEPENDENT DIMENSION FUNCTIONS(1)

BY

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1. **Introduction.** This paper is a study of metric-dependent dimension functions for metric spaces. Let X be a metric space with metric ρ. We introduced in a previous paper [15] two dimension functions d_1 and d_2 of (X, ρ) which by definition appear to depend on ρ. We showed however, that $d_1(X, \rho)=\dim X$ (the covering dimension of X). On the other hand d_2 does depend on the particular metric ρ, and there exists (X, ρ) with $d_2(X, \rho)<\dim X$.

DEFINITION 1. The empty set $\varnothing$ has $d_2\varnothing=-1$. $d_2(X, \rho)\leqq n$ if (X, ρ) satisfies the condition:

(D_2) For any $n+1$ pairs of closed sets $C_1, C_1'; \dots; C_{n+1}, C_{n+1}'$ with $\rho(C_i, C_i')>0$, $i=1,\dots, n+1$, there exist closed sets B_i, $i=1,\dots,n+1$, such that (i) B_i separates C_i and C_i' for each i and (ii) $\bigcap_{i=1}^{n+1} B_i=\varnothing$.

If $d_2(X, \rho)\leqq n$ and the statement $d_2(X, \rho)\leqq n-1$ is false, we set $d_2(X, \rho)=n$.

This definition stems from Eilenberg-Otto's characterization of dimension [3]:

A metric space X has $\dim X\leqq n$ if and only if the following condition is satisfied:

(D_2') For any $n+1$ pairs of closed sets $C_1, C_1'; \dots; C_{n+1}, C_{n+1}'$ with $C_i\cap C_i'=\varnothing$, $i=1,\dots,n+1$, there exist closed sets B_i, $i=1,\dots,n+1$, such that (i) B_i separates C_i and C_i' for each i and (ii) $\bigcap_{i=1}^{n+1} B_i=\varnothing$.

This characterization of (covering) dimension is still true even when X is only a normal space (cf. Hemmingsen [5, Theorem 6.1] or Morita [10, Theorem 3.1]). All spaces considered in this paper are T_1. To clarify the situation of d_2 we introduce the following two apparently metric-dependent dimension functions which are similar to d_2.

DEFINITION 2. First we set $d_3\varnothing=-1$. $d_3(X, \rho)\leqq n$ if (X, ρ) satisfies the condition:

(D_3) For any finite number m of pairs of closed sets $C_1, C_1'; \dots; C_m, C_m'$ with $\rho(C_i, C_i')>0$, $i=1,\dots,m$, there exist closed sets B_i, $i=1,\dots,m$, such that (i) B_i separates C_i and C_i' for each i and (ii) the order of $\{B_i : i=1,\dots,m\}$, ord $\{B_i\}$, is at most n.

If $d_3(X, \rho)\leqq n$ and the statement $d_3(X, \rho)\leqq n-1$ is false, then we set $d_3(X, \rho)=n$.

DEFINITION 3. First we set $d_4\varnothing=-1$. $d_4(X, \rho)\leqq n$ if (X, ρ) satisfies the condition:

(D_4) For any countable number of pairs of closed sets $C_1, C_1'; C_2, C_2'; \dots$ with $\rho(C_i, C_i')>0$, $i=1,2,\dots$, there exist closed sets B_i, $i=1,2,\dots$, such that (i) B_i separates C_i and C_i' for each i and (ii) ord $\{B_i : i=1,2,\cdots\}\leqq n$.

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If $d_4(X, \rho) \leqq n$ and the statement $d_4(X, \rho) \leqq n-1$ is false, then we set $d_4(X, \rho)=n$.

Let (D_3') (respectively (D_4')) be the condition which is obtained from (D_3) (resp. from (D_4)) when "$\rho(C_i, C_i')>0$" is replaced by "$C_i \cap C_i' = \varnothing$". It is evident that (D_4') implies (D_3'), say $(D_4') \to (D_3')$, and $(D_3') \to (D_2')$. It is also true that $(D_4) \to (D_3) \to (D_2)$. Morita [10] proved that $(D_2') \to (D_3') \to (D_4')$ even when X is only a normal space. Then it is natural to ask whether or not $(D_2) \to (D_3) \to (D_4)$. The answer is no for each implication. It will be shown that $d_4(X, \rho)=\dim X$ for any (X, ρ) (Theorem 2 below). Moreover we shall construct in this paper a space (R, ρ) such that $d_2(R, \rho)=2$, $d_3(R, \rho)=3$ and $d_4(R, \rho)=4$. It is to be noticed that (R, ρ) is topologically complete and ρ is totally bounded. Our dimension functions are closely related to so-called *metric dimension* which is defined as follows:

DEFINITION 4. First we set $\mu \dim \varnothing = -1$. $\mu \dim (X, \rho) \leqq n$ if (X, ρ) satisfies the condition:

(D_0) There exists a sequence of open coverings $\mathscr{U}_i$ of X such that (i) ord $\mathscr{U}_i \leqq n+1$ for each i and (ii) mesh $\mathscr{U}_i$ $(=\sup \{\rho(U) : U \in \mathscr{U}_i\})$ converges to zero.

If $\mu \dim (X, \rho) \leqq n$ and the statement $\mu \dim (X, \rho) \leqq n-1$ is false, then we set $\mu \dim (X, \rho)=n$.

Here we note that whether $\mu \dim (X, \rho)=\dim X$ or not had been a serious problem in dimension theory and that the gap between $\mu \dim$ and dim played an important role when the study of dimension theory moved to general metric spaces from separable metric spaces (cf. Sitnikov [19], Nagata [17], [18], Nagami [13], Vopěnka [22], Dowker-Hurewicz [2] and Katětov [8]).

We prove that $d_3(X, \rho) \leqq \mu \dim (X, \rho)$ for any (X, ρ) and that $d_3(X, \rho)=\mu \dim (X, \rho)$ when ρ is totally bounded (Theorems 4 and 5 below). Thus the space (R, ρ) mentioned before offers an example such that $d_2(R, \rho)<\mu \dim (R, \rho)<\dim R$. Sitnikov [19] was the first to construct a space (Y, ρ) such that $\mu \dim (Y, \rho)<\dim Y$.

In every Cantor n-manifold (K_n, ρ), $n \geqq 3$, we shall construct subspaces (X_n, ρ) and (Y_n, ρ) such that

(i) $\dim X_n=\dim Y_n \geqq n-1$ and

(ii) $d_2(X_n, \rho)=\mu \dim (Y_n, \rho)=[n/2]$.

To prove $\dim X_n$ or $\dim Y_n \geqq n-1$ we need the following theorem (Theorem 1 below) which is interesting in itself:

If A_i, $i=1, 2, \ldots$, are disjoint closed sets of K_n with $\dim A_i \leqq n-1$ for every i, then $\dim (K_n-\bigcup A_i) \geqq n-1$.

Sitnikov [20] proved that $\dim (K_n-\bigcup A_i) \geqq n-1$ if $K_n=I^n$ (n-cube) without the condition $\dim A_i \leqq n-1$ and with $A_i \neq I^n$ for $i=1, 2, \ldots$. Then it is natural to ask whether our present theorem for K_n is still true without any hypothesis on $\dim A_i$, and with $A_i \neq K_n$ for $i=1, 2, \ldots$. We give a negative answer for this question. (See Figure 2.)

We give for each $n \geqq 2$ a metric space Z_n which allows equivalent metrics ρ_m, $m=[(n+1)/2], [(n+1)/2]+1, \ldots, n$, such that $d_2(Z_n, \rho_m)=\mu \dim (Z_n, \rho_m)=m$. This space not only illustrates the dependence of μ dim and d_2 on the metric but plays a role in the construction of our final example R which is mentioned above.

The final section lists four unsolved problems.

2. Dimension of the complement of a disjoint collection of sets.

LEMMA 1. *Let X be a hereditarily normal space and Y a subset of X with* $\dim (X-Y)<n$. *Then for any n pairs of disjoint closed sets of X, $C_1, C_1'; \dots; C_n, C_n'$, there exist closed sets of X, $B_1, \dots, B_n$, such that $\bigcap B_i \subset Y$ and B_i separates C_i and C_i' for each i.*

Proof. Let $D_1, D_1'; \dots; D_n, D_n'$ be open sets of X such that $C_i \subset D_i$, $C_i' \subset D_i'$ and $\bar{D}_i \cap \bar{D}_i' = \varnothing$ for each i. By Hemmingsen [5, Theorem 6.1] or Morita [10, Theorem 3.1] there exist relatively open sets $U_1, \dots, U_n$ of $X-Y$ such that

(i) $\bar{D}_i - Y \subset U_i \subset \bar{U}_i - Y \subset (X-\bar{D}_i') - Y$, $i=1, \dots, n$,

(ii) $\bigcap (\bar{U}_i - U_i) \subset Y$.

If we set $G_i = C_i \cup U_i$ and $H_i = C_i' \cup ((X-Y)-\bar{U}_i)$, then $\bar{G}_i \cap H_i = G_i \cap \bar{H}_i = \varnothing$. By the hereditary normality of X there exists an open set V_i of X such that $G_i \subset V_i \subset \bar{V}_i \subset X - H_i$. Set $B_i = \bar{V}_i - V_i$. Then B_i, $i=1, \dots, n$, satisfy the required condition.

LEMMA 2. *Let X be a compact Hausdorff space and let H and K be disjoint closed sets of X such that no connected set meets both H and K. Then there exist disjoint open sets H_1 and K_1 such that $H \subset H_1$, $K \subset K_1$ and $H_1 \cup K_1 = X$.*

This can be proved by a method analogous to the one in Moore [9, Theorem 44, p. 15] with the consideration of Hocking-Young [6, Theorem 2-9, p. 44].

LEMMA 3. *A connected compact Hausdorff space cannot be decomposed into a countably infinite or finite (but more than one) union of disjoint closed subsets.*

This can be proved by a method analogous to the one in Moore [9, Theorem 56, p. 23] with the aid of Lemma 2.

DEFINITION 5. Let X be a normal space. A system of pairs $C_1, C_1'; \dots; C_n, C_n'$ is called a *defining* system of X if (i) C_i and C_i' are disjoint closed sets of X for each i and (ii) for arbitrary closed sets B_i, $i=1, \dots, n$, separating C_i and C_i' we have $\bigcap B_i \neq \varnothing$.

LEMMA 4. *Let X be a compact Hausdorff space, F a closed set of X and f a mapping (continuous transformation) of F into the $(n-1)$-sphere S^{n-1}. Consider S^{n-1} as the surface of the n-cube $I^n = \{(x_1, \dots, x_n) : -1 \leqq x_i \leqq 1\}$. Let $C_1, C_1'; \dots; C_n, C_n'$ be n pairs of opposite faces of I^n defined by:*

$$C_i = \{(x_1, \dots, x_n) : x_i = -1\}, \qquad C_i' = \{(x_1, \dots, x_n) : x_i = 1\},$$

$i=1, \dots, n$. *If the system $f^{-1}(C_1), f^{-1}(C_1'); \dots; f^{-1}(C_n), f^{-1}(C_n')$ is not defining, then f has an extension $f^*: X \to S^{n-1}$.*

Proof. Let $B_1, \ldots, B_n$ be closed sets of X such that B_i separates $f^{-1}(C_i)$ and $f^{-1}(C_i')$ for every i and such that $\bigcap B_i = \varnothing$. By Morita [10, Lemma 1.2] we can assume that every B_i is a G_δ. Let $f(x) = (f_1(x), \ldots, f_n(x))$, where each f_i is a mapping into $[-1, 1]$. Let $g_i \colon X \to [-1, 1]$ be an extension of $f_i | f^{-1}(C_i) \cup f^{-1}(C_i')$ such that $g_i(x) = 0$ if and only if $x \in B_i$ and such that $|g_i(x)| = 1$ if and only if $x \in f^{-1}(C_i) \cup f^{-1}(C_i')$. Let $g(x) = (g_1(x), \ldots, g_n(x))$. Then g is a mapping of X into I^n and $g(F) \subset S^{n-1}$. If $x \in F$, then $f(x)$ and $g(x)$ cannot be a pair of opposite points on S^{n-1}. Hence f is homotopic to $g|F$. Let p be the original point $(0, \ldots, 0)$ of I^n. Then $p \notin g(X)$. Let $r \colon I^n - p \to S^{n-1}$ be a retraction. Then rg maps X into S^{n-1}. By the same argument as in Hurewicz-Wallman [7, Chapter VI], f has an extension f^* over X whose values are still in S^{n-1}.

THEOREM 1. *Let X be a compact hereditarily normal space with* $\dim X = n$, $n \geqq 1$, *and $A_1, A_2, \cdots$ be a sequence of disjoint closed sets of X such that* $\dim A_i \leqq n-1$ *for each i. Then*

$$\dim \left(X - \bigcup A_i\right) \geqq n-1.$$

Proof. *First step.* Since $\dim X = n$ there exist by Morita [10, Theorem 5.1] a closed set F of X, a mapping f of F into S^{n-1} and a closed set $Y \supset F$ such that (i) f cannot be extended over Y and (ii) if Z is any proper closed subset of Y with $F \subset Z$, then f is extendable over Z. We may even assume that Y is actually a Cantor n-manifold (cf. Hurewicz-Wallman [7, pp. 99–100]). Since $\dim (Y - \bigcup A_i) \geqq n-1$ implies $\dim (X - \bigcup A_i) \geqq n-1$, we assume hereafter X is Y itself with the above minimal property.

Second step. Since $\dim A_1 \leqq n-1$, there exists a mapping $f' \colon F \cup A_1 \to S^{n-1}$ with $f'|F = f$. Since S^{n-1} is a neighborhood extensor for normal spaces by Hanner [4, Theorem 13.2], there exist an open set U_1 with $U_1 \supset F \cup A_1$ and a mapping $f_1 \colon \overline{U}_1 \to S^{n-1}$ with $f_1 | F \cup A_1 = f'$. Continuing such procedure, we have a sequence of open sets $U_1, U_2, \cdots$ and a sequence of mappings $f_i \colon \overline{U}_i \to S^{n-1}$ such that (i) $F \cup (\bigcup_{j \leqq i} A_j) \subset U_i$ and $\overline{U}_i \subset U_{i+1}$, for every i and (ii) f_i is an extension of f_{i-1} for every i, where $f_0 = f$.

Define $g \colon \bigcup U_i \to S^{n-1}$ in such a way that $g|\overline{U}_i = f_i$ for each i. Then g is an extension of f over $\bigcup U_i$. Let $\varphi \colon X \to [0, 1]$ be a mapping such that

(i) $\varphi(x) = 0$ if and only if $x \notin \bigcup U_i$,

(ii) $\varphi(x) = 1$ if $x \in F$,

(iii) $\overline{U}_i \subset \{x : \varphi(x) > 2^{-i}\} \subset U_{i+1}$ for every i.

Consider S^{n-1} as the surface of the solid n-ball I^n of radius 1 whose center is the origin p. We define $h \colon X \to I^n$ as follows:

(i) $h(x) = p$ if $x \notin \bigcup U_i$,

(ii) $h(x) = \varphi(x) g(x)$ if $x \in \bigcup U_i$, where $g(x)$ is considered as a vector from p to $g(x)$.

Then h is continuous and $h|F = f$. Moreover $h^{-1}(p) \cap (\bigcup A_i) = \varnothing$, which will be a meaningful fact later.

Third step. Here we reconsider that I^n is the n-cube expressed as

$$\{(x_1, \ldots, x_n) : -1 \leqq x_i \leqq 1, i = 1, \ldots, n\}$$

whose surface is S^{n-1} and whose origin $(0, \ldots, 0)$ is p. Consider the solid pyramid P in I^n whose base is $B=\{(x_1, \ldots, x_n) : x_n = -1\}$ and whose apex is p. The $n-1$ pairs of opposite sides of P may be denoted by (S_i, T_i), $i=1, \ldots, n-1$, where S_i is spanned by

$$S_i' = \{(x_1, \ldots, x_n) : x_i = x_n = -1\}$$

and p, and T_i is spanned by

$$T_i' = \{(x_1, \ldots, x_n) : x_i = 1, x_n = -1\}$$

and p. Then

$$C_i = h^{-1}(S_i) - h^{-1}(p), \quad C_i' = h^{-1}(T_i) - h^{-1}(p), \qquad i = 1, \ldots, n-1,$$

are $n-1$ pairs of disjoint closed sets of $X' = X - h^{-1}(p)$. Figure 1 will help us to treat the situation.

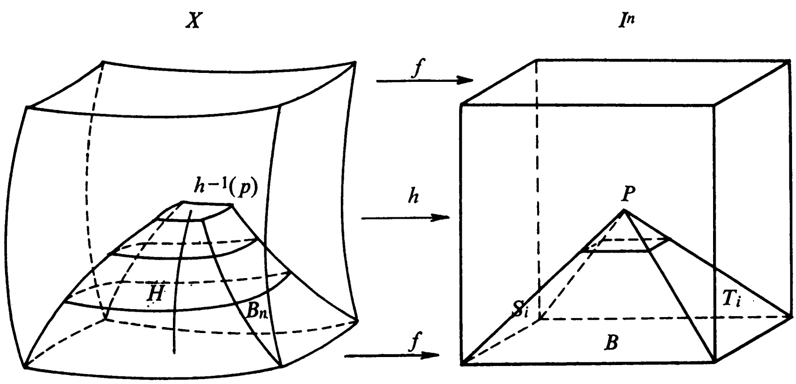

FIGURE 1

Fourth step. Assume that $\dim (X - \bigcup A_i) < n-1$. Then

$$\dim (X - ((\bigcup A_i) \cup h^{-1}(p))) < n-1,$$

since $h^{-1}(p)$ is a G_δ. By Lemma 1 there exist closed sets B_i of $X - h^{-1}(p)$, $i=1, \ldots, n-1$, such that (i) B_i separates C_i and C_i' for each i and (ii)

$$\left(\bigcap_{i=1}^{n-1} B_i\right) \cap (X - \bigcup A_i) = \varnothing.$$

The latter condition implies $H = \bigcap_{i=1}^{n-1} B_i \subset \bigcup A_i$. Let us consider the compact set $H \cup h^{-1}(p)$ and the two disjoint subsets $H \cap h^{-1}(B)$ and $h^{-1}(p)$. Suppose that $H \cap h^{-1}(B) \neq \varnothing$ and there exists a connected closed set $K \subset H \cup h^{-1}(p)$ such that

$K \cap H \cap h^{-1}(B) \neq \varnothing$ and $h^{-1}(p) \cap K \neq \varnothing$. Then for some i, $K \cap A_i \neq \varnothing$. Since $K \subset h^{-1}(p) \cup A_1 \cup A_2 \cup \cdots$, we have a contradiction by Lemma 3.

Fifth step. By Lemma 2 we can now conclude that there exist disjoint compact sets H_1 and H_2 such that (i) $H_1 \cup H_2 = h^{-1}(B) \cup H \cup h^{-1}(p)$, (ii) $h^{-1}(p) \subset H_1$ and (iii) $h^{-1}(B) \subset H_2$, whether $H \cap h^{-1}(B) = \varnothing$ or not. Hence there exists a closed set B_n of X separating $h^{-1}(p)$ and $h^{-1}(B)$ without touching H. Let c be a number with $0 < c < 1$, Q_c the intersection of P and the hyperplane $\{(x_1, \ldots, x_n) : x_n = -c\}$, P_c the intersection of P and $\{(x_1, \ldots, x_n) : x_n \leqq -c\}$ and R_c the surface of P_c. Then there exists a number b with $0 < b < 1$ such that

$$h^{-1}(\overline{P - P_b}) \cap B_n = \varnothing.$$

If we confine our attention to the set $h^{-1}(P_b)$, there are closed sets

$$B_1 \cap h^{-1}(P_b), \ldots, B_n \cap h^{-1}(P_b)$$

which separate pairs

$$h^{-1}(S_1 \cap P_b),\ h^{-1}(T_1 \cap P_b); \ldots; h^{-1}(S_{n-1} \cap P_b),\ h^{-1}(T_{n-1} \cap P_b);$$
$$h^{-1}(B),\ h^{-1}(Q_b),$$

respectively. Denote this system of pairs by α. Since

$$\bigcap_{i=1}^{n} (B_i \cap h^{-1}(P_b)) \subset \bigcap_{i=1}^{n} B_i = \varnothing,$$

α is not defining. Then by Lemma 4 there exists a mapping $k_1 : h^{-1}(P_b) \to R_b$ such that $k_1 | h^{-1}(R_b) = h | h^{-1}(R_b)$. Let $k : X \to I^n$ be a mapping such that

(i) $k | X - h^{-1}(P_b) = h | X - h^{-1}(P_b)$,

(ii) $k | h^{-1}(P_b) = k_1$.

Let s be an inner point of P_b and r a retraction of $I^n - \{s\}$ onto S^{n-1}. Then $rk : X \to S^{n-1}$ is an extension of f, a contradiction. Thus we have $\dim (X - \bigcup A_i) \geqq n-1$ and the proof is completed.

COROLLARY 1 (SITNIKOV [20]). *Let $n \geqq 1$. Let A_i, $i = 1, 2, \ldots$, be a disjoint sequence of closed sets of I^n at least two of which are not empty. Then*

$$\dim (I^n - \bigcup A_i) \geqq n-1.$$

Proof. Let S^{n-1} be the surface of $I^n = \{(x_1, \ldots, x_n) : -1 \leqq x_i \leqq 1, i = 1, \ldots, n\}$. Let $f : S^{n-1} \to S^{n-1}$ be the identity mapping. Since it is impossible that $I^n - S^{n-1}$ is contained in one A_i, we have one of the following two cases:

(i) There exists i such that $(I^n - S^{n-1}) - A_i \neq \varnothing$ and $A_j \subset S^{n-1}$ for any $j \neq i$.

(ii) There exist i and j with $i \neq j$ such that

$$(I^n - S^{n-1}) \cap A_i \neq \varnothing \quad \text{and} \quad (I^n - S^{n-1}) \cap A_j \neq \varnothing.$$

The first case yields $\dim(I^n - \bigcup A_i) = n$. If the second case happens, then there exists a number ε with $0 < \varepsilon < 1$ such that

$$I_\varepsilon^n = \{(x_1, \ldots, x_n) : |x_i| \leqq \varepsilon,\ i = 1, \ldots, n\}$$

meets A_i and A_j. Then by Lemma 3 there exists a point q in $I_\varepsilon^n - \bigcup A_i$. Then we can apply the same argument on f and q as in the proof of Theorem 1 and we get $\dim(I^n - \bigcup A_i) \geqq n-1$.

COROLLARY 2. *Let X be a connected metric space such that every point has a neighborhood homeomorphic to I^n, $n \geqq 1$. Let A_i, $i = 1, 2, \ldots$, be a disjoint sequence of closed sets of X at least two of which are not empty. Then*

$$\dim(X - \bigcup A_i) \geqq n-1.$$

Proof. Consider a closed covering $\{F_\alpha\}$ of X such that each F_α is homeomorphic to I^n. If each F_α is contained in some A_i, then each A_i has to be open, which contradicts the fact that X is connected. Hence (i) there exists F_α which meets at least two of the A_i's, (ii) there exists F_β such that $F_\beta \cap A_i \neq \varnothing$, $F_\beta - A_i \neq \varnothing$ and $F_\beta \cap A_j = \varnothing$ for $j \neq i$, or (iii) there exists F_γ such that $F_\gamma \cap A_i = \varnothing$ for every i. The first case yields $\dim(X - \bigcup A_i) \geqq \dim(F_\alpha - \bigcup A_i) \geqq n-1$. The second case yields $\dim(X - \bigcup A_i) \geqq \dim(F_\beta - A_i) \geqq n-1$. The third case yields $\dim(X - \bigcup A_i) \geqq \dim F_\gamma \geqq n-1$.

Figure 2 gives a Cantor 2-manifold X such that a proposition for X analogous to Corollary 1 fails. In fact $\dim X = 2$, yet $\dim(X - \bigcup A_{ij}) = 0$ since $X - \bigcup A_{ij}$ is a subset of the Cantor discontinuum.

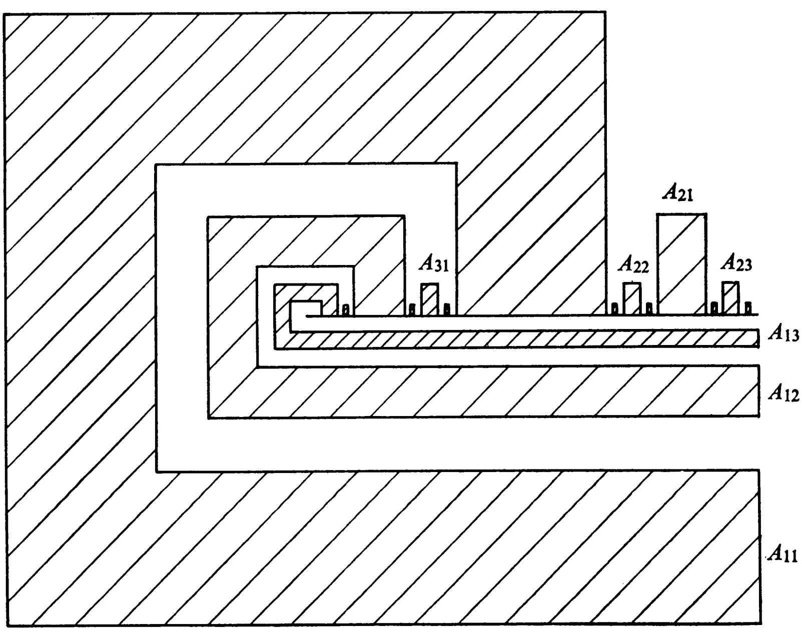

FIGURE 2

3. Relations among the various functions.

LEMMA 5 (MORITA [10, THEOREM 3.4]). *Let X be a normal space with* $\dim X \leqq n$. *Then X satisfies the condition* (D_4').

LEMMA 6. *If a metric space X has a σ-locally finite open base $\mathcal{U}$ such that* $\operatorname{ord}\{\bar{U}-U : U \in \mathcal{U}\} \leqq n$, *then* $\dim X \leqq n$. *($\mathcal{U}$ is called σ-locally finite if $\mathcal{U}$ can be decomposed into a countable number of locally finite subcollections.)*

This is a slight modification of Morita [12, Theorem 8.7].

THEOREM 2. $\dim X = d_4(X, \rho)$ *for any* (X, ρ).

Proof. By Lemma 5 $\dim X \geqq d_4(X, \rho)$. To prove $\dim X \leqq d_4(X, \rho)$ assume $d_4(X, \rho) \leqq n$. Let us show the existence of such $\mathcal{U}$ as in Lemma 6. By Stone [21] there exists an open base $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$ of X, $\mathcal{V}_i = \{V_\alpha : \alpha \in \Lambda_i\}$, $i = 1, 2, \ldots$, such that each $\mathcal{V}_i$ is discrete. $\mathcal{V}_i$ is called discrete if $\bar{\mathcal{V}}_i = \{\bar{V}_\alpha : \alpha \in \Lambda_i\}$ is a locally finite disjoint collection. Set $V_i = \bigcup \{V_\alpha : \alpha \in \Lambda_i\}$, $i = 1, 2, \ldots$, and

$$F_{ij} = \{x : \rho(x, X - V_i) \geqq 1/j\}, \qquad j = 1, 2, \ldots.$$

Then by $d_4(X, \rho) \leqq n$ there exist open sets U_{ij}, $i, j = 1, 2, \ldots$, such that

(i) $V_i \supset \bar{U}_{ij} \supset U_{ij} \supset F_{ij}$ for each i and j and

(ii) $\operatorname{ord}\{\bar{U}_{ij} - U_{ij} : i, j = 1, 2, \cdots\} \leqq n$.

Set

$$\mathcal{U}_{ij} = \{V_\alpha \cap U_{ij} : \alpha \in \Lambda_i\}.$$

Then $\mathcal{U}_{ij}$ is discrete and hence locally finite. $\mathcal{U} = \bigcup_{i,j=1}^{\infty} \mathcal{U}_{ij}$ is an open base for X such that $\operatorname{ord}\{\bar{U} - U : U \in \mathcal{U}\} \leqq n$. Hence by Lemma 6 we have $\dim X \leqq n$ and the theorem is proved.

LEMMA 7. *If $\{F_\alpha\}$ is a locally finite closed collection of a paracompact Hausdorff space, then there exists an open collection $\{G_\alpha\}$ such that* (i) $G_\alpha \supset F_\alpha$ *for each α and* (ii) $\operatorname{ord}\{F_\alpha\} = \operatorname{ord}\{G_\alpha\}$.

This can easily be seen with the aid of Morita [11, Theorem 1.3].

THEOREM 3 (KATĚTOV [8]). $\dim X \leqq 2\mu \dim (X, \rho)$ *for any* (X, ρ).

Proof. Suppose $\mu \dim (X, \rho) \leqq n$. Let $\mathcal{U}_i = \{U_\alpha : \alpha \in \Lambda_i\}$, $i = 1, 2, \ldots$, be a sequence of open coverings of X such that

(i) mesh $\mathcal{U}_i < 2^{-i}$ for each i and

(ii) $\operatorname{ord} \mathcal{U}_i \leqq n+1$ for each i.

By an easy observation we can assume that each $\mathcal{U}_i$ is locally finite. Let $\mathcal{G} = \{G_1, \ldots, G_m\}$ be an arbitrary finite open covering of X. Set

$$D_i = \bigcup \{\{x : \rho(x, X - G_j) > 2^{-i+1}\} : j = 1, \ldots, m\}, \qquad i = 1, 2, \ldots.$$

Then (i) $D_1 \subset \bar{D}_1 \subset D_2 \subset \bar{D}_2 \subset D_3 \cdots$, (ii) each D_i is open and (iii) $\bigcup_{i=1}^{\infty} D_i = X$. Let

$\mathscr{F}_i=\{F_\alpha : \alpha \in \Lambda_i\}$ be a closed covering of X such that $F_\alpha \subset U_\alpha$ for each $\alpha \in \Lambda_i$. Set

$$\Lambda_i' = \{\alpha : F_\alpha \cap \bar{D}_i = \varnothing\},$$
$$W_i = X - \bigcup \{F_\alpha : \alpha \in \Lambda_i'\}.$$

Then W_i is an open set with $\bar{D}_i \subset W_i \subset \bar{W}_i \subset D_{i+1}$ for every i. Since every point of $\bar{W}_i - W_i$ is contained in some $F_\alpha \in \mathscr{F}_i$ with $\alpha \in \Lambda_i'$, ord $\{F_\alpha \cap (\bar{W}_i - W_i) : \alpha \notin \Lambda_i'\} \leqq n$. (This type of argument comes from Morita [10, Theorem 3.3].) Since $\{F_\alpha : \alpha \in \Lambda_i - \Lambda_i'\}$ covers $\bar{W}_i - W_i$, there exists by Lemma 7 an open collection $\mathscr{V}_i=\{V_\alpha : \alpha \in \Lambda_i - \Lambda_i'\}$ of X such that

(i) $F_\alpha \cap (\bar{W}_i - W_i) \subset V_\alpha \subset (D_{i+1} - \bar{D}_i) \cap U_\alpha$ for each $\alpha \in \Lambda_i - \Lambda_i'$ and

(ii) ord $\{V_\alpha : \alpha \in \Lambda_i - \Lambda_i'\} \leqq n$.

Then $\mathscr{V}_i$ refines $\mathscr{G}$. Moreover $V \in \mathscr{V}_i$, $V' \in \mathscr{V}_j$, $i \neq j$, imply $V \cap V' = \varnothing$. We set

$$\mathscr{H}_i = \{U_\alpha \cap (W_i - \bar{W}_{i-1}) : \alpha \in \Lambda_i\}, \quad \text{where } W_{-1} = \varnothing.$$

Then $H \in \mathscr{H}_i$, $H' \in \mathscr{H}_j$, $i \neq j$, imply $H \cap H' = \varnothing$. If we set

$$\mathscr{H} = \left(\bigcup_{i=1}^{\infty} \mathscr{H}_i\right) \cup \left(\bigcup_{i=1}^{\infty} \mathscr{V}_i\right),$$

then it is easy to see that $\mathscr{H}$ is an open covering of X such that (i) $\mathscr{H}$ refines $\mathscr{G}$ and (ii) ord $\mathscr{H} \leqq 2n+1$. Thus we have dim $X \leqq 2n$ and the theorem is proved.

THEOREM 4. *$d_3(X, \rho) \leqq \mu \dim (X, \rho)$ for any (X, ρ).*

Proof. Suppose $\mu \dim (X, \rho) \leqq n$. Let $C_1, C_1'; \cdots; C_m, C_m'$ be a finite number of pairs of closed sets of X such that there exists a positive number ε such that $\rho(C_i, C_i') > \varepsilon$ for each i. By $\mu \dim (X, \rho) \leqq n$ there exists a locally finite open covering $\mathscr{U}=\{U_\alpha : \alpha \in \Lambda\}$ of X such that (i) ord $\mathscr{U} \leqq n+1$ and (ii) mesh $\mathscr{U} < \varepsilon/2$. Let $\mathscr{F}=\{F_\alpha : \alpha \in \Lambda\}$ be a locally finite closed covering of X such that $F_\alpha \subset U_\alpha$ for each $\alpha \in \Lambda$. Set

$$\Lambda' = \{\alpha : F_\alpha \cap C_1 = \varnothing\},$$
$$W = X - \bigcup \{F_\alpha : \alpha \in \Lambda'\},$$
$$B_1 = \bar{W} - W.$$

Then B_1 separates C_1 and C_1' and

$$\mathscr{U}_1 = \{U_\alpha : \alpha \in \Lambda - \Lambda'\} \cup \{U_\alpha - B_1 : \alpha \in \Lambda'\}$$

is an open covering of X such that

(i) $\mathscr{U}_1$ refines $\mathscr{U}$,

(ii) $\mathscr{U} | X - B_1 = \mathscr{U}_1 | X - B_1$,

(iii) ord $(x, \mathscr{U}_1) \leqq$ ord $(x, \mathscr{U}) - 1$ for each $x \in B_1$, where ord $(x, \mathscr{U}_1)$ is the order of $\mathscr{U}_1$ at x.

This can be verified by the same argument as in the proof of the previous theorem. Continuing this procedure, we get closed sets B_i, $i=2,\dots,m$, separating C_i and C_i' respectively and open coverings $\mathscr{U}_2,\dots,\mathscr{U}_m$ such that

(i) $\mathscr{U}_{i+1}$ refines $\mathscr{U}_i$ for $i=1,\dots,m-1$,

(ii) $\mathscr{U}_i|X-B^i=\mathscr{U}_{i-1}|X-B_i$ for each i,

(iii) $\operatorname{ord}(x,\mathscr{U}_i)\leqq\operatorname{ord}(x,\mathscr{U}_{i-1})-1$ for each $x\in B_i$, $i=2,\dots,m$.

If $x\in B_{i_1}\cap\cdots\cap B_{i_{n+1}}$, $i_1<i_2<\cdots<i_{n+1}$, then

$$\begin{aligned}1 &\leqq \operatorname{ord}(x,\mathscr{U}_{i_{n+1}}) \leqq \operatorname{ord}(x,\mathscr{U}_{i_{n+1}-1})-1\\ &\leqq \operatorname{ord}(x,\mathscr{U}_{i_n})-1 \leqq \operatorname{ord}(x,\mathscr{U}_{i_n-1})-2\\ &\leqq\cdots\leqq \operatorname{ord}(x,\mathscr{U}_{i_1})-n \leqq \operatorname{ord}(x,\mathscr{U})-(n+1).\end{aligned}$$

Thus we have $\operatorname{ord}(x,\mathscr{U})\geqq n+2$, a contradiction. We have therefore

$$\operatorname{ord}\{B_i : i = 1,\dots,m\} \leqq n,$$

and the theorem is proved.

THEOREM 5. $d_3(X,\rho)=\mu\dim(X,\rho)$ *for a totally bounded metric space* (X,ρ).

Proof. Suppose that $d_3(X,\rho)\leqq n$. For an arbitrary positive number ε there exists a finite set of points $x_1,\dots,x_m$ such that

$$\{U_i = S_\varepsilon(x_i) : i = 1,\dots,m\}$$

covers X. Set

$$V_i = S_{2\varepsilon}(x_i), \qquad i = 1,\dots,m.$$

Then there exist open sets $W_1,\dots,W_m$ such that

(i) $\overline{U}_i\subset W_i\subset\overline{W}_i\subset V_i$ for each i,

(ii) $B_i=\overline{W}_i-W_i$ separates $X-V_i$ and $\overline{U}_i$ for each i,

(iii) $\operatorname{ord}\{B_i : i=1,\dots,m\}\leqq n$.

By Lemma 7 there exist open sets $G_1,\dots,G_m$ of X such that (i) $B_i\subset G_i\subset V_i$ for each i and (ii) $\operatorname{ord}\{G_i\}\leqq n$. Set

$$\begin{aligned}\mathscr{W}_i &= \{W_{i1} = W_i,\ W_{i2} = X-\overline{W}_i\},\\ \mathscr{W} &= \bigwedge_{i=1}^m \mathscr{W}_i = \{W_{1i_1}\cap\cdots\cap W_{mi_m} : i_1,\dots,i_m = 1, 2\}.\end{aligned}$$

Since

$$\bigcap_{i=1}^m W_{i2} = \bigcap_{i=1}^m (X-\overline{W}_i) = X-\bigcup_{i=1}^m \overline{W}_i = X-X = \varnothing,$$

$\mathscr{W}$ refines $\{W_1,\dots,W_m\}$ and hence $\mathscr{W}$ refines $\{V_1,\dots,V_m\}$. Moreover

$$\bigcup\{W : W\in\mathscr{W}\} = X-\bigcup_{i=1}^m B_i$$

and $\operatorname{ord}\mathscr{W}\leqq 1$. If we set

$$\mathscr{G} = \mathscr{W}\cup\{G_1,\dots,G_m\},$$

then $\mathscr{G}$ is an open covering of X such that (i) mesh $\mathscr{G} \leqq 4\varepsilon$ and (ii) ord $\mathscr{G} \leqq n+1$. Thus we have μ dim $(X, \rho) \leqq n$. If we combine $d_3(X, \rho) \geqq \mu$ dim (X, ρ) just proved with Theorem 4, we get $d_3(X, \rho) = \mu$ dim (X, ρ) and the theorem is proved.

Since $d_2(X, \rho) \leqq d_3(X, \rho) \leqq d_4(X, \rho)$ are trivially true, we have now $d_2 \leqq d_3 \leqq \mu$ dim $\leqq d_4 =$ dim $\leqq 2\mu$ dim, symbolically.

Remark 1. When (X, ρ) is locally compact, all of these dimension functions coincide with each other, of course. On the other hand there exists a space (X, ρ) which is not locally compact at any point yet $d_2(X, \rho) = d_3(X, \rho) = \mu$ dim (X, ρ) $=$ dim X. Let us consider $I^3 = \{(x_1, x_2, x_3) : 0 \leqq x_i \leqq 1, i = 1, 2, 3\}$. Let ρ be a metric of I^3. Set $B = \{(x_1, x_2, x_3) : x_1 = 0\}$. Let C be the set of all points in I^3 whose coordinates are all rational. If we set $X = B \cup C$, then (X, ρ) satisfies the condition as follows: $2 = d_2(B, \rho) \leqq d_2(X, \rho) \leqq d_3(X, \rho) \leqq \mu$ dim $(X, \rho) \leqq$ dim $X = 2$.

Remark 2. It is to be noticed that even if (X, ρ) is any metric space, (i) dim $X = 1$ implies $d_2(X, \rho) = 1$ and (ii) $d_2(X, \rho) = 0$ implies dim $X = 0$ (see Nagami-Roberts [15]).

4. Spaces (X_n, ρ) with $d_2 = [n/2]$ and dim $\geqq n-1$.

Lemma 8. *Let $F_1, F_2, \ldots$ be a sequence of closed sets of a metric space X with* dim $F_i = n_i$. *Let $C_1, C_1'; \ldots; C_m, C_m'$ be m disjoint pairs of closed sets of X. Then there exist closed sets $B_1, \ldots, B_m$ such that*

(i) *for each i B_i separates C_i and C_i',*

(ii) *for each j and for each sequence* $1 \leqq i_1 < i_2 < \cdots < i_t \leqq \min\{n+1, m\}$,

$$\dim (B_{i_1} \cap \cdots \cap B_{i_t} \cap F_j) \leqq n_j - t.$$

See Morita [12, Theorem 9.1].

Construction of (X_n, ρ). Let (K_n, ρ) be a Cantor n-manifold with $n \geqq 3$. Put $m = [n/2] + 1$. By compactness of K_n there exists a sequence of m disjoint pairs of closed sets of K_n, say $C_{11}, C_{11}'; \ldots; C_{1m}, C_{1m}' : C_{21}, C_{21}'; \ldots; C_{2m}, C_{2m}' : \ldots$, such that for any m disjoint pairs of closed sets $C_1, C_1'; \ldots; C_m, C_m'$ there exists i with $C_j \subset C_{ij}$ and $C_j' \subset C_{ij}'$ for $j = 1, \ldots, m$. By Lemma 8 there exist closed sets $B_{11}, \ldots, B_{1m}$ such that

(i) for each i B_{1i} separates C_{1i} and C_{1i}',

(ii) dim $B_1 \leqq n-m$ where $B_1 = \bigcap_{j=1}^m B_{1j}$.

By repeated application of Lemma 8 there exist closed sets $B_{21}, \ldots, B_{2m}$ such that

(i) for each i B_{2i} separates C_{2i} and C_{2i}',

(ii) dim $B_2 \leqq n-m$ where $B_2 = \bigcap_{j=1}^m B_{2j}$,

(iii) $B_1 \cap B_2 = \varnothing$.

Continuing such process we get finally a sequence of closed sets B_{ij}, $i = 1, 2, \ldots$, $j = 1, \ldots, m$, which have the following property:

(i) B_{ij} separates C_{ij} and C_{ij}' for each i and j.

(ii) dim $B_i \leqq n-m$ for $i = 1, 2, \ldots$, where $B_i = \bigcap_{j=1}^m B_{ij}$.

(iii) $B_i \cap B_j = \varnothing$ if $i \neq j$.

If we set

$$X_n = K_n - \bigcup_{i=1}^{\infty} B_i,$$

then we have the space (X_n, ρ).

ASSERTION 1. $d_2(X_n, \rho) \leqq [n/2]$.

Proof. Let $C_1, C_1'; \ldots; C_m, C_m'$ be m pairs of closed sets of X_n such that $\rho(C_i, C_i') > 0$ for $i = 1, \ldots, m$. Since their closures $\bar{C}_1, \bar{C}_1'; \ldots; \bar{C}_m, \bar{C}_m'$ in K_n constitute m disjoint pairs of closed sets of K_n, there exists i such that $\bar{C}_j \subset C_{ij}$ and $\bar{C}_j' \subset C_{ij}'$ for $j = 1, \ldots, m$. Then $B_{i1} \cap X_n, \ldots, B_{im} \cap X_n$ are closed sets of X_n such that $B_{ij} \cap X_n$ separates C_j and C_j' for $j = 1, \ldots, m$.

$$\bigcap_{j=1}^{m} (B_{ij} \cap X_n) = B_i \cap X_n = \varnothing,$$

and hence we have $d_2(X_n, \rho) \leqq m - 1 = [n/2]$.

ASSERTION 2. $d_2(X_n, \rho) \geqq [n/2]$.

Proof. If G is a nonempty open set of K_n, then $\dim G = n$. Since

$$\dim \left(\bigcup B_i\right) \leqq n - m < n, \qquad G - \left(\bigcup B_i\right) \neq \varnothing$$

and hence $G \cap X_n \neq \varnothing$. Thus X_n is dense in K_n. Assume $d_2(X_n, \rho) = t < [n/2]$. Take a defining system $\bar{D}_1, \bar{D}_1'; \ldots; \bar{D}_{t+1}, \bar{D}_{t+1}'$ of K_n such that

(i) each D_i and D_i' are open in K_n,

(ii) for any closed sets A_i, $i = 1, \ldots, t+1$, separating $\bar{D}_i$ and $\bar{D}_i'$,

$$\dim \left(\bigcap_{i=1}^{t+1} A_i\right) \geqq n - (t+1).$$

Set $C_i = \bar{D}_i \cap X_n$ and $C_i' = \bar{D}_i' \cap X_n$. Then it is easy to see that $\bar{C}_i = \bar{D}_i$ and $\bar{C}_i' = \bar{D}_i'$, since X_n is dense in K_n. Set

$$\varepsilon = \min \{\rho(C_i, C_i') : i = 1, \ldots, t+1\}.$$

Take open sets U_i, $i = 1, \ldots, t+1$, of K_n such that

(i)
$$\{x : x \in X_n, \rho(x, C_i) < \varepsilon/4\} \subset U_i \cap X_n \subset \bar{U}_i \cap X_n$$
$$\subset X_n - \{x : x \in X_n, \rho(x, C_i') < \varepsilon/4\} \quad \text{for each } i,$$

(ii)
$$\bigcap_{i=1}^{t+1} ((\bar{U}_i - U_i) \cap X_n) = \varnothing.$$

It is easy to see that $\bar{U}_i - U_i$ thus chosen separates $\bar{C}_i = \bar{D}_i$ and $\bar{C}_i' = \bar{D}_i'$ for each i. Hence

$$\dim B \geqq n - (t+1) \geqq n - [n/2] \quad \text{where} \quad B = \bigcap_{i=1}^{t+1} (\bar{U}_i - U_i).$$

On the other hand

$$\dim B \leqq \dim (\bigcup B_i) \leqq n-m = n-[n/2]-1$$

because $B \cap X_n = \varnothing$, which is a contradiction.

ASSERTION 3. $\dim X_n \geqq n-1$.

Proof. Since $\dim B_i \leqq n-m = n-[n/2]-1 \leqq n-1$, we have $\dim X_n \geqq n-1$ at once by Theorem 1.

5. **Spaces (Y_n, ρ) with μ dim $=[n/2]$ and dim $\geqq n-1$.**

LEMMA 9. *(X, ρ) has $\mu \dim (X, \rho) \leqq n$ if and only if there exists a sequence of locally finite closed coverings $\mathscr{F}_i$, $i=1, 2, \ldots$, such that*

(i) mesh $\mathscr{F}_i < 1/i$ *for any i,*

(ii) ord $\mathscr{F}_i \leqq n+1$ *for any i.*

This is verified at once by Lemma 7.

LEMMA 10. *Let X be a metric space with* $\dim X \leqq n$ *and $B_1, B_2, \ldots$ a sequence of closed sets of X with* $\dim B_i = n_i$, *where $B_1 = X$. Let ε be an arbitrary positive number. Then there exists a locally finite closed covering $\mathscr{F} = \{F_\alpha : \alpha \in \Lambda\}$ which satisfies the following conditions*:

(i) mesh $\mathscr{F} < \varepsilon$.

(ii) *For any i* ord $\mathscr{F} | B_i \leqq n_i + 1$.

(iii) *For any i, any $j \leqq n_i + 2$ and any j different indices $\alpha(1), \ldots, \alpha(j)$ of Λ,*

$$\dim \bigcap_{k=1}^{j} (F_{\alpha(k)} \cap B_i) \leqq n_i - j + 1.$$

This is proved essentially in Nagami [13, Theorem 3.6].

CONSTRUCTION OF (Y_n, ρ). Let (K_n, ρ) be a Cantor n-manifold, $n \geqq 3$. Set $m = [n/2] + 2$. By Lemma 10 there exists a locally finite closed covering $\mathscr{F}_1 = \{F_\alpha : \alpha \in \Lambda_1\}$ of K_n such that (i) mesh $\mathscr{F}_1 < 1$, (ii) ord $\mathscr{F}_1 \leqq n+1$, and (iii) $\dim B_1 \leqq n-m+1$ where $B_1 = \{x : \text{ord}\,(x, \mathscr{F}_1) \geqq m\}$ which is closed by the local finiteness of $\mathscr{F}_1$. Then ord $\mathscr{F}_1 | K_n - B_1 < m$.

By Lemma 10 again there exists a locally finite closed covering

$$\mathscr{F}_2 = \{F_\alpha : \alpha \in \Lambda_2\}$$

of K_n such that (i) mesh $\mathscr{F}_2 < 1/2$, (ii) ord $\mathscr{F}_2 \leqq n+1$, (iii) $\dim B_2 \leqq n-m+1$ where $B_2 = \{x : \text{ord}\,(x, \mathscr{F}_2) \geqq m\}$, and (iv) $\dim \bigcap_{k=1}^{j} (F_{\alpha(k)} \cap B_1) \leqq \dim B_1 - j + 1$ for any $j \leqq \dim B_1 + 2$ and any j different indices $\alpha(1), \ldots, \alpha(j)$ of Λ_2. To show that the last condition (iv) implies $B_1 \cap B_2 = \varnothing$, set $\dim B_1 = n_1$. Take n_1+2 different indices $\alpha(1), \ldots, \alpha(n_1+2)$ of Λ_2. Then

$$\dim \bigcap_{k=1}^{n_1+2} (F_{\alpha(k)} \cap B_1) \leqq n_1 - (n_1+2) + 1 = -1.$$

Hence we have

$$B_1 \cap \{x : \text{ord}\,(x, \mathscr{F}_2) \geqq n_1 + 2\} = \varnothing.$$

Since

$$\begin{aligned} n_1+2 &\leqq (n-m+1)+2 \leqq n-([n/2]+2)+3 \\ &= n-[n/2]+1 \leqq (2[n/2]+1)-[n/2]+1 \\ &= [n/2]+2 = m, \end{aligned}$$

we have $B_1 \cap B_2 = \varnothing$.

Repeating such procedure we have a sequence of locally finite closed coverings $\mathscr{F}_i$, $i=1, 2, \ldots$, which satisfy the following conditions:

(i) For each i, mesh $\mathscr{F}_i < 1/i$.

(ii) For each i, dim $B_i \leqq n-m+1$ where B_i is a closed set defined by

$$B_i = \{x : \operatorname{ord}(x, \mathscr{F}_i) \geqq m\}.$$

(iii) B_i, $i=1, 2, \ldots$, are mutually disjoint.

We set $Y_n = K_n - \bigcup B_i$. Then (Y_n, ρ) is the desired space.

ASSERTION 1. dim $Y_n \geqq n-1$.

Proof. Since

$$\dim B_i \leqq n-m+1 = n-[n/2]-1 \leqq n-1,$$

the assertion is true by Theorem 1.

ASSERTION 2. $\mu \dim (Y_n, \rho) \leqq [n/2]$.

Proof. Since $\operatorname{ord} \mathscr{F}_i | Y_n \leqq \operatorname{ord} \mathscr{F}_i | K_n - B_i \leqq m-1 = ([n/2]+2)-1 = [n/2]+1$, the assertion is true by Lemma 9.

ASSERTION 3. dim $Y_n \leqq n-1$ when n is odd.

Proof. Since dim $Y_n \leqq 2\mu \dim (Y_n, \rho)$ by Theorem 3, we have

$$\dim Y_n \leqq 2[n/2] = 2((n-1)/2) = n-1.$$

ASSERTION 4. $\mu \dim (Y_n, \rho) \geqq [n/2]$.

Proof. Assume the contrary. Then

$$\dim Y_n \leqq 2\mu \dim (Y_n, \rho) \leqq 2([n/2]-1) \leqq n-2,$$

a contradiction.

Thus (Y_n, ρ) satisfies (i) dim $Y_n \geqq n-1$ and (ii) $\mu \dim (Y_n, \rho) = [n/2]$. Furthermore when n is odd, dim $Y_n = n-1$.

REMARK 3. It is to be noted that for X_n and Y_n obtained by replacing K_n with I^n, dim X_n = dim $Y_n = n-1$ for any n, because of the fact that $I^n - X_n$ and $I^n - Y_n$ are dense in I^n, and the invariance theorem of domain.

REMARK 4. Note that the existence of a sequence of open coverings $\mathscr{U}_i$, $i=1, 2, \ldots$, with ord $\mathscr{U}_i \leqq n+1$ and lim mesh $\mathscr{U}_i = 0$ does not characterize dimension. Thus it is natural to seek an additional condition upon $\mathscr{U}_i$ with which the existence of the sequence does characterize dimension. Dowker-Hurewicz [2], Nagata [17] and Nagami [14] considered such a condition. This type of characterization theorem is one of the main foundations on which modern dimension theory has been built up. Vopěnka [22] gave a simple condition: "$\mathscr{U}_{i+1} <$ (refines) $\mathscr{U}_i$ for each i". Recently Nagami-Roberts [16, Theorem 3] refined Vopěnka's theorem, weakening the mesh condition. But our proof contains an error. The definition of V_α in [16, line 15, p. 157] is not adequate. Let us take this opportunity to give a correct proof as follows:

THEOREM 6. *A metric space X has* $\dim X \leqq n$ *if there exists a sequence $\mathcal{U}_1 > \mathcal{U}_2 > \cdots$ of open coverings $\mathcal{U}_i$ of X such that*

(i) *for each $x \in X$, $\{\mathrm{St}(x, \mathcal{U}_i^\Delta) : i=1,2,\dots\}$ is a local base of x,*

(ii) $\operatorname{ord} \mathcal{U}_i \leqq n+1$.

Proof. Set

$$\mathcal{U}_i = \{U(\alpha_i) : \alpha_i \in A_i\}, \qquad i = 1, 2, \dots.$$

Let $f_i^{i+1} : A_{i+1} \to A_i$ be a function such that $f_i^{i+1}(\alpha_{i+1}) = \alpha_i$ yields $U(\alpha_{i+1}) \subset U(\alpha_i)$. For each pair $i<j$ let $f_i^j = f_i^{i+1} \cdots f_{j-1}^j$ and f_i^i be the identity mapping. Let $\mathcal{G}$ be an arbitrary finite open covering of X. Set

$$X_i = \bigcup \{U(\alpha_i) : \mathrm{St}(U(\alpha_i), \mathcal{U}_i) \text{ refines } \mathcal{G}\}.$$

Then by the condition (i) $\{X_1, X_2, \dots\}$ is an open covering of X. Set $X_0 = \varnothing$. Set

$$B_i = \{\alpha_i : U(\alpha_i) \cap X_i \neq \varnothing\},$$

$$C_i = \left\{\alpha_i : \alpha_i \in B_i,\ U(\alpha_i) \cap \left(\bigcup_{j<i} X_j\right) = \varnothing\right\},$$

$$D_i = \left\{\alpha_i : \alpha_i \in B_i,\ U(\alpha_i) \cap \left(\bigcup_{j<i} X_j\right) \neq \varnothing\right\}.$$

Then $B_i \subset A_i$, $B_1 = C_1$, $B_i = C_i \cup D_i$ and $C_i \cap D_i = \varnothing$.

For every $i<j$ and every $\alpha_i \in C_i$ set

$$D_j(\alpha_i) = \left(\bigcap_{k=i+1}^{j} (f_k^j)^{-1}(D_k)\right) \cap (f_i^j)^{-1}(\alpha_i).$$

Then

(i) $f_k^j(D_j(\alpha_i)) \subset D_k(\alpha_i)$, $i < k \leqq j$,

(ii) $D_j = \bigcup \{D_j(\alpha_i) : \alpha_i \in C_i,\ i<j\}$.

For every $\alpha_i \in C_i$ let

$$V(\alpha_i) = (U(\alpha_i) \cap X_i) \cup \left(\bigcup \{U(\alpha_j) \cap X_j : \alpha_j \in D_j(\alpha_i),\ i<j\}\right).$$

Let us show that

$$\mathcal{V} = \{V(\alpha_i) : \alpha_i \in C_i,\ i = 1, 2, \dots\}$$

is an open covering of X such that $\mathcal{V}$ refines $\mathcal{G}$ and $\operatorname{ord} \mathcal{V} \leqq n+1$, which will prove $\dim X \leqq n$.

Let x be an arbitrary point of X. Since $X_0 = \varnothing$, there exists i with $x \in X_i - \bigcup_{j<i} X_j$. Take $\alpha_i \in B_i$ with $x \in U(\alpha_i)$. When $\alpha_i \in C_i$, $x \in U(\alpha_i) \cap X_i \subset V(\alpha_i)$. When $\alpha_i \in D_i$, there exist $j<i$ and $\alpha_j \in C_j$ such that $\alpha_i \in D_i(\alpha_j)$. Then $x \in U(\alpha_i) \cap X_i \subset V(\alpha_j)$. Thus $\mathcal{V}$ is an open covering of X.

Let i be an arbitrary positive integer and α_i an arbitrary index in C_i. Since $\varnothing \neq U(\alpha_i) \cap X_i \subset V(\alpha_i) \subset U(\alpha_i)$, there exists $\beta_i \in A_i$ such that $U(\beta_i) \cap U(\alpha_i) \cap X_i \neq \varnothing$ and St $(U(\beta_i), \mathscr{U}_i)$ refines $\mathscr{G}$. Thus $V(\alpha_i)$ refines $\mathscr{G}$ and hence $\mathscr{V}$ refines $\mathscr{G}$.

To prove ord $\mathscr{V} \leqq n+1$ assume the contrary. Then there exist a point x and $n+2$ indices $\alpha^1, \ldots, \alpha^{n+2}$ such that

(i) $\alpha^i \in C_{m_i}$, $i=1, \ldots, n+2$,

(ii) $x \in V(\alpha^i)$, $i=1, \ldots, n+2$.

Let k be the smallest integer such that $x \in X_k - \bigcup_{j<k} X_j$. Every m_i is less than or equal to k. For every α^i there exist $j(i)$ with $j(i) \geqq k$ and $\beta^i \in D_{j(i)}(\alpha^i)$ such that $x \in U(\beta^i)$. Set $\gamma^i = f_k^{j(i)}(\beta^i)$. Then (i) $x \in U(\gamma^i)$, (ii) $\gamma^i \in D_k(\alpha^i)$ if $m_i < k$ and (iii) $\gamma^i = \alpha^i$ if $m_i = k$. Since γ^i, $i=1, \ldots, n+2$, are all different from one another by our construction, ord $(x, \mathscr{U}_k) \geqq n+2$, a contradiction. Hence ord $\mathscr{V} \leqq n+1$ and the proof is finished.

6. Spaces (Z_n, σ_i) illustrating the dependence of μ dim and d_2 on the metric.

LEMMA 11. *If (X, ρ) is a metric space with* dim $X=n$, *then there exists an equivalent metric ρ' to ρ such that $d_2(X, \rho')=n$.*

Proof. Since dim $X=n$, there exists a defining system of n pairs $C_1, C_1'; \cdots; C_n, C_n'$. Let $f_1, \ldots, f_n$ be real-valued mappings of X such that

(i) $0 \leqq f_i(x) \leqq 1$ for any i and any $x \in X$,

(ii) $f_i(x)=0$ for any i and any $x \in C_i$,

(iii) $f_i(x)=1$ for any i and any $x \in C_i'$.

Set

$$\rho'(x, y) = \rho(x, y) + \sum_{i=1}^{n} |f_i(x) - f_i(y)|.$$

Then ρ' is an equivalent metric to ρ and $\rho'(C_i, C_i') > 0$ for each i. Thus we have $d_2(X, \rho') \geqq n$ and hence $d_2(X, \rho')=n$.

CONSTRUCTION OF Z_n, $n \geqq 2$. Set $m=[(n+1)/2]+1$. In every (I^i, ρ_i), $i=m, m+1, \ldots, n+1$, we construct (Y_i, ρ_i) as in the preceding section. Then μ dim $(Y_i, \rho_i) \leqq [i/2] \leqq [(n+1)/2]$ and dim $Y_i = i-1$ for $i=m, \ldots, n+1$. We assume here that $\rho_i(I^i) \leqq 1$ for $i=m, \ldots, n+1$. Take a metric ρ_i' equivalent to ρ_i as in Lemma 11 such that $d_2(Y_i, \rho_i')=i-1$. Then μ dim $(Y_i, \rho_i') = d_3(Y_i, \rho_i') = i-1$ are automatically true for $i=m, \ldots, n+1$. By the construction of ρ_i' in Lemma 11 ρ_i' satisfies $\rho_i'(Y_i) \leqq i+1$.

Z_n is merely the disjoint sum of $Y_m, Y_{m+1}, \ldots, Y_{n+1}$. The topology of Z_n is defined in such a way that a subset G of Z_n is open if and only if $G \cap Y_i$ is open in Y_i for $i=m, \ldots, n+1$. Then Z_n is a metric space. Define for $i=m, \ldots, n+1$ the metrics σ_i of Z_n as follows:

(i) $\sigma_i | Y_j = \rho_j$ if $i \neq j$.

(ii) $\sigma_i | Y_i = \rho_i'$.

(iii) $\sigma_i(x, y) = n+2$ if for any $j=m, \ldots, n+1$, x and y are not in the same Y_j.

$\sigma_m, \ldots, \sigma_{n+1}$ are equivalent metrics which give the preassigned topology of Z_n.

ASSERTION 1. $\dim Z_n = n$.

Proof. $\dim Z_n = \max\{\dim Y_i : i = m, \ldots, n+1\} = n$.

ASSERTION 2. $\mu \dim (Z_n, \sigma_i) = d_2(Z_n, \sigma_i) = d_3(Z_n, \sigma_i) = i-1$ for $i = [(n+1)/2]+1, \ldots, n+1$.

Proof. If $j \neq i$, then $\mu \dim (Y_j, \sigma_i) = \mu \dim (Y_j, \rho_j) \leqq [(n+1)/2]$. Since

$$\mu \dim (Y_i, \sigma_i) = d_2(Y_i, \sigma_i) = d_3(Y_i, \sigma_i) = \mu \dim (Y_i, \rho_i') = i-1 \geqq [(n+1)/2],$$

we have

$$\begin{aligned} i-1 &= d_2(Z_n, \sigma_i) \leqq d_3(Z_n, \sigma_i) \leqq \mu \dim (Z_n, \sigma_i) \\ &= \max\{\mu \dim (Y_m, \rho_m), \ldots, \mu \dim (Y_{i-1}, \rho_{i-1}), \mu \dim (Y_i, \rho_i'), \\ &\qquad \mu \dim (Y_{i+1}, \rho_{i+1}), \ldots, \mu \dim (Y_{n+1}, \rho_{n+1})\} = i-1. \end{aligned}$$

Thus the assertion is proved.

7. **A space (R, ρ) with $d_2 = 2$, $\mu \dim = 3$, $\dim = 4$.**

First let us construct a space (S, σ) with $d_2(S, \sigma) = 2$ and $\mu \dim (S, \sigma) = \dim S = 3$.

CONSTRUCTION OF (S, σ). (S, σ) will be a subset of

$$(I^4 = \{(x_1, \ldots, x_4) : 0 \leqq x_i \leqq 1, i = 1, \ldots, 4\}, \sigma),$$

where σ is Euclidean metric on I^4. Let C_{ij}, C_{ij}', $i = 1, 2, \ldots, j = 1, 2, 3$, be disjoint pairs of closed sets of I^4 such that for any three disjoint pairs of closed sets $C_1, C_1'; C_2, C_2'; C_3, C_3'$, there exists i with $C_j \subset C_{ij}$ and $C_j' \subset C_{ij}'$ for $j = 1, 2, 3$. Let π be a prime number with $5 \leqq \pi$. Consider an open covering $\mathscr{D}(\pi)$ of the unit interval $[0, 1]$ consisting of overlapping intervals $[0, 2/\pi)$, $((\pi-2)/\pi, 1]$ and $((2k-1)/\pi, (2k+2)/\pi)$, $k = 1, \ldots, (\pi-3)/2$. Define an open covering $\mathscr{E}(\pi)$ of I^4 as follows:

$$\begin{aligned} \mathscr{E}(\pi) &= \{D_1 \times D_2 \times D_3 \times D_4 : D_1, \ldots, D_4 \in \mathscr{D}(\pi)\} \\ &= \{E_\lambda : \lambda \in \Lambda(\pi)\}. \end{aligned}$$

Let π_{ij}, $i = 1, 2, \ldots, j = 1, 2, 3$, be prime numbers which are different from each other and satisfy the following conditions:

(i) $5 \leqq \pi_{ij}$ for every i and j.

(ii) $\max\{\operatorname{mesh} \mathscr{E}(\pi_{ij}) : j = 1, 2, 3\} < \min\{\sigma(C_{ij}, C_{ij}') : j = 1, 2, 3\}$ for every i.

Let U_{ij} be the sum of all elements of $\mathscr{E}(\pi_{ij})$ which meet C_{ij}. Set $B_{ij} = \overline{U}_{ij} - U_{ij}$ and $B_i = \bigcap_{j=1}^3 B_{ij}$. Then B_{ij} separates C_{ij} and C_{ij}'. Set

$$S = I^4 - \bigcup B_i.$$

Then (S, σ) satisfies the required equalities.

ASSERTION 1. $B_i \cap B_k = \varnothing$ if $i \neq k$.

Proof. Set

$$L_{ij} = \{a/\pi_{ij} : a = 1, \ldots, \pi_{ij}-1\}.$$

Then $L_{ij} \cap L_{kl} \neq \varnothing$ if and only if $i = k$ and $j = l$. If $x = (x_1, \ldots, x_4)$ is a point of B_{ij}, then for some t, $x_t \in L_{ij}$. Hence $B_i \cap B_k = \varnothing$ if $i \neq k$.

ASSERTION 2. B_i does not meet the 2-dimensional edge of I^4. B_i meets the surface of I^4 at only a finite number of points. B_i is the sum of a finite number of segments.

This is evident from the above observation.

ASSERTION 3. B_i is the disjoint sum of a finite number of simple closed curves and a finite number of simple arcs.

Proof. If three different lines l_1, l_2, l_3 lying in B_i have a common point, then they lie in some hyperplane $H : x_j = \text{constant}$. Since H is 3-dimensional, it is now easy to see that $H \cap B_i$ cannot contain l_1, l_2, l_3 at the same time because (i) $\mathscr{E}(\pi_{ij})|H$, $=1, 2, 3$, are collections of bordered blicks and (ii) π_{ij}, $j=1, 2, 3$, are different from each other.

ASSERTION 4. $d_2(S, \sigma)=2$ and $\dim S=3$.

The first equality was proved in §4. As for the second equality see Remark 3.

ASSERTION 5. $\mu \dim (S, \sigma)=3$.

Proof. To show $\mu \dim (S, \sigma)>2$, assume that $\mu \dim (S, \sigma) \leqq 2$. Then there exists a finite closed (in S) covering $\mathscr{F}=\{F\}$ of S which satisfies the following conditions:

(i) $\{G(F)=\text{interior of } F \text{ with respect to } S : F \in \mathscr{F}\}$ covers S.

(ii) $\text{mesh}\, \mathscr{F} < 1$.

(iii) $\text{ord}\, \mathscr{F} \leqq 3$.

The proof for the existence of such $\mathscr{F}$ is left to the reader. Cf. Lemma 7 and also use the total boundedness of (S, σ). Set

$$\begin{aligned} \mathscr{F}_1 &= \{F : F \in \mathscr{F}, \bar{F} \cap \{x : x_1 = 0\} \neq \varnothing\}, \\ M_1 &= \text{boundary in } I^4 \text{ of } \bigcup \{\bar{F} : F \in \mathscr{F}_1\}. \end{aligned}$$

Let F be an arbitrary element of $\mathscr{F}_1$. Let G' be an open set of I^4 with $G' \cap S = G(F)$. Since $\dim \bigcup B_i = 1$, S is dense in I^4. Hence $G' - \bar{F} \neq \varnothing$ yields $(G' - \bar{F}) \cap S \neq \varnothing$, a contradiction. Thus $G' \subset \bar{F}$, which implies $G(F) \cap M_1 = \varnothing$. Take an arbitrary point x from $M_1 \cap S$. Since $x \notin G(F)$ for any F in $\mathscr{F}_1$, there exists an element $F_0 \in \mathscr{F} - \mathscr{F}_1$ such that $x \in G(F_0)$ by the condition (i) imposed upon $\mathscr{F}$. Hence

$$\text{ord}\, \mathscr{F}_1 | M_1 \cap S \leqq \text{ord}\, \mathscr{F} - 1 \leqq 2.$$

Set

$$\begin{aligned} \mathscr{F}_2 &= \{F : F \in \mathscr{F}_1, \bar{F} \cap \{x : x_2 = 0\} \neq \varnothing\}, \\ M_2 &= \text{boundary in } M_1 \text{ of } \bigcup \{\bar{F} \cap M_1 : F \in \mathscr{F}_2\}. \end{aligned}$$

Take an arbitrary point x' from $M_2 \cap S$. Let $y^1, y^2, \ldots$ be a sequence of points of $M_1 - \bigcup \{\bar{F} \cap M_1 : F \in \mathscr{F}_2\}$ with $\lim y^i = x'$. Since $\mathscr{F}_1$ is finite and $\bar{\mathscr{F}}_1 = \{\bar{F} : F \in \mathscr{F}_1\}$ covers M_1, we assume here without loss of generality that the sequence $\{y^i\}$ is contained in one $\bar{F}_1$ with $F_1 \in \mathscr{F}_1 - \mathscr{F}_2$. For any i let z^i be a point of F_1 with $\sigma(y^i, z^i) < \sigma(y^i, x')$. Since $\lim z^i = x'$, $x' \in F_1$. Therefore

$$\text{ord}\, \mathscr{F}_2 | M_2 \cap S \leqq \text{ord}\, \mathscr{F}_1 | M_1 \cap S - 1 \leqq 1.$$

Set

$$\begin{aligned} \mathscr{F}_3 &= \{F : F \in \mathscr{F}_2, \bar{F} \cap \{x : x_3 = 0\} \neq \varnothing\}, \\ M_3 &= \text{boundary in } M_2 \text{ of } \bigcup \{\bar{F} \cap M_2 : F \in \mathscr{F}_3\}. \end{aligned}$$

Since $\operatorname{ord} \overline{\mathscr{F}}_2|M_2 \cap S = \operatorname{ord} \mathscr{F}_2|M_2 \cap S \leqq 1$,

$$M_3 \cap S = \varnothing.$$

Set

$$T = \{x : x \in M_2, \operatorname{ord}(x, \overline{\mathscr{F}}_2) \geqq 2\}.$$

Then T is a closed set of I^4 such that

$$M_3 \subset T \subset M_2 \cap (\bigcup B_i).$$

Let K_1 and K_2 be mutually separated relatively open sets of M_2 such that

$$M_2 - M_3 = K_1 \cup K_2,$$
$$K_1 \supset M_2 \cap \{x : x_3 = 0\},$$
$$K_2 \supset M_2 \cap \{x : x_3 = 1\}.$$

Let P, P', Q or Q' be the union of all components of T which meet $\{x : x_3 = 0\}$, $\{x : x_3 = 1\}$, $\{x : x_4 = 0\}$ or $\{x : x_4 = 1\}$, respectively. Then these four sets are closed. Let us show for instance P is closed. Let x^0 be an arbitrary point of the closure of P and $C_1, C_2, \ldots$ a sequence of components of T such that

(i) each C_i intersects $\{x : x_3 = 0\}$,

(ii) each C_i contains a point z^i with $\lim z^i = x^0$.

Since $x^0 \in \liminf C_i$, $\limsup C_i$ is connected by [6, Theorem 2-101]. Since $\limsup C_i$ intersects $\{x : x_3 = 0\}$ and $\limsup C_i \subset T$, $\limsup C_i \subset P$. Especially $x^0 \in P$ and hence P is closed.

By Assertions 2 and 3 $P \cup P'$ and $Q \cap Q'$ are disjoint closed sets of T such that there is no continuum in T between them. Hence by Lemma 2 there exists a subset V of T such that

(i) V is open and closed in T,

(ii) $Q \cap Q' \subset V$,

(iii) $V \cap (P \cup P') = \varnothing$.

Since $Q \cap Q' \cap \{x : x_3 = 0, 1\} = \varnothing$, there exists a subset W of M_2 such that

(i) W is open in M_2,

(ii) $W \cap T = V$,

(iii) $\overline{W} \cap \{x : x_3 = 0, 1\} = \varnothing$.

Then

$$(\overline{W} - W) \cap T = \varnothing,$$
$$Q \cap Q' \subset W,$$
$$\overline{W} \cap (P \cup P' \cup \{x : x_3 = 0, 1\}) = \varnothing.$$

Set

$$M = (M_3 - W) \cup (\overline{W} - W),$$
$$G_1 = K_1 - \overline{W},$$
$$G_2 = (K_2 \cup W) - (\overline{W} - W).$$

Then

$$M_2 - M = G_1 \cup G_2,$$
$$G_1 \cap G_2 = \varnothing,$$
$$G_1 \supset M_2 \cap \{x : x_3 = 0\},$$
$$G_2 \supset M_2 \cap \{x : x_3 = 1\}.$$

Since G_1 and G_2 are open in M_2, M separates $M_2 \cap \{x : x_3 = 0\}$ and $M_2 \cap \{x : x_3 = 1\}$ in M_2.

Let us show that no component of M meets both $\{x : x_4 = 0\}$ and $\{x : x_4 = 1\}$. Take an arbitrary element F from $\mathscr{F}_2$. Set

$$U(F) = M_2 - \bigcup \{\bar{F}' : F' \in \mathscr{F}_2, F' \neq F\}.$$

Then $\{U(F) : F \in \mathscr{F}_2\}$ is a disjoint collection of open sets of M_2. Since

$$M_2 - T = \bigcup \{U(F) : F \in \mathscr{F}_2\}$$

and $(\overline{W} - W) \cap T = \varnothing$, $\overline{W} - W$ is the sum of the disjoint collection:

$$\mathscr{H} = \{(\overline{W} - W) \cap U(F) = H(F) : F \in \mathscr{F}_2\}.$$

Since

$$H(F) = (\overline{W} - W) - \bigcup \{U(F') : F' \neq F, F' \in \mathscr{F}_2\},$$

$H(F)$ is closed and hence $\mathscr{H}$ is a disjoint collection of closed sets. Since

$$\text{mesh } \mathscr{H} \leqq \text{mesh } \mathscr{F}_2 < 1,$$

no $H(F)$ meets both $\{x : x_4 = 0\}$ and $\{x : x_4 = 1\}$. Now M is the sum of the disjoint closed sets:

$$M \cap B_i, \quad i = 1, 2, \ldots, H(F) \in \mathscr{H}.$$

By our construction no $M \cap B_i$ meets both $\{x : x_4 = 0\}$ and $\{x : x_4 = 1\}$ since $Q \cap Q' \cap M = \varnothing$. Therefore no component of M meets both $\{x : x_4 = 0\}$ and $\{x : x_4 = 1\}$ by Lemma 3.

Consider the closed set:

$$X = \{x : x_4 = 0, 1\} \cup M.$$

Let X_1 be the sum of $\{x : x_4 = 0\}$ and all components of M which meet $\{x : x_4 = 0\}$. Let X_2 be the sum of $\{x : x_4 = 1\}$ and all components of M which meet $\{x : x_4 = 1\}$. Then X_1 and X_2 are closed by the same argument as in the proof for the closedness of P. With the aid of Lemma 2 we can find a closed set N of M_2 which separates $\{x : x_4 = 0\}$ and $\{x : x_4 = 1\}$ such that $N \cap M = \varnothing$. Thus two pairs of opposite sides of M_2 are not defining, which shows in turn three pairs of opposite sides of M_1 are not defining as can easily be seen. At last four pairs of opposite faces of I^4 are not defining, a contradiction. Hence $2 < \mu \dim (S, \sigma)$. Since

$$\mu \dim (S, \sigma) \leqq \dim S = 3, \qquad \mu \dim (S, \sigma) = 3.$$

ASSERTION 6. $d_3(S, \sigma)=3$.

Proof. Since σ is totally bounded, $d_3(S, \sigma)=\mu \dim (S, \sigma)=3$ by Theorem 5.

CONSTRUCTION OF (R, ρ). Take the space (Z_4, σ_3) constructed in the preceding section. Then $\dim Z_4=4$ and $d_2(Z_4, \sigma_3)=d_3(Z_4, \sigma_3)=\mu \dim (Z_4, \sigma_3)=2$. R is the disjoint union of Z_4 and S just constructed. The metric ρ on R is defined as follows:

$$\rho|Z_4 = \sigma_3,$$
$$\rho|S = \sigma,$$
$$\rho(x, y) = \max \{\sigma_3(Z_4), \sigma(S)\} \ (\leqq 6) \text{ if } \{x, y\} \text{ is contained in neither } Z_4 \text{ nor } S.$$

Then it is evident that $d_2(R, \rho)=2$, $d_3(R, \rho)=\mu \dim (R, \rho)=3$ and $\dim R=4$.

8. Problems.

Problem 1. Is it true that $\dim X \leqq 2d_2(X, \rho)$ for all (separable) metric spaces (X, ρ)?

Problem 2. Let (X, ρ) be a metric space with $d_2(X, \rho) < \dim X$ and k an arbitrary integer with

$$d_2(X, \rho) \leqq k \leqq \dim X.$$

Can X allow an equivalent metric σ with $d_2(X, \sigma)=k$?

REMARK 5. Recently Roberts and his student Slaughter solved a problem analogous to Problem 2 for the case when d_2 is replaced by μ dim. (**Added in proof.** This paper has been accepted for publication in Fundamenta Mathematicae.)

Problem 3. Find a necessary and sufficient condition on X with which $d_2(X, \rho)$ (or $\mu \dim (X, \rho)$) $=\dim X$ for any metric ρ agreeing with the preassigned topology of X.

REMARK 6. It is reported by Alexandroff [1] that K. Sitnikov got a sufficient condition: If X is a subset of the n-dimensional Euclidean space (R^n, ρ) such that $\dim X=\dim \bar{X}$, then $\mu \dim (X, \rho)=\dim X$.

Problem 4. Is there a space (X, ρ) with $d_3(X, \rho) < \mu \dim (X, \rho)$?

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