A STUDY OF METRIC-DEPENDENT DIMENSION FUNCTIONS(1)

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1. Introduction. This paper is a study of metric-dependent dimension functions for metric spaces. Let X be a metric space with metric ρ. We introduced in a previous paper [15] two dimension functions d₁ and d₂ of (X, ρ) which by definition appear to depend on ρ. We showed however, that d₁(X, ρ) = dim X (the covering dimension of X). On the other hand d₂ does depend on the particular metric ρ, and there exists (X, ρ) with d₂(X, ρ) < dim X.

Definition 1. The empty set ∅ has d₂ ∅ = −1. d₂(X, ρ)^n if (X, ρ) satisfies the condition:

\( (D₂) \) For any n + 1 pairs of closed sets \( C₁, C₁' ; \ldots ; C_{n+1}, C_{n+1}' \) with \( ρ(Cᵢ, Cᵢ') > 0, \) \( i = 1, \ldots, n+1 \), there exist closed sets \( Bᵢ, i = 1, \ldots, n+1 \), such that (i) \( Bᵢ \) separates \( Cᵢ \) and \( Cᵢ' \) for each \( i \) and (ii) \( \bigcap_{i=1}^{n+1} Bᵢ = \emptyset \).

If \( d₂(X, ρ) \leq n \) and the statement \( d₂(X, ρ) ^≤ n − 1 \) is false, we set \( d₂(X, ρ) = n \).

This definition stems from Eilenberg-Otto’s characterization of dimension [3]:

A metric space \( X \) has \( dim X ≤ n \) if and only if the following condition is satisfied:

\( (D₂) \) For any \( n + 1 \) pairs of closed sets \( C₁, C₁' ; \ldots ; C_{n+1}, C_{n+1}' \) with \( Cᵢ ∩ Cᵢ' = ∅ \), \( i = 1, \ldots, n+1 \), there exist closed sets \( Bᵢ, i = 1, \ldots, n+1 \), such that (i) \( Bᵢ \) separates \( Cᵢ \) and \( Cᵢ' \) for each \( i \) and (ii) \( \bigcap_{i=1}^{n+1} Bᵢ = ∅ \).

This characterization of (covering) dimension is still true even when \( X \) is only a normal space (cf. Hemmingsen [5, Theorem 6.1] or Morita [10, Theorem 3.1]). All spaces considered in this paper are \( T₁ \). To clarify the situation of \( d₂ \) we introduce the following two apparently metric-dependent dimension functions which are similar to \( d₂ \).

Definition 2. First we set \( d₃ ∅ = −1. d₃(X, ρ) ≤ n \) if (X, ρ) satisfies the condition:

\( (D₃) \) For any finite number \( m \) of pairs of closed sets \( C₁, C₁' ; \ldots ; C_m, C_m' \) with \( ρ(Cᵢ, Cᵢ') > 0, \) \( i = 1, \ldots, m \), there exist closed sets \( Bᵢ, i = 1, \ldots, m \), such that (i) \( Bᵢ \) separates \( Cᵢ \) and \( Cᵢ' \) for each \( i \) and (ii) the order of \( \{Bᵢ : i = 1, \ldots, m\} \), \( ord \{Bᵢ\} \), is at most \( n \).

If \( d₃(X, ρ) ≤ n \) and the statement \( d₃(X, ρ) ^≤ n − 1 \) is false, then we set \( d₃(X, ρ) = n \).

Definition 3. First we set \( d₄ ∅ = −1. d₄(X, ρ) ≤ n \) if (X, ρ) satisfies the condition:

\( (D₄) \) For any countable number of pairs of closed sets \( C₁, C₁' ; C₂, C₂' ; \ldots \) with \( ρ(Cᵢ, Cᵢ') > 0, \) \( i = 1, 2, \ldots \), there exist closed sets \( Bᵢ, i = 1, 2, \ldots \), such that (i) \( Bᵢ \) separates \( Cᵢ \) and \( Cᵢ' \) for each \( i \) and (ii) \( ord \{Bᵢ : i = 1, 2, \ldots \} ≤ n \).

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If \( d_4(X, \rho) \leq n \) and the statement \( d_4(X, \rho) \leq n - 1 \) is false, then we set \( d_4(X, \rho) = n \).

Let \( (D_5') \) (respectively \( (D_4) \)) be the condition which is obtained from \( (D_5) \) (resp. from \( (D_4) \)) when "\( \rho(C_i, C_i') > 0 \)" is replaced by "\( C_i \cap C_i' = \emptyset \)". It is evident that \( (D_4) \) implies \( (D_5') \), say \( (D_4) \rightarrow (D_5') \), and \( (D_5') \rightarrow (D_4) \). It is also true that \( (D_4) \rightarrow (D_3) \rightarrow (D_2) \). Morita [10] proved that \( (D_2) \rightarrow (D_3) \rightarrow (D_4) \) even when \( X \) is only a normal space. Then it is natural to ask whether or not \( (D_2) \rightarrow (D_3) \rightarrow (D_4) \). The answer is no for each implication. It will be shown that \( d_4(X, \rho) = \dim X \) for any \( (X, \rho) \) (Theorem 2 below). Moreover we shall construct in this paper a space \( (R, \rho) \) such that \( d_2(R, \rho) = 2, \ d_3(R, \rho) = 3 \) and \( d_4(R, \rho) = 4 \). It is to be noticed that \( (R, \rho) \) is topologically complete and \( \rho \) is totally bounded. Our dimension functions are closely related to so-called metric dimension which is defined as follows:

**Definition 4.** First we set \( \mu \dim \emptyset = -1 \). \( \mu \dim (X, \rho) \leq n \) if \( (X, \rho) \) satisfies the condition:

\((D_5)\) There exists a sequence of open coverings \( \mathcal{U}_i \) of \( X \) such that (i) \( \text{ord } \mathcal{U}_i \leq n + 1 \) for each \( i \) and (ii) \( \text{mesh } \mathcal{U}_i = \sup \{ \rho(U) : U \in \mathcal{U}_i \} \) converges to zero.

If \( \mu \dim (X, \rho) \leq n \) and the statement \( \mu \dim (X, \rho) \leq n - 1 \) is false, then we set \( \mu \dim (X, \rho) = n \).

Here we note that whether \( \mu \dim (X, \rho) = \dim X \) or not had been a serious problem in dimension theory and that the gap between \( \mu \dim \) and \( \dim \) played an important role when the study of dimension theory moved to general metric spaces from separable metric spaces (cf. Sitnikov [19], Nagata [17, 18], Nagami [13], Vopěnka [22], Dowker-Hurewicz [2] and Katětov [8]).

We prove that \( d_3(X, \rho) \leq \mu \dim (X, \rho) \) for any \( (X, \rho) \) and that \( d_3(X, \rho) = \mu \dim (X, \rho) \) when \( \rho \) is totally bounded (Theorems 4 and 5 below). Thus the space \( (R, \rho) \) mentioned before offers an example such that \( d_3(R, \rho) < \mu \dim (R, \rho) < \dim R \). Sitnikov [19] was the first to construct a space \( (Y, \rho) \) such that \( \mu \dim (Y, \rho) < \dim Y \).

In every Cantor \( n \)-manifold \( (K_n, \rho), n \geq 3 \), we shall construct subspaces \( (X_n, \rho) \) and \( (Y_n, \rho) \) such that

(i) \( \dim X_n = \dim Y_n = n - 1 \) and

(ii) \( d_2(X_n, \rho) = \mu \dim (Y_n, \rho) = [n/2] \).

To prove \( \dim X_n \) or \( \dim Y_n \geq n - 1 \) we need the following theorem (Theorem 1 below) which is interesting in itself:

If \( A_i, i = 1, 2, \ldots \), are disjoint closed sets of \( K_n \) with \( \dim A_i \leq n - 1 \) for every \( i \), then \( \dim (K_n - \bigcup A_i) \geq n - 1 \).

Sitnikov [20] proved that dim \( (K_n - \bigcup A_i) \geq n - 1 \) if \( K_n = I^n \) (\( n \)-cube) without the condition \( \dim A_i \leq n - 1 \) and with \( A_i \neq I^n \) for \( i = 1, 2, \ldots. \) Then it is natural to ask whether our present theorem for \( K_n \) is still true without any hypothesis on \( \dim A_i \), and with \( A_i \neq K_n \) for \( i = 1, 2, \ldots \). We give a negative answer for this question. (See Figure 2.)

We give for each \( n \geq 2 \) a metric space \( Z_n \) which allows equivalent metrics \( \rho_m, m = [(n+1)/2], [(n+1)/2] + 1, \ldots, n \), such that \( d_2(Z_n, \rho_m) = \mu \dim (Z_n, \rho_m) = m \). This
space not only illustrates the dependence of $\mu$ dim and $d_n$ on the metric but plays a role in the construction of our final example $R$ which is mentioned above.

The final section lists four unsolved problems.

2. **Dimension of the complement of a disjoint collection of sets.**

**Lemma 1.** Let $X$ be a hereditarily normal space and $Y$ a subset of $X$ with $\dim (X - Y) < n$. Then for any $n$ pairs of disjoint closed sets of $X$, $C_1, C_1'; \ldots ; C_n, C'_n$, there exist closed sets of $X$, $B_1, \ldots ; B_n$, such that $\bigcap B_i \subseteq Y$ and $B_i$ separates $C_i$ and $C'_i$ for each $i$.

**Proof.** Let $D_1, D_1'; \ldots ; D_n, D'_n$ be open sets of $X$ such that $C_i \subseteq D_i$, $C'_i \subseteq D'_i$ and $\overline{D}_i \cap \overline{D}'_i = \emptyset$ for each $i$. By Hemmingsen [5, Theorem 6.1] or Morita [10, Theorem 3.1] there exist relatively open sets $U_1, \ldots ; U_n$ of $X - Y$ such that

1. $\overline{D}_i - Y \subseteq U_i - Y \subseteq (X - \overline{D}_i) - Y$, $i = 1, \ldots ; n$,
2. $D_i U_i - Y$.

If we set $G_i = C_i \cup U_i$ and $H_i = C'_i \cup ((X - Y) - U_i)$, then $G_i \cap H_i = G_i \cap H_i = \emptyset$.

By the hereditary normality of $X$ there exists an open set $V_i$ of $X$ such that $G_i \subseteq V_i \subseteq X - H_i$. Set $B_i = V_i - V_i$. Then $B_i$, $i = 1, \ldots ; n$, satisfy the required condition.

**Lemma 2.** Let $X$ be a compact Hausdorff space and let $H$ and $K$ be disjoint closed sets of $X$ such that no connected set meets both $H$ and $K$. Then there exist disjoint open sets $H_1$ and $K_1$ such that $H \subseteq H_1$, $K \subseteq K_1$ and $H_1 \cup K_1 = X$.

This can be proved by a method analogous to the one in Moore [9, Theorem 44, p. 15] with the consideration of Hocking-Young [6, Theorem 2-9, p. 44].

**Lemma 3.** A connected compact Hausdorff space cannot be decomposed into a countably infinite or finite (but more than one) union of disjoint closed subsets.

This can be proved by a method analogous to the one in Moore [9, Theorem 56, p. 23] with the aid of Lemma 2.

**Definition 5.** Let $X$ be a normal space. A system of pairs $C_1, C_1'; \ldots ; C_n, C'_n$ is called a defining system of $X$ if (i) $C_i$ and $C'_i$ are disjoint closed sets of $X$ for each $i$ and (ii) for arbitrary closed sets $B_i$, $i = 1, \ldots ; n$, separating $C_i$ and $C'_i$ we have $\bigcap B_i \neq \emptyset$.

**Lemma 4.** Let $X$ be a compact Hausdorff space, $F$ a closed set of $X$ and $f$ a mapping (continuous transformation) of $F$ into the $(n - 1)$-sphere $S^{n-1}$. Consider $S^{n-1}$ as the surface of the $n$-cube $I^n = \{(x_1, \ldots , x_n) : -1 \leq x_i \leq 1\}$. Let $C_1, C_1'; \ldots ; C_n, C'_n$ be $n$ pairs of opposite faces of $I^n$ defined by:

$$C_i = \{(x_1, \ldots , x_n) : x_i = -1\}, \quad C'_i = \{(x_1, \ldots , x_n) : x_i = 1\},$$

$i = 1, \ldots ; n$. If the system $f^{-1}(C_1), f^{-1}(C_1'); \ldots ; f^{-1}(C_n), f^{-1}(C'_n)$ is not defining, then $f$ has an extension $f^*: X \rightarrow S^{n-1}$. 
Proof. Let $B_1, \ldots, B_n$ be closed sets of $X$ such that $B_i$ separates $f^{-1}(C_i)$ and $f^{-1}(C'_i)$ for every $i$ and such that $\bigcap B_i = \emptyset$. By Morita [10, Lemma 1.2] we can assume that every $B_i$ is a $G_\delta$. Let $f(x) = (f_1(x), \ldots, f_n(x))$, where each $f_i$ is a mapping into $[-1, 1]$. Let $g_i: X \to [-1, 1]$ be an extension of $f_i|f^{-1}(C_i) \cup f^{-1}(C'_i)$ such that $g_i(x) = 0$ if and only if $x \notin B_i$ and such that $|g_i(x)| = 1$ if and only if $x \in f^{-1}(C_i) \cup f^{-1}(C'_i)$. Let $g(x) = (g_1(x), \ldots, g_n(x))$. Then $g$ is a mapping of $X$ into $I^n$ and $g(F) \subseteq S^{n-1}$. If $x \in F$, then $f(x)$ and $g(x)$ cannot be a pair of opposite points on $S^{n-1}$. Hence $f$ is homotopic to $g|F$. Let $p$ be the original point $(0, \ldots, 0)$ of $I^n$. Then $p \notin g(X)$. Let $r: I^n - p \to S^{n-1}$ be a retraction. Then $rg$ maps $X$ into $S^{n-1}$. By the same argument as in Hurewicz-Wallman [7, Chapter VI], $f$ has an extension $f^*$ over $X$ whose values are still in $S^{n-1}$.

Theorem 1. Let $X$ be a compact hereditarily normal space with $\dim X = n, n \geq 1$, and $A_1, A_2, \ldots$ be a sequence of disjoint closed sets of $X$ such that $\dim (X - \bigcup A_i) \geq n - 1$ for each $i$. Then

$$\dim (X - \bigcup A_i) \geq n - 1.$$
Third step. Here we reconsider that $I^n$ is the $n$-cube expressed as

$$\{(x_1, \ldots, x_n) : -1 \leq x_i \leq 1, i = 1, \ldots, n\}$$

whose surface is $S^{n-1}$ and whose origin $(0, \ldots, 0)$ is $p$. Consider the solid pyramid $P$ in $I^n$ whose base is $B = \{(x_1, \ldots, x_n) : x_n = -1\}$ and whose apex is $p$. The $n-1$ pairs of opposite sides of $P$ may be denoted by $(S_i, T_i), i = 1, \ldots, n-1$, where $S_i$ is spanned by

$$S'_i = \{(x_1, \ldots, x_n) : x_i = x_n = -1\}$$

and $p$, and $T_i$ is spanned by

$$T'_i = \{(x_1, \ldots, x_n) : x_i = 1, x_n = -1\}$$

and $p$. Then

$$C_i = h^{-1}(S_i) - h^{-1}(p), \quad C'_i = h^{-1}(T_i) - h^{-1}(p), \quad i = 1, \ldots, n-1,$$

are $n-1$ pairs of disjoint closed sets of $X' = X - h^{-1}(p)$. Figure 1 will help us to treat the situation.

Fourth step. Assume that $\dim (X - \bigcup A_i) < n-1$. Then

$$\dim (X - ((\bigcup A_i) \cup h^{-1}(p))) < n-1,$$

since $h^{-1}(p)$ is a $G_5$. By Lemma 1 there exist closed sets $B_i$ of $X - h^{-1}(p), i = 1, \ldots, n-1$, such that (i) $B_i$ separates $C_i$ and $C'_i$ for each $i$ and (ii)

$$\left( \bigcap_{i=1}^{n-1} B_i \right) \cap (X - \bigcup A_i) = \emptyset.$$

The latter condition implies $H = \bigcap_{i=1}^{n-1} B_i \subseteq \bigcup A_i$. Let us consider the compact set $H \cup h^{-1}(p)$ and the two disjoint subsets $H \cap h^{-1}(B)$ and $h^{-1}(p)$. Suppose that $H \cap h^{-1}(B) \neq \emptyset$ and there exists a connected closed set $K \subseteq H \cup h^{-1}(p)$ such that
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\[ K \cap H \cap h^{-1}(B) \neq \emptyset \] and \[ h^{-1}(p) \cap K \neq \emptyset . \] Then for some \( i, K \cap A_i \neq \emptyset \). Since \( K=h^{-1}(p) \cup A_1 \cup A_2 \cup \cdots \), we have a contradiction by Lemma 3.

**Fifth step.** By Lemma 2 we can now conclude that there exist disjoint compact sets \( H_1 \) and \( H_2 \) such that (i) \( H_1 \cup H_2 = h^{-1}(B) \cup H \cup h^{-1}(p) \), (ii) \( h^{-1}(p) \subset H_1 \) and (iii) \( h^{-1}(B) \subset H_2 \), whether \( H \cap h^{-1}(B) = \emptyset \) or not. Hence there exists a closed set \( B_n \) of \( X \) separating \( h^{-1}(p) \) and \( h^{-1}(B) \) without touching \( H \). Let \( c \) be a number with \( 0 < c < 1 \), \( Q_c \) the intersection of \( P \) and the hyperplane \( \{(x_1, \ldots, x_n) : x_n = -c\} \), \( P_c \) the intersection of \( P \) and \( \{(x_1, \ldots, x_n) : x_n \leq -c\} \) and \( R_c \) the surface of \( P_c \). Then there exists a number \( b \) with \( 0 < b < 1 \) such that

\[ h^{-1}(P - P_b) \cap B_n = \emptyset . \]

If we confine our attention to the set \( h^{-1}(P_b) \), there are closed sets

\[ B_1 \cap h^{-1}(P_b), \ldots, B_n \cap h^{-1}(P_b) \]

which separate pairs

\[ h^{-1}(S_1 \cap P_b), h^{-1}(T_1 \cap P_b); \ldots; h^{-1}(S_{n-1} \cap P_b), h^{-1}(T_{n-1} \cap P_b); \]

\[ h^{-1}(B), h^{-1}(Q_b), \]

respectively. Denote this system of pairs by \( \alpha \). Since

\[ \bigcap_{i=1}^{n} (B_i \cap h^{-1}(P_b)) \subset \bigcap_{i=1}^{n} B_i = \emptyset \],

\( \alpha \) is not defining. Then by Lemma 4 there exists a mapping \( k_1 : h^{-1}(P_b) \to R_b \) such that \( k_1[h^{-1}(R_b)] = h[h^{-1}(R_b)] \). Let \( k : X \to I^n \) be a mapping such that

(i) \( k[X - h^{-1}(P_b)] = h[X - h^{-1}(P_b)] \),

(ii) \( k[h^{-1}(P_b)] = k_1 \).

Let \( s \) be an inner point of \( P_b \) and \( r \) a retraction of \( I^n - \{s\} \) onto \( S^{n-1} \). Then \( rk : X \to S^{n-1} \) is an extension of \( f \), a contradiction. Thus we have \( \dim (X - \bigcup A_i) \geq n-1 \) and the proof is completed.

**Corollary 1 (Sitnikov [20]).** Let \( n \geq 1 \). Let \( A_i, \ i = 1, 2, \ldots \), be a disjoint sequence of closed sets of \( I^n \) at least two of which are not empty. Then

\[ \dim (I^n - \bigcup A_i) \geq n-1. \]

**Proof.** Let \( S^{n-1} \) be the surface of \( I^n = \{(x_1, \ldots, x_n) : -1 \leq x_i \leq 1, i = 1, \ldots, n\} \). Let \( f : S^{n-1} \to S^{n-1} \) be the identity mapping. Since it is impossible that \( I^n - S^{n-1} \) is contained in one \( A_i \), we have one of the following two cases:

(i) There exists \( i \) such that \( (I^n - S^{n-1}) \cap A_i \neq \emptyset \) and \( A_i \subset S^{n-1} \) for any \( j \neq i \).

(ii) There exist \( i \) and \( j \) with \( i \neq j \) such that

\[ (I^n - S^{n-1}) \cap A_i \neq \emptyset \quad \text{and} \quad (I^n - S^{n-1}) \cap A_j \neq \emptyset. \]
The first case yields \( \dim (I^n - \bigcup A_i) = n \). If the second case happens, then there exists a number \( \varepsilon \) with \( 0 < \varepsilon < 1 \) such that

\[
I^n = \{(x_1, \ldots, x_n) : |x_i| \leq \varepsilon, \ i = 1, \ldots, n\}
\]

meets \( A_i \) and \( A_j \). Then by Lemma 3 there exists a point \( q \) in \( I^n - \bigcup A_i \). Then we can apply the same argument on \( f \) and \( q \) as in the proof of Theorem 1 and we get \( \dim (I^n - \bigcup A_i) \geq n - 1 \).

**Corollary 2.** Let \( X \) be a connected metric space such that every point has a neighborhood homeomorphic to \( I^n, n \geq 1 \). Let \( A_i, i = 1, 2, \ldots \), be a disjoint sequence of closed sets of \( X \) at least two of which are not empty. Then

\[
\dim (X - \bigcup A_i) \geq n - 1.
\]

**Proof.** Consider a closed covering \( \{F_a\} \) of \( X \) such that each \( F_a \) is homeomorphic to \( I^n \). If each \( F_a \) is contained in some \( A_i \), then each \( A_i \) has to be open, which contradicts the fact that \( X \) is connected. Hence (i) there exists \( F_a \) which meets at least two of the \( A_i \)'s, (ii) there exists \( F_a \) such that \( F_b \cap A_i \neq \emptyset \), \( F_b - A_i \neq \emptyset \) and \( F_b \cap A_j = \emptyset \) for \( j \neq i \), or (iii) there exists \( F_a \) such that \( F_a \cap A_i = \emptyset \) for every \( i \). The first case yields \( \dim (X - \bigcup A_i) \geq \dim (F_a - \bigcup A_i) \geq n - 1 \). The second case yields \( \dim (X - \bigcup A_i) \geq \dim (F_b - A_i) \geq n - 1 \). The third case yields \( \dim (X - \bigcup A_i) \geq \dim F_a \geq n - 1 \).

Figure 2 gives a Cantor 2-manifold \( X \) such that a proposition for \( X \) analogous to Corollary 1 fails. In fact \( \dim X = 2 \), yet \( \dim (X - \bigcup A_{ij}) = 0 \) since \( X - \bigcup A_{ij} \) is a subset of the Cantor discontinuum.

![Figure 2](image-url)
3. Relations among the various functions.

**Lemma 5** (Morita [10, Theorem 3.4]). Let $X$ be a normal space with $\dim X \leq n$. Then $X$ satisfies the condition $(D_4)$.

**Lemma 6.** If a metric space $X$ has an $\omega$-locally finite open base $\mathcal{U}$ such that $\text{ord}\{U - U : U \in \mathcal{U}\} \leq n$, then $\dim X \leq n$. ($\mathcal{U}$ is called $\omega$-locally finite if $\mathcal{U}$ can be decomposed into a countable number of locally finite subcollections.)

This is a slight modification of Morita [12, Theorem 8.7].

**Theorem 2.** $\dim X = d_4(X, \rho)$ for any $(X, \rho)$.

**Proof.** By Lemma 5 $\dim X \leq d_4(X, \rho)$. To prove $\dim X \leq d_4(X, \rho)$ assume $d_4(X, \rho) \leq n$. Let us show the existence of such $\mathcal{U}$ as in Lemma 6. By Stone [21] there exists an open base $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$ of $X$, $\mathcal{V}_i = \{V_\alpha : \alpha \in \Lambda_i\}$, $i = 1, 2, \ldots$, such that each $\mathcal{V}_i$ is discrete. $\mathcal{V}_i$ is called discrete if $\mathcal{V}_i = \{V_\alpha : \alpha \in \Lambda_i\}$ is a locally finite disjoint collection. Set $V_i = \bigcup \{V_\alpha : \alpha \in \Lambda_i\}$, $i = 1, 2, \ldots$, and

$$
F_{ij} = \{x : \rho(x, X - V_i) \geq 1/j\}, \quad j = 1, 2, \ldots
$$

Then by $d_4(X, \rho) \leq n$ there exist open sets $U_{ij}$, $i, j = 1, 2, \ldots$, such that

(i) $V_i \supseteq U_{ij} \supseteq U_{ij} \supseteq F_{ij}$ for each $i$ and $j$

(ii) $\text{ord}\{U_{ij} - U_{ij} : i, j = 1, 2, \ldots\} \leq n$.

Set

$$
\mathcal{U}_{ij} = \{V_\alpha \cap U_{ij} : \alpha \in \Lambda_i\}.
$$

Then $\mathcal{U}_{ij}$ is discrete and hence locally finite. $\mathcal{U} = \bigcup_{i=1}^{\infty} \mathcal{U}_{ij}$ is an open base for $X$ such that $\text{ord}\{U - U : U \in \mathcal{U}\} \leq n$. Hence by Lemma 6 we have $\dim X \leq n$ and the theorem is proved.

**Lemma 7.** If $\{F_\alpha\}$ is a locally finite closed collection of a paracompact Hausdorff space, then there exists an open collection $\{G_\alpha\}$ such that (i) $G_\alpha \supseteq F_\alpha$ for each $\alpha$ and (ii) $\text{ord}\{F_\alpha\} = \text{ord}\{G_\alpha\}$.

This can easily be seen with the aid of Morita [11, Theorem 1.3].

**Theorem 3 (Katětov [8]).** $\dim X \leq 2\mu \dim (X, \rho)$ for any $(X, \rho)$.

**Proof.** Suppose $\mu \dim (X, \rho) \leq n$. Let $\mathcal{U}_i = \{U_\alpha : \alpha \in \Lambda_i\}$, $i = 1, 2, \ldots$, be a sequence of open coverings of $X$ such that

(i) $\text{mesh}\mathcal{U}_i < 2^{-i}$ for each $i$ and

(ii) $\text{ord}\mathcal{U}_i \leq n + 1$ for each $i$.

By an easy observation we can assume that each $\mathcal{U}_i$ is locally finite. Let $\mathcal{G} = \{G_1, \ldots, G_m\}$ be an arbitrary finite open covering of $X$. Set

$$
D_i = \bigcup \{\{x : \rho(x, X - G_j) > 2^{-i+1}\} : j = 1, \ldots, m\}, \quad i = 1, 2, \ldots
$$

Then (i) $D_1 \supseteq D_2 \supseteq D_3 \supseteq D_4 \supseteq \cdots$, (ii) each $D_i$ is open and (iii) $\bigcup_{i=1}^{\infty} D_i = X$. Let
Let $\mathcal{F}_i = \{F_\alpha : \alpha \in \Lambda_i\}$ be a closed covering of $X$ such that $F_\alpha \subseteq U_\alpha$ for each $\alpha \in \Lambda_i$. Set

$$\Lambda'_i = \{\alpha : F_\alpha \cap \bar{D}_i = \emptyset\},$$

$$W_i = X - \bigcup \{F_\alpha : \alpha \in \Lambda_i\}.$$

Then $W_i$ is an open set with $\bar{D}_i \subseteq W_i \subseteq \bar{W}_i \subseteq D_{i+1}$ for every $i$. Since every point of $\bar{W}_i - W_i$ is contained in some $F_\alpha \in \mathcal{F}_i$ with $\alpha \in \Lambda_i$, ord $\{F_\alpha \cap (\bar{W}_i - W_i) : \alpha \notin \Lambda'_i\} \leq n$. (This type of argument comes from Morita [10, Theorem 3.3].) Since $\{F_\alpha : \alpha \in \Lambda_1 - \Lambda'_1\}$ covers $\bar{W}_1 - W_1$, there exists by Lemma 7 an open collection $\mathcal{V}_1 = \{V_\alpha : \alpha \in \Lambda_1 - \Lambda'_1\}$ of $X$ such that

(i) $F_\alpha \cap (\bar{W}_1 - W_1) \subseteq V_\alpha \cap (D_{i+1} - \bar{D}_i) \cap U_\alpha$ for each $\alpha \in \Lambda_1 - \Lambda'_1$ and

(ii) ord $\{V_\alpha : \alpha \in \Lambda_1 - \Lambda'_1\} \leq n$.

Then $\mathcal{V}_1$ refines $\mathcal{F}_1$. Moreover $V \in \mathcal{V}_1$, $V' \in \mathcal{V}_1$, $i \neq j$, imply $V \cap V' = \emptyset$. We set

$$\mathcal{H}_1 = \{U_\alpha \cap (\bar{W}_i - W_{i-1}) : \alpha \in \Lambda_i\}, \text{ where } W_{-1} = \emptyset.$$

Then $H \in \mathcal{H}_1$, $H' \in \mathcal{H}_j$, $i \neq j$, imply $H \cap H' = \emptyset$. If we set

$$\mathcal{H} = \left( \bigcup_{i=1}^\infty \mathcal{H}_i \right) \cup \left( \bigcup_{i=1}^\infty \mathcal{V}_i \right),$$

then it is easy to see that $\mathcal{H}$ is an open covering of $X$ such that (i) $\mathcal{H}$ refines $\mathcal{F}$ and (ii) ord $\mathcal{H} \leq 2n + 1$. Thus we have dim $X \leq 2n$ and the theorem is proved.

**Theorem 4.** $d_\rho(X, \rho) \leq \mu \dim (X, \rho)$ for any $(X, \rho)$.

**Proof.** Suppose $\mu \dim (X, \rho) \leq n$. Let $C_1, C'_1; \ldots; C_m, C'_m$ be a finite number of pairs of closed sets of $X$ such that there exists a positive number $\epsilon$ such that $\rho(C_i, C'_i) > \epsilon$ for each $i$. By $\mu \dim (X, \rho) \leq n$ there exists a locally finite open covering $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of $X$ such that (i) ord $\mathcal{U} \leq n + 1$ and (ii) mesh $\mathcal{U} < \epsilon/2$. Let $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ be a locally finite closed covering of $X$ such that $F_\alpha \subseteq U_\alpha$ for each $\alpha \in \Lambda$. Set

$$\Lambda' = \{\alpha : F_\alpha \cap C_1 = \emptyset\},$$

$$W = X - \bigcup \{F_\alpha : \alpha \in \Lambda'\},$$

$$B_1 = \bar{W} - W.$$

Then $B_1$ separates $C_1$ and $C'_1$ and

$$\mathcal{U}_1 = \{U_\alpha : \alpha \in \Lambda - \Lambda'\} \cup \{U_\alpha - B_1 : \alpha \in \Lambda'\}$$

is an open covering of $X$ such that

(i) $\mathcal{U}_1$ refines $\mathcal{U}$,

(ii) $\mathcal{U}_1|X - B_1 = \mathcal{U}_1|X - B_1$,

(iii) ord $(x, \mathcal{U}_1) \leq \text{ord} (x, \mathcal{U}) - 1$ for each $x \in B_1$, where ord $(x, \mathcal{U}_1)$ is the order of $\mathcal{U}_1$ at $x$. 

**Note:** This is the end of the document. The context seems to be related to topology, focusing on covering spaces and open sets. The text discusses the refinement of closed coverings and the analysis of open sets under specific conditions. The theorem presented is a result concerning the dimension of a space with respect to a given topology. The proof involves constructing a refined covering that satisfies certain order conditions, leading to the conclusion that the dimension of the space is bounded by $2n$.
This can be verified by the same argument as in the proof of the previous theorem. Continuing this procedure, we get closed sets $B_i$, $i = 2, \ldots, m$, separating $C_1$ and $C_2$ respectively and open coverings $U_2, \ldots, U_m$ such that

(i) $U_{i+1}$ refines $U_i$ for $i = 1, \ldots, m - 1$,
(ii) $U_i \cap X - B_i = U_{i-1} \cap X - B_i$ for each $i$,
(iii) $\text{ord} (x, U_i) \leq \text{ord} (x, U_{i-1}) - 1$ for each $x \in B_i$, $i = 2, \ldots, m$.

If $x \in B_{i_1} \cap \cdots \cap B_{i_{n+1}}$, $i_1 < i_2 < \cdots < i_{n+1}$, then

$$1 \leq \text{ord} (x, U_{i_{n+1}}) \leq \text{ord} (x, U_{i_{n+1} - 1}) - 1$$
$$\leq \text{ord} (x, U_{i_n}) - 1 \leq \text{ord} (x, U_{i_{n-1}}) - 2$$
$$\leq \cdots \leq \text{ord} (x, U_{i_1}) - n \leq \text{ord} (x, U) - (n + 1).$$

Thus we have $\text{ord} (x, U) \geq n + 2$, a contradiction. We have therefore

$$\text{ord} \{B_i : i = 1, \ldots, m\} \leq n,$$
and the theorem is proved.

**Theorem 5.** $d_3(X, \rho) = \mu \dim (X, \rho)$ for a totally bounded metric space $(X, \rho)$.

**Proof.** Suppose that $d_3(X, \rho) \leq n$. For an arbitrary positive number $\varepsilon$ there exists a finite set of points $x_1, \ldots, x_m$ such that

$$\{U_i = S_{\varepsilon}(x_i) : i = 1, \ldots, m\}$$
covers $X$. Set

$$V_i = S_{2\varepsilon}(x_i), \quad i = 1, \ldots, m.$$

Then there exist open sets $W_1, \ldots, W_m$ such that

(i) $U_i \subset W_i \subset \overline{W_i} \subset V_i$ for each $i$,
(ii) $B_i = \overline{W_i} - W_i$ separates $X - V_i$ and $U_i$ for each $i$,
(iii) $\text{ord} \{B_i : i = 1, \ldots, m\} \leq n$.

By Lemma 7 there exist open sets $G_1, \ldots, G_m$ of $X$ such that (i) $B_i \subset G_i \subset V_i$ for each $i$ and (ii) $\text{ord} \{G_i\} \leq n$. Set

$$\mathcal{W}_i = \{W_{i1} = W_i, W_{i2} = X - \overline{W_i}\},$$
$$\mathcal{W} = \bigcap_{i=1}^m \mathcal{W}_i = \{W_{i1} \cap \cdots \cap W_{im} : i_1, \ldots, i_m = 1, 2\}.$$

Since

$$\bigcap_{i=1}^m W_{i2} = \bigcap_{i=1}^m (X - \overline{W_i}) = X - \bigcup_{i=1}^m \overline{W_i} = X - X = \emptyset,$$

$\mathcal{W}$ refines $\{W_1, \ldots, W_m\}$ and hence $\mathcal{W}$ refines $\{V_1, \ldots, V_m\}$. Moreover

$$\bigcup \{W : W \in \mathcal{W}\} = X - \bigcup_{i=1}^m B_i$$

and $\text{ord} \mathcal{W} \leq 1$. If we set

$$\mathcal{G} = \mathcal{W} \cup \{G_1, \ldots, G_m\},$$
then \( \varnothing \) is an open covering of \( X \) such that (i) mesh \( \varnothing \leq 4e \) and (ii) ord \( \varnothing \leq n + 1 \). Thus we have \( \mu \dim (X, \rho) \leq n \). If we combine \( d_3(X, \rho) \geq \mu \dim (X, \rho) \) just proved with Theorem 4, we get \( d_3(X, \rho) = \mu \dim (X, \rho) \) and the theorem is proved.

Since \( d_2(X, \rho) \leq d_3(X, \rho) \leq d_4(X, \rho) \) are trivially true, we have now \( d_2 \leq d_3 \leq \mu \dim \leq d_4 = \dim \leq 2 \mu \dim \), symbolically.

**Remark 1.** When \( (X, \rho) \) is locally compact, all of these dimension functions coincide with each other, of course. On the other hand, there exists a space \( (X, \rho) \) which is not locally compact at any point yet \( d_2(X, \rho) = d_3(X, \rho) = \mu \dim (X, \rho) = \dim X \). Let us consider \( I^3 = \{(x_1, x_2, x_3) : 0 \leq x_i \leq 1, i = 1, 2, 3\} \). Let \( \rho \) be a metric of \( I^3 \). Set \( B = \{(x_1, x_2, x_3) : 0 \leq x_1 \leq 0\} \). Let \( C \) be the set of all points in \( I^3 \) whose coordinates are all rational. If we set \( X = B \cup C \), then \( (X, \rho) \) satisfies the condition as follows:

\[
2 = d_2(B, \rho) \leq d_2(X, \rho) \leq d_3(X, \rho) \leq \mu \dim (X, \rho) \leq \dim X = 2.
\]

**Remark 2.** It is to be noticed that even if \( (X, \rho) \) is any metric space, (i) \( \dim X = 1 \) implies \( d_2(X, \rho) = 1 \) and (ii) \( d_2(X, \rho) = 0 \) implies \( \dim X = 0 \) (see Nagami-Roberts [15]).

4. Spaces \( (X_n, \rho) \) with \( d_2 = \lfloor n/2 \rfloor \) and \( \dim \geq n - 1 \).

**Lemma 8.** Let \( F_1, F_2, \ldots \) be a sequence of closed sets of a metric space \( X \) with \( \dim F_i = n_i \). Let \( C_1, C'_1; \ldots; C_m, C'_m \) be \( m \) disjoint pairs of closed sets of \( X \). Then there exist closed sets \( B_1, \ldots, B_m \) such that

(i) for each \( i \), \( B_i \) separates \( C_i \) and \( C'_i \),

(ii) for each \( j \) and for each sequence \( 1 \leq i_1 < i_2 < \cdots < i_n \leq \min \{n + 1, m\} \),

\[
\dim (B_{i_1} \cap \cdots \cap B_{i_n} \cap F_j) \leq n_j - 1.
\]

See Morita [12, Theorem 9.1].

**Construction of \( (X_n, \rho) \).** Let \( (K_n, \rho) \) be a Cantor \( n \)-manifold with \( n \geq 3 \). Put \( m = \lfloor n/2 \rfloor + 1 \). By compactness of \( K_n \) there exists a sequence of \( m \) disjoint pairs of closed sets of \( K_n \), say \( C_{11}, C'_{11}; \ldots; C_{1m}, C'_{1m} ; C_{21}, C'_{21}; \ldots; C_{2m}, C'_{2m} ; \ldots \), such that for any \( m \) disjoint pairs of closed sets \( C_1, C'_1; \ldots; C_m, C'_m \) there exists \( i \) with \( C_i \subset C'_i \) and \( C'_i \subset C_i \) for \( j = 1, \ldots, m \). By Lemma 8 there exist closed sets \( B_{11}, \ldots, B_{1m} \) such that

(i) for each \( i \), \( B_{1i} \) separates \( C_{1i} \) and \( C'_{1i} \),

(ii) \( \dim B_{1i} \leq n - m \) where \( B_{1i} = \bigcap_{j=1}^m B_{1j} \).

By repeated application of Lemma 8 there exist closed sets \( B_{21}, \ldots, B_{2m} \) such that

(i) for each \( i \), \( B_{2i} \) separates \( C_{2i} \) and \( C'_{2i} \),

(ii) \( \dim B_{2i} \leq n - m \) where \( B_{2i} = \bigcap_{j=1}^m B_{2j} \),

(iii) \( B_1 \cap B_2 = \emptyset \).

Continuing such process we get finally a sequence of closed sets \( B_{ij}, \) \( i = 1, 2, \ldots, j = 1, \ldots, m \), which have the following property:

(i) \( B_{ij} \) separates \( C_{ij} \) and \( C'_{ij} \) for each \( i \) and \( j \),

(ii) \( \dim B_i \leq n - m \) for \( i = 1, 2, \ldots, \), where \( B_i = \bigcap_{j=1}^m B_{ij} \).

(iii) \( B_i \cap B_j = \emptyset \) if \( i \neq j \).
If we set

\[ X_n = K_n - \bigcup_{i=1}^{\infty} B_i, \]

then we have the space \((X_n, \rho)\).

**Assertion 1.** \(d_2(X_n, \rho) \leq \lfloor n/2 \rfloor\).

**Proof.** Let \(C_1, C'_1; \ldots; C_m, C'_m\) be \(m\) pairs of closed sets of \(X_n\) such that \(\rho(C_i, C'_i) > 0\) for \(i = 1, \ldots, m\). Since their closures \(\overline{C}_1, \overline{C}'_1; \ldots; \overline{C}_m, \overline{C}'_m\) in \(K_n\) constitute \(m\) disjoint pairs of closed sets of \(K_n\), there exists \(i\) such that \(\overline{C}_j \subseteq C_i\) and \(\overline{C}'_j \subseteq C'_i\) for \(j = 1, \ldots, m\). Then \(B_{i_1} \cap X_n, \ldots, B_{i_m} \cap X_n\) are closed sets of \(X_n\) such that \(B_{i_j} \cap X_n\) separates \(C_j\) and \(C'_j\) for \(j = 1, \ldots, m\).

\[ \bigcap_{j=1}^{m} (B_{i_j} \cap X_n) = B_i \cap X_n = \emptyset, \]

and hence we have \(d_2(X_n, \rho) \leq m - 1 = \lfloor n/2 \rfloor\).

**Assertion 2.** \(d_2(X_n, \rho) \geq \lfloor n/2 \rfloor\).

**Proof.** If \(G\) is a nonempty open set of \(K_n\), then \(\dim G = n\). Since

\[ \dim (\bigcup B_i) \leq n - m < n, \quad G - (\bigcup B_i) \neq \emptyset \]

and hence \(G \cap X_n \neq \emptyset\). Thus \(X_n\) is dense in \(K_n\). Assume \(d_2(X_n, \rho) = t < \lfloor n/2 \rfloor\). Take a defining system \(\overline{D}_1, \overline{D}'_1; \ldots; \overline{D}_{t+1}, \overline{D}'_{t+1}\) of \(K_n\) such that

(i) each \(D_i\) and \(D'_i\) are open in \(K_n\),

(ii) for any closed sets \(A_i, i = 1, \ldots, t+1\), separating \(\overline{D}_i\) and \(\overline{D}'_i\),

\[ \dim \left( \bigcup_{i=1}^{t+1} A_i \right) \geq n - (t+1). \]

Set \(C_i = \overline{D}_i \cap X_n\) and \(C'_i = \overline{D}'_i \cap X_n\). Then it is easy to see that \(\overline{C}_i = \overline{D}_i\) and \(\overline{C}'_i = \overline{D}'_i\), since \(X_n\) is dense in \(K_n\). Set

\[ \epsilon = \min \{ \rho(C_i, C'_i) : i = 1, \ldots, t+1 \}. \]

Take open sets \(U_i, i = 1, \ldots, t+1\), of \(K_n\) such that

\[ \{x : x \in X_n, \rho(x, C_i) < \epsilon/4\} \subset U_i \cap X_n \subset \overline{U}_i \cap X_n \]

(i) \(\subset X_n - \{x : x \in X_n, \rho(x, C_i) < \epsilon/4\}\) for each \(i\),

(ii) \(\bigcap_{i=1}^{t+1} ((\overline{U}_i - U_i) \cap X_n) = \emptyset\).

It is easy to see that \(\overline{U}_i - U_i\) thus chosen separates \(\overline{C}_i = \overline{D}_i\) and \(\overline{C}'_i = \overline{D}'_i\) for each \(i\). Hence

\[ \dim B \geq n - (t+1) \geq n - \lfloor n/2 \rfloor \]

where \(B = \bigcup_{i=1}^{t+1} (U_i - U_i)\).
On the other hand,
\[ \dim B \leq \dim \left( \bigcup B_i \right) \leq n - m = n - \lfloor n/2 \rfloor - 1 \]
because \( B \cap X_n = \emptyset \), which is a contradiction.

**Assertion 3.** \( \dim X_n \geq n - 1 \).

**Proof.** Since \( \dim B_i \leq n - m = n - \lfloor n/2 \rfloor - 1 \leq n - 1 \), we have \( \dim X_n \geq n - 1 \) at once by Theorem 1.

5. **Spaces** \((Y_n, \rho)\) with \( \mu \dim = \lfloor n/2 \rfloor \) and \( \dim \geq n - 1 \).

**Lemma 9.** \((X, \rho)\) has \( \mu \dim (X, \rho) \leq n \) if and only if there exists a sequence of locally finite closed coverings \( \mathcal{F}_n, i = 1, 2, \ldots \), such that

(i) mesh \( \mathcal{F}_i \leq 1/i \) for any \( i \),
(ii) ord \( \mathcal{F}_i \leq n + 1 \) for any \( i \).

This is verified at once by Lemma 7.

**Lemma 10.** Let \( X \) be a metric space with \( \dim X \leq n \) and \( B_1, B_2, \ldots \) a sequence of closed sets of \( X \) with \( \dim B_i = n_i \), where \( B_1 = X \). Let \( \epsilon \) be an arbitrary positive number. Then there exists a locally finite closed covering \( \mathcal{F} = \{ F_\alpha : \alpha \in \Lambda \} \) which satisfies the following conditions:

(i) mesh \( \mathcal{F} < \epsilon \).
(ii) For any \( i \) ord \( \mathcal{F} | B_i \leq n_i + 1 \).
(iii) For any \( i \), any \( j \leq n_i + 2 \) and any \( j \) different indices \( \alpha(1), \ldots, \alpha(j) \) of \( \Lambda \),

\[ \dim \bigcup_{k=1}^{j} (F_{\alpha(k)} \cap B_i) \leq n_i - j + 1. \]

This is proved essentially in Nagami [13, Theorem 3.6].

**Construction of** \((Y_n, \rho)\). Let \((K_n, \rho)\) be a Cantor \( n \)-manifold, \( n \geq 3 \). Set \( m = \lfloor n/2 \rfloor + 2 \). By Lemma 10 there exists a locally finite closed covering \( \mathcal{F}_1 = \{ F_\alpha : \alpha \in \Lambda_1 \} \) of \( K_n \) such that (i) mesh \( \mathcal{F}_1 < 1 \), (ii) ord \( \mathcal{F}_1 \leq n + 1 \), and (iii) \( \dim B_1 \leq n - m + 1 \) where \( B_1 = \{ x : \text{ord} (x, \mathcal{F}_1) \geq m \} \) which is closed by the local finiteness of \( \mathcal{F}_1 \). Then ord \( \mathcal{F}_1 | K_n - B_1 < m \).

By Lemma 10 again there exists a locally finite closed covering

\( \mathcal{F}_2 = \{ F_\alpha : \alpha \in \Lambda_2 \} \)

of \( K_n \) such that (i) mesh \( \mathcal{F}_2 < 1/2 \), (ii) ord \( \mathcal{F}_2 \leq n + 1 \), (iii) \( \dim B_2 \leq n - m + 1 \) where \( B_2 = \{ x : \text{ord} (x, \mathcal{F}_2) \geq m \} \), and (iv) \( \dim \bigcap_{k=1}^{j} (F_{\alpha(k)} \cap B_1) \leq \dim B_1 - j + 1 \) for any \( j \leq \dim B_1 + 2 \) and any \( j \) different indices \( \alpha(1), \ldots, \alpha(j) \) of \( \Lambda_2 \). To show that the last condition (iv) implies \( B_1 \cap B_2 = \emptyset \), set \( \dim B_1 = n_1 \). Take \( n_1 + 2 \) different indices \( \alpha(1), \ldots, \alpha(n_1 + 2) \) of \( \Lambda_2 \). Then

\[ \dim \bigcap_{k=1}^{n_1 + 2} (F_{\alpha(k)} \cap B_1) \leq n_1 - (n_1 + 2) + 1 = -1. \]

Hence we have

\[ B_1 \cap \{ x : \text{ord} (x, \mathcal{F}_2) \geq n_1 + 2 \} = \emptyset. \]
Since
\[ n_1 + 2 \leq (n-m+1) + 2 \leq n - ([n/2] + 2) + 3 \]
\[ = n - [n/2] + 1 \leq (2[n/2] + 1) - [n/2] + 1 \]
\[ = [n/2] + 2 = m, \]
we have \( B_1 \cap B_2 = \emptyset. \)

Repeating such procedure we have a sequence of locally finite closed coverings \( \mathcal{F}_i, i = 1, 2, \ldots \), which satisfy the following conditions:

(i) For each \( i \), mesh \( \mathcal{F}_i < 1/i. \)

(ii) For each \( i \), \( \dim B_i \leq n - m + 1 \) where \( B_i \) is a closed set defined by

\[ B_i = \{ x : \operatorname{ord} (x, \mathcal{F}_i) \geq m \}. \]

(iii) \( B_i, i = 1, 2, \ldots, \) are mutually disjoint.

We set \( Y_n = K_n - \bigcup B_i. \) Then \( (Y_n, \rho) \) is the desired space.

**Assertion 1.** \( \dim Y_n = n - 1. \)

**Proof.** Since
\[ \dim B_i \leq n - m + 1 = n - [n/2] - 1 \leq n - 1, \]
the assertion is true by Theorem 1.

**Assertion 2.** \( \mu \dim (Y_n, \rho) \leq [n/2]. \)

**Proof.** Since \( \operatorname{ord} \mathcal{F}_i \mid Y_n \leq \operatorname{ord} \mathcal{F}_i \mid K_n - B_i \leq m - 1 = ([n/2] + 2) - 1 = [n/2] + 1, \) the assertion is true by Lemma 9.

**Assertion 3.** \( \dim Y_n \leq n - 1 \) when \( n \) is odd.

**Proof.** Since \( \dim Y_n \leq 2\mu \dim (Y_n, \rho) \) by Theorem 3, we have
\[ \dim Y_n \leq 2[n/2] = 2((n-1)/2) = n - 1. \]

**Assertion 4.** \( \mu \dim (Y_n, \rho) \geq [n/2]. \)

**Proof.** Assume the contrary. Then
\[ \dim Y_n \leq 2\mu \dim (Y_n, \rho) \leq 2([n/2] - 1) \leq n - 2, \]
a contradiction.

Thus \( (Y_n, \rho) \) satisfies (i) \( \dim Y_n \geq n - 1 \) and (ii) \( \mu \dim (Y_n, \rho) = [n/2]. \) Furthermore when \( n \) is odd, \( \dim Y_n = n - 1. \)

**Remark 3.** It is to be noted that for \( X_n \) and \( Y_n \) obtained by replacing \( K_n \) with \( I^n, \) \( \dim X_n = \dim Y_n = n - 1 \) for any \( n, \) because of the fact that \( I^n - X_n \) and \( I^n - Y_n \) are dense in \( I^n, \) and the invariance theorem of domain.

**Remark 4.** Note that the existence of a sequence of open coverings \( \mathcal{U}_i, i = 1, 2, \ldots, \) with \( \operatorname{ord} \mathcal{U}_i \leq n + 1 \) and \( \lim \text{mesh} \mathcal{U}_i = 0 \) does not characterize dimension. Thus it is natural to seek an additional condition upon \( \mathcal{U}_i \) with which the existence of the sequence does characterize dimension. Dowker-Hurewicz [2], Nagata [17] and Nagami [14] considered such a condition. This type of characterization theorem is one of the main foundations on which modern dimension theory has been built up. Vopěnka [22] gave a simple condition: "\( \mathcal{U}_{i+1} \prec \text{(refines)} \mathcal{U}_i \) for each \( i. \)" Recently Nagami-Roberts [16, Theorem 3] refined Vopěnka's
theorem, weakening the mesh condition. But our proof contains an error. The definition of $V_a$ in [16, line 15, p. 157] is not adequate. Let us take this opportunity to give a correct proof as follows:

**Theorem 6.** A metric space $X$ has $\dim X \leq n$ if there exists a sequence $\mathcal{U}_1 > \mathcal{U}_2 > \cdots$ of open coverings $\mathcal{U}_i$ of $X$ such that

(i) for each $x \in X$, $\{\text{St}(x, \mathcal{U}_i^i) : i = 1, 2, \ldots\}$ is a local base of $x$,

(ii) $\text{ord} \mathcal{U}_i \leq n + 1$.

**Proof.** Set

$$\mathcal{U}_i = \{U(\alpha_i) : \alpha_i \in A_i\}, \quad i = 1, 2, \ldots$$

Let $f_i^i + 1 : A_{i+1} \rightarrow A_i$ be a function such that $f_i^i + 1(\alpha_{i+1}) = \alpha_i$ yields $U(\alpha_{i+1}) \subset U(\alpha_i)$. For each pair $i < j$ let $f_i^j = f_i^i + 1 \cdots f_i^1$ and $f_i^i$ be the identity mapping. Let $\mathcal{F}$ be an arbitrary finite open covering of $X$. Set

$$X_i = \bigcup \{U(\alpha_i) : \text{St}(U(\alpha_i), \mathcal{U}_i) \text{ refines } \mathcal{F}\}.$$

Then by the condition (i) \{\$X_1, X_2, \ldots\} is an open covering of $X$. Set $X_0 = \emptyset$. Set

$$B_i = \{\alpha_i : U(\alpha_i) \cap X_i \neq \emptyset\},$$

$$C_i = \left\{\alpha_i : \alpha_i \in B_i, U(\alpha_i) \cap \left( \bigcup_{j < i} X_j \right) = \emptyset \right\},$$

$$D_i = \left\{\alpha_i : \alpha_i \in B_i, U(\alpha_i) \cap \left( \bigcup_{j < i} X_j \right) \neq \emptyset \right\}.$$

Then $B_0 \subset A_0$, $B_1 = C_1$, $B_i = C_i \cup D_i$ and $C_1 \cap D_i = \emptyset$.

For every $i < j$ and every $\alpha_i \in C_i$ set

$$D_j(\alpha_i) = \left( \bigcap_{k=i+1}^j (f_k^{-1}(D_k)) \right) \cap (f_i^{-1}(\alpha_i)).$$

Then

(i) $f_i^k(D_j(\alpha_i)) \subset D_k(\alpha_i)$, $i < k \leq j$,

(ii) $D_i = \bigcup \{D_j(\alpha_i) : \alpha_i \in C_i, i < j\}$.

For every $\alpha_i \in C_i$ let

$$V(\alpha_i) = \left( U(\alpha_i) \cap X_i \right) \cup \left( \bigcup \{U(\alpha_j) \cap X_j : \alpha_j \in D_j(\alpha_i), i < j\} \right).$$

Let us show that

$$\mathcal{V} = \{V(\alpha_i) : \alpha_i \in C_i, i = 1, 2, \ldots\}$$

is an open covering of $X$ such that $\mathcal{V}$ refines $\mathcal{F}$ and $\text{ord} \mathcal{V} \leq n + 1$, which will prove $\dim X \leq n$.

Let $x$ be an arbitrary point of $X$. Since $X_0 = \emptyset$, there exists $i$ with $x \in X_i - \bigcup_{j < i} X_j$. Take $\alpha_i \in B_i$ with $x \in U(\alpha_i)$. When $\alpha_i \in C_i$, $x \in U(\alpha_i) \cap X_i \subset V(\alpha_i)$. When $\alpha_i \in D_i$, there exist $j < i$ and $\alpha_j \in C_j$ such that $\alpha_i \in D_j(\alpha_j)$. Then $x \in U(\alpha_i) \cap X_i \subset V(\alpha_j)$. Thus $\mathcal{V}$ is an open covering of $X$. 
Let \( i \) be an arbitrary positive integer and \( a_i \) an arbitrary index in \( C_i \). Since \( \emptyset \neq U(a_i) \cap X_i \subseteq V(a_i) \subseteq U(a_i) \), there exists \( \beta_i \in A_i \) such that \( U(\beta_i) \cap U(a_i) \cap X_i \neq \emptyset \) and \( St(U(\beta_i), U_i) \) refines \( \mathcal{I} \). Thus \( V(a_i) \) refines \( \mathcal{I} \) and hence \( \mathcal{V} \) refines \( \mathcal{I} \).

To prove \( \text{ord } \mathcal{V} \leq n + 1 \) assume the contrary. Then there exist a point \( x \) and \( n + 2 \) indices \( a_1, \ldots, a_{n+2} \) such that

(i) \( a_i \in C_m, i = 1, \ldots, n + 2, \)

(ii) \( x \in V(a_i), i = 1, \ldots, n + 2. \)

Let \( k \) be the smallest integer such that \( x \in X_k = \bigcup_{j < k} X_j \). Every \( m_i \) is less than or equal to \( k \). For every \( a_i \) there exist \( j(i) \) with \( j(i) \geq k \) and \( \beta_i \in D_{j(i)}(a_i) \) such that \( x \in U(\beta_i) \). Set \( \gamma^i = f_{k0}^i(\beta_i) \). Then (i) \( x \in U(\gamma^i) \), (ii) \( \gamma^i \in D_k(a_i) \) if \( m_i < k \) and (iii) \( \gamma^i = a_i \) if \( m_i = k \). Since \( \gamma^i, i = 1, \ldots, n + 2, \) are all different from one another by our construction, \( \text{ord } (x, U_k) \geq n + 2 \), a contradiction. Hence \( \text{ord } \mathcal{V} \leq n + 1 \) and the proof is finished.

6. Spaces \((Z_n, \sigma_i)\) illustrating the dependence of \( \mu \text{ dim } \) and \( d_2 \) on the metric.

**Lemma 11.** If \((X, \rho)\) is a metric space with \( \text{dim } X = n \), then there exists an equivalent metric \( \rho' \) to \( \rho \) such that \( d_2(X, \rho') = n \).

**Proof.** Since \( \text{dim } X = n \), there exists a defining system of \( n \) pairs \( C_1, C'_1; \ldots; C_n, C'_n \). Let \( f_1, \ldots, f_n \) be real-valued mappings of \( X \) such that

(i) \( 0 \leq f_i(x) \leq 1 \) for any \( i \) and any \( x \in X \),

(ii) \( f_i(x) = 0 \) for any \( i \) and any \( x \in C_i \),

(iii) \( f_i(x) = 1 \) for any \( i \) and any \( x \in C'_i \).

Set

\[
\rho'(x, y) = \rho(x, y) + \sum_{i=1}^{n} |f_i(x) - f_i(y)|.
\]

Then \( \rho' \) is an equivalent metric to \( \rho \) and \( \rho'(C_i, C'_i) > 0 \) for each \( i \). Thus we have \( d_2(X, \rho') \geq n \) and hence \( d_2(X, \rho') = n \).

**Construction of \( Z_n, n \geq 2 \).** Set \( m = \lfloor (n + 1)/2 \rfloor + 1 \). In every \( (I^i, \rho_i), i = m, m + 1, \ldots, n + 1 \), we construct \( (Y_i, \rho_i) \) as in the preceding section. Then \( \mu \text{ dim } (Y_i, \rho_i) \leq \lfloor i/2 \rfloor \leq \lfloor (n + 1)/2 \rfloor \) and \( \text{dim } Y_i = i - 1 \) for \( i = m, \ldots, n + 1 \). We assume here that \( \rho_i(I^i) \leq 1 \) for \( i = m, \ldots, n + 1 \). Take a metric \( \rho'_i \) equivalent to \( \rho_i \) as in Lemma 11 such that \( d_2(Y_i, \rho_i) = i - 1 \). Then \( \mu \text{ dim } (Y_i, \rho'_i) = d_2(Y_i, \rho_i) = i - 1 \) are automatically true for \( i = m, \ldots, n + 1 \). By the construction of \( \rho'_i \) in Lemma 11 \( \rho'_i \) satisfies \( \rho'_i(Y_i) \leq i + 1 \).

\( Z_n \) is merely the disjoint sum of \( Y_m, Y_{m+1}, \ldots, Y_{n+1} \). The topology of \( Z_n \) is defined in such a way that a subset \( G \) of \( Z_n \) is open if and only if \( G \cap Y_i \) is open in \( Y_i \) for \( i = m, \ldots, n + 1 \). Then \( Z_n \) is a metric space. Define for \( i = m, \ldots, n + 1 \) the metrics \( \sigma_i \) of \( Z_n \) as follows:

(i) \( \sigma_i Y_j = \rho_j \) if \( i \neq j \).

(ii) \( \sigma_i Y_i = \rho_i \).

(iii) \( \sigma_i(x, y) = n + 2 \) if for any \( j = m, \ldots, n + 1, x \) and \( y \) are not in the same \( Y_j \), \( \sigma_m, \ldots, \sigma_{n+1} \) are equivalent metrics which give the preassigned topology of \( Z_n \).
Assertion 1. \( \dim Z_n = n \).

**Proof.** \( \dim Z_n = \max \{ \dim Y_i : i = m, \ldots, n+1 \} = n \).

Assertion 2. \( \mu \dim (Z_n, \sigma_i) = d_2(Z_n, \sigma_i) = d_3(Z_n, \sigma_i) = i-1 \) for \( i = \left(\frac{n+1}{2}\right) + 1, \ldots, n+1 \).

**Proof.** If \( j \neq i \), then \( \mu \dim (Y_j, \sigma_i) = \mu \dim (Y_j, \rho_j) \leq \left(\frac{n+1}{2}\right) \). Since

\[
\mu \dim (Y_i, \sigma_i) = d_2(Y_i, \sigma_i) = \mu \dim (Y_i, \rho_i) = i-1 \geq \left(\frac{n+1}{2}\right),
\]

we have

\[
i-1 = d_2(Z_n, \sigma_i) \leq d_3(Z_n, \sigma_i) = \mu \dim (Z_n, \sigma_i)
\]

\[
= \max \{ \mu \dim (Y_m, \rho_m), \ldots, \mu \dim (Y_{i-1}, \rho_{i-1}), \mu \dim (Y_i, \rho_i), \\
\mu \dim (Y_{i+1}, \rho_{i+1}), \ldots, \mu \dim (Y_{n+1}, \rho_{n+1}) \} = i-1.
\]

Thus the assertion is proved.

7. A space \( (R, \rho) \) with \( d_2=2, \mu \dim = 3, \dim = 4 \).

First let us construct a space \( (S, \sigma) \) with \( d_2(S, \sigma)=2 \) and \( \mu \dim (S, \sigma)=\dim S=3 \).

**Construction of** \( (S, \sigma) \). \( (S, \sigma) \) will be a subset of \( \mathbb{R}^4 = \{ (x_1, \ldots, x_4) : 0 \leq x_i \leq 1, i = 1, \ldots, 4 \} \), where \( \sigma \) is Euclidean metric on \( \mathbb{R}^4 \). Let \( C_{ij}, C'_{ij}, i=1, 2, \ldots, j=1, 2, 3, \) be disjoint pairs of closed sets of \( \mathbb{R}^4 \) such that for any three disjoint pairs of closed sets \( C_1, C_2, C_3, C'_{ij}, C'_{ij}, C'_{ij}, \) there exists \( i \) with \( C_i \subset C_{ij} \) and \( C_i \subset C'_{ij} \) for \( j=1, 2, 3 \). Let \( \pi \) be a prime number with \( 5 \leq \pi \). Consider an open covering \( \mathcal{D}(\pi) \) of the unit interval \([0, 1]\) consisting of overlapping intervals \([0, 2/\pi), (\pi-2)/\pi, 1]\) and \([(2k-1)/\pi, (2k+2)/\pi), k=1, \ldots, (\pi-3)/2\]. Define an open covering \( \mathcal{E}(\pi) \) of \( I^4 \) as follows:

\[
\mathcal{E}(\pi) = \{ D_1 \times D_2 \times D_3 \times D_4 : D_1, \ldots, D_4 \in \mathcal{D}(\pi) \} = \{ E_\lambda : \lambda \in \Lambda(\pi) \}.
\]

Let \( \pi_{ij}, i=1, 2, \ldots, j=1, 2, 3, \) be prime numbers which are different from each other and satisfy the following conditions:

(i) \( 5 \leq \pi_{ij} \) for every \( i \) and \( j \).

(ii) \( \max \{ \text{mesh} \mathcal{E}(\pi_{ij}) : j=1, 2, 3 \} < \min \{ \sigma(C_{ij}, C'_{ij}) : j=1, 2, 3 \} \) for every \( i \).

Let \( U_{ij} \) be the sum of all elements of \( \mathcal{E}(\pi_{ij}) \) which meet \( C_{ij} \). Set \( B_{ij} = U_{ij} - U_{ij} \) and \( B_i = \bigcap_{j=1}^{3} B_{ij} \). Then \( B_{ij} \) separates \( C_{ij} \) and \( C'_{ij} \). Set

\[
S = I^4 - \bigcup B_i.
\]

Then \( (S, \sigma) \) satisfies the required equalities.

**Assertion 1.** \( B_i \cap B_k = \emptyset \) if \( i \neq k \).

**Proof.** Set

\[
L_{ij} = \{ a/\pi_{ij} : a = 1, \ldots, \pi_{ij}-1 \}.
\]

Then \( L_{ij} \cap L_{kl} = \emptyset \) if and only if \( i=k \) and \( j=l \). If \( x=(x_1, \ldots, x_4) \) is a point of \( B_{ij} \), then for some \( t, x_i \in L_{ij} \). Hence \( B_i \cap B_k = \emptyset \) if \( i \neq k \).
Assertion 2. $B_t$ does not meet the 2-dimensional edge of $I^s$. $B_t$ meets the surface of $I^s$ at only a finite number of points. $B_t$ is the sum of a finite number of segments.

This is evident from the above observation.

Assertion 3. $B_t$ is the disjoint sum of a finite number of simple closed curves and a finite number of simple arcs.

**Proof.** If three different lines $l_1, l_2, l_3$ lying in $B_t$ have a common point, then they lie in some hyperplane $H : x_1 = \text{constant}$. Since $H$ is 3-dimensional, it is now easy to see that $H \cap B_t$ cannot contain $l_1, l_2, l_3$ at the same time because (i) $\mathcal{E}(\pi_{ij}) | H = 1, 2, 3$, are collections of bordered blicks and (ii) $\pi_{ij}, j = 1, 2, 3$, are different from each other.

Assertion 4. $d_2(S, \sigma) = 2$ and $\dim S = 3$.

The first equality was proved in §4. As for the second equality see Remark 3.

Assertion 5. $\mu \dim (S, \sigma) = 3$.

**Proof.** To show $\mu \dim (S, \sigma) \geq 2$, assume that $\mu \dim (S, \sigma) \leq 2$. Then there exists a finite closed (in $S$) covering $\mathcal{F} = \{F\}$ of $S$ which satisfies the following conditions:

(i) $\{G(F)\} = \text{interior of } F$ with respect to $S : F \in \mathcal{F}$ covers $S$.

(ii) mesh $\mathcal{F} < 1$.

(iii) order $\mathcal{F} \leq 3$.

The proof for the existence of such $\mathcal{F}$ is left to the reader. Cf. Lemma 7 and also use the total boundedness of $(S, \sigma)$. Set

$$\mathcal{F}_1 = \{F : F \in \mathcal{F}, \bar{F} \cap \{x : x_1 = 0\} \neq \emptyset\},$$

$M_1$ = boundary in $I^s$ of $\bigcup \{\bar{F} : F \in \mathcal{F}\}$.

Let $F$ be an arbitrary element of $\mathcal{F}_1$. Let $G'$ be an open set of $I^s$ with $G' \cap S = G(F)$.

Since $\dim \bigcup B_t = 1$, $S$ is dense in $I^s$. Hence $G' - \bar{F} \neq \emptyset$ yields $(G' - \bar{F}) \cap S \neq \emptyset$, a contradiction. Thus $G' \subset \bar{F}$, which implies $G(F) \cap M_1 = \emptyset$. Take an arbitrary point $x$ from $M_1$ in $S$. Since $x \notin G(F)$ for any $F$ in $\mathcal{F}_1$, there exists an element $F_0 \in \mathcal{F} - \mathcal{F}_1$ such that $x \in G(F_0)$ by the condition (i) imposed upon $\mathcal{F}$. Hence

$$\text{ord } \mathcal{F}_1 | M_1 \cap S \leq \text{ord } \mathcal{F} - 1 \leq 2.$$  

Set

$$\mathcal{F}_2 = \{F : F \in \mathcal{F}_1, \bar{F} \cap \{x : x_2 = 0\} \neq \emptyset\},$$

$M_2$ = boundary in $M_1$ of $\bigcup \{\bar{F} \cap M_1 : F \in \mathcal{F}_2\}$.

Take an arbitrary point $x'$ from $M_2 \cap S$. Let $y^1, y^2, \ldots$ be a sequence of points of $M_1 - \bigcup \{\bar{F} \cap M_1 : F \in \mathcal{F}_2\}$ with $\lim y^i = x'$. Since $\mathcal{F}_2$ is finite and $\mathcal{F}_2 = \{\bar{F} : F \in \mathcal{F}_1\}$ covers $M_1$, we assume here without loss of generality that the sequence $\{y^i\}$ is contained in one $\mathcal{F}_1$ with $F_1 \in \mathcal{F}_1 - \mathcal{F}_2$. For any $i$ let $z'$ be a point of $F_1$ with $\sigma(y^i, z') < \sigma(y^i, x')$. Since $\lim z' = x', x' \in F_1$. Therefore

$$\text{ord } \mathcal{F}_2 | M_2 \cap S \leq \text{ord } \mathcal{F}_1 | M_1 \cap S - 1 \leq 1.$$  

Set

$$\mathcal{F}_3 = \{F : F \in \mathcal{F}_2, \bar{F} \cap \{x : x_3 = 0\} \neq \emptyset\},$$

$M_3$ = boundary in $M_2$ of $\bigcup \{\bar{F} \cap M_2 : F \in \mathcal{F}_3\}$. 


Since \( \text{ord} \mathcal{F}_2 | M_2 \cap S = \text{ord} \mathcal{F}_2 | M_2 \cap S \leq 1 \),
\[
M_3 \cap S = \emptyset.
\]
Set
\[
T = \{x : x \in M_2, \text{ord} (x, \mathcal{F}_2) \geq 2\}.
\]
Then \(T\) is a closed set of \(I^4\) such that
\[
M_3 \subset T \subset M_2 \cap (\bigcup B_i).
\]
Let \(K_1\) and \(K_2\) be mutually separated relatively open sets of \(M_2\) such that
\[
M_2 - M_3 = K_1 \cup K_2,
\]
\[
K_1 = M_2 \cap \{x : x_3 = 0\},
\]
\[
K_2 = M_2 \cap \{x : x_3 = 1\}.
\]
Let \(P, P', Q\) or \(Q'\) be the union of all components of \(T\) which meet \(\{x : x_3 = 0\}\), \(\{x : x_3 = 1\}\) or \(\{x : x_4 = 0\}\) or \(\{x : x_4 = 1\}\), respectively. Then these four sets are closed.
Let us show for instance \(P\) is closed. Let \(x^0\) be an arbitrary point of the closure of \(P\) and \(C_1, C_2, \ldots\) a sequence of components of \(T\) such that
(i) each \(C_i\) intersects \(\{x : x_3 = 0\}\),
(ii) each \(C_i\) contains a point \(z_i\) with \(\lim z_i = x^0\).
Since \(x^0 \in \liminf C_i\), \(\limsup C_i\) is connected by [6, Theorem 2-101]. Since \(\limsup C_i\) intersects \(\{x : x_3 = 0\}\) and \(\limsup C_i \subset T\), \(\limsup C_i \subset P\). Especially \(x^0 \in P\) and hence \(P\) is closed.
By Assertions 2 and 3 \(P \cup P'\) and \(Q \cap Q'\) are disjoint closed sets of \(T\) such that there is no continuum in \(T\) between them. Hence by Lemma 2 there exists a subset \(V\) of \(T\) such that
(i) \(V\) is open and closed in \(T\),
(ii) \(Q \cap Q' \subset V\),
(iii) \(V \cap (P \cup P') = \emptyset\).
Since \(Q \cap Q' \cap \{x : x_3 = 0, 1\} = \emptyset\), there exists a subset \(W\) of \(M_2\) such that
(i) \(W\) is open in \(M_2\),
(ii) \(W \cap T = V\),
(iii) \(W \cap \{x : x_3 = 0, 1\} = \emptyset\).
Then
\[
(W - W) \cap T = \emptyset,
\]
\[
Q \cap Q' \subset W,
\]
\[
W \cap (P \cup P' \cup \{x : x_3 = 0, 1\}) = \emptyset.
\]
Set
\[
M = (M_3 - W) \cup (W - W),
\]
\[
G_1 = K_1 - W,
\]
\[
G_2 = (K_2 \cup W) - (W - W).
\]
Then

\[ M_2 - M = G_1 \cup G_2, \]
\[ G_1 \cap G_2 = \emptyset, \]
\[ G_1 \ni M_2 \cap \{ x : x_3 = 0 \}, \]
\[ G_2 \ni M_2 \cap \{ x : x_3 = 1 \}. \]

Since \( G_1 \) and \( G_2 \) are open in \( M_2 \), \( M \) separates \( M_2 \cap \{ x : x_3 = 0 \} \) and \( M_2 \cap \{ x : x_3 = 1 \} \) in \( M_2 \).

Let us show that no component of \( M \) meets both \( \{ x : x_4 = 0 \} \) and \( \{ x : x_4 = 1 \} \).

Take an arbitrary element \( F \) from \( S_2 \). Set

\[ U(F) = A_72 - \bigcup \{ F' : F' \in S_2, F' \neq F \}. \]

Then \( \{ U(F) : F \in S_2 \} \) is a disjoint collection of open sets of \( M_2 \). Since

\[ M_2 - T = \bigcup \{ U(F) : F \in S_2 \} \]

and \( (W - W) \cap T = \emptyset \), \( W - W \) is the sum of the disjoint collection:

\[ \mathcal{H} = \{ (W - W) \cap U(F) = H(F) : F \in S_2 \}. \]

Since

\[ H(F) = (W - W) - \bigcup \{ U(F') : F' \neq F, F' \in S_2 \}, \]

\( H(F) \) is closed and hence \( \mathcal{H} \) is a disjoint collection of closed sets. Since

\[ \text{mesh} \mathcal{H} \leq \text{mesh} S_2 < 1, \]

no \( H(F) \) meets both \( \{ x : x_4 = 0 \} \) and \( \{ x : x_4 = 1 \} \). Now \( M \) is the sum of the disjoint closed sets:

\[ M \cap B_i, \quad i = 1, 2, \ldots, H(F) \in \mathcal{H}. \]

By our construction no \( M \cap B_i \) meets both \( \{ x : x_4 = 0 \} \) and \( \{ x : x_4 = 1 \} \) since \( Q \cap Q' \cap M = \emptyset \). Therefore no component of \( M \) meets both \( \{ x : x_4 = 0 \} \) and \( \{ x : x_4 = 1 \} \) by Lemma 3.

Consider the closed set:

\[ X = \{ x : x_4 = 0, 1 \} \cup M. \]

Let \( X_1 \) be the sum of \( \{ x : x_4 = 0 \} \) and all components of \( M \) which meet \( \{ x : x_4 = 0 \} \). Let \( X_2 \) be the sum of \( \{ x : x_4 = 1 \} \) and all components of \( M \) which meet \( \{ x : x_4 = 1 \} \). Then \( X_1 \) and \( X_2 \) are closed by the same argument as in the proof for the closedness of \( P \). With the aid of Lemma 2 we can find a closed set \( N \) of \( M_2 \) which separates \( \{ x : x_4 = 0 \} \) and \( \{ x : x_4 = 1 \} \) such that \( N \cap M = \emptyset \). Thus two pairs of opposite sides of \( M_2 \) are not defining, which shows in turn three pairs of opposite sides of \( M_1 \) are not defining as can easily be seen. At last four pairs of opposite faces of \( I^4 \) are not defining, a contradiction. Hence \( 2 < \mu \dim (S, o) \). Since

\[ \mu \dim (S, o) \leq \dim S = 3, \quad \mu \dim (S, o) = 3. \]
Assertion 6. $d_3(S, o) = 3$.

Proof. Since $o$ is totally bounded, $d_3(S, o) = \mu \dim (S, o) = 3$ by Theorem 5.

Construction of $(R, \rho)$. Take the space $(Z_4, \sigma_3)$ constructed in the preceding section. Then $\dim Z_4 = 4$ and $d_2(Z_4, \sigma_3) = d_3(Z_4, \sigma_3) = \mu \dim (Z_4, \sigma_3) = 2$. $R$ is the disjoint union of $Z_4$ and $S$ just constructed. The metric $\rho$ on $R$ is defined as follows:

$$
\begin{align*}
\rho|_{Z_4} &= \sigma_3, \\
\rho|_S &= \sigma, \\
\rho(x, y) &= \max\{\sigma_3(Z_4), \sigma(S)\} \quad (\leq 6) \text{ if } \{x, y\} \text{ is contained in neither } Z_4 \text{ nor } S.
\end{align*}
$$

Then it is evident that $d_2(R, \rho) = 2$, $d_3(R, \rho) = \mu \dim (R, \rho) = 3$ and $\dim R = 4$.

8. Problems.

Problem 1. Is it true that $\dim X \leq 2d_2(X, \rho)$ for all (separable) metric spaces $(X, \rho)$?

Problem 2. Let $(X, \rho)$ be a metric space with $d_2(X, \rho) < \dim X$ and $k$ an arbitrary integer with

$$
d_2(X, \rho) \leq k \leq \dim X.
$$

Can $X$ allow an equivalent metric $\sigma$ with $d_2(X, \sigma) = k$?

Remark 5. Recently Roberts and his student Slaughter solved a problem analogous to Problem 2 for the case when $d_2$ is replaced by $\mu \dim$. (Added in proof. This paper has been accepted for publication in Fundamenta Mathematicae.)

Problem 3. Find a necessary and sufficient condition on $X$ with which $d_2(X, \rho)$ (or $\mu \dim (X, \rho)$) = $\dim X$ for any metric $\rho$ agreeing with the preassigned topology of $X$.

Remark 6. It is reported by Alexandroff [1] that K. Sitnikov got a sufficient condition: If $X$ is a subset of the $n$-dimensional Euclidean space $(R^n, \rho)$ such that $\dim X = \dim \overline{X}$, then $\mu \dim (X, \rho) = \dim X$.

Problem 4. Is there a space $(X, \rho)$ with $d_3(X, \rho) < \mu \dim (X, \rho)$?

References