M-SEMIREGULAR SUBALGEBRAS IN HYPERFINITE FACTORS

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1. **Introduction.** The general study of algebras of operators on Hilbert space has led to the investigation of rings of operators, also called W^*-algebras or von Neumann algebras. If the center of a ring (*center* in the algebraic sense) consists only of scalar multiples of the identity, then the ring is a factor. Since every ring can be decomposed into factors [6], the study of rings is, in a sense, reduced to a study of factors. In this paper we are concerned with the maximal abelian subalgebras of type II_1 factors, or continuous factors which have a finite trace defined on them [2]. For the present, we restrict ourselves to the study of hyperfinite factors, that is, those which are generated by a sequence of factors $\mathfrak{M}_n$ of type I_n, with $\mathfrak{M}_{n_1} \subsetneqq \mathfrak{M}_{n_2} \subsetneqq \cdots$. (The factor $\mathfrak{M}_n$ is isomorphic to the algebra of n by n matrices.) Since all hyperfinite factors are algebraically isomorphic [5, §4.7], while the concept of a subring of a finite factor is purely algebraic [5, §1.6], any construction used will yield general results.

Dixmier has defined three types of maximal abelian subalgebras $\boldsymbol{R}$ in a factor $\mathfrak{A}$: regular, semiregular, and singular [3]. These depend on the properties of $N(\boldsymbol{R})$, the ring generated by $\{V : V\boldsymbol{R}V^*=\boldsymbol{R},\ V \text{ unitary},\ V\in\mathfrak{A}\}$. In other words, $N(\boldsymbol{R})$ is the normalizer of $\boldsymbol{R}$ in $\mathfrak{A}$. Later, Anastasio defined an additional type, M-semiregular ($M=1, 2, 3, \ldots$), which coincides with the semiregular type when $M=1$. Extending the notation $N(\boldsymbol{D})$ to any subring $\boldsymbol{D}\subset\mathfrak{A}$, and letting $N^j(\boldsymbol{D}) = N[N^{j-1}(\boldsymbol{D})]$, we have a chain $\boldsymbol{R}\subsetneqq N(\boldsymbol{R})\subsetneqq N^2(\boldsymbol{R})\subsetneqq \cdots \subsetneqq N^t(\boldsymbol{R})=\mathfrak{A}$. We say that a maximal abelian subalgebra $\boldsymbol{R}$ is *M-semiregular* if $N^k(\boldsymbol{R})$ is not a factor for $k<M$, but $N^M(\boldsymbol{R})$ is a factor [1]. Anastasio constructed infinite sequences of non-isomorphic 2-semiregular and 3-semiregular subalgebras in a hyperfinite factor. (The 1-semiregular case had already been done [7].) In this paper we propose to show the existence of M-semiregular subalgebras for every positive integer $M\neq 1$.

We use the notation and results of [7]. Let $\mathfrak{M}_p$ be the full 2^p by 2^p matrix algebra over the complex numbers, and $\{{}^pE_{ij} : i, j=0, 1, \ldots, 2^p-1\}$ the matrix units which generate it. By letting ${}^pE_{ij}={}^{p+1}E_{2i,2j}+{}^{p+1}E_{2i+1,2j+1}$, we imbed $\mathfrak{M}_p$ in $\mathfrak{M}_{p+1}$. Then $\bigcup_{p=1}^{\infty}\mathfrak{M}_p=\mathfrak{M}$ is a $*$-algebra. The normalized matrix trace on $\mathfrak{M}$ makes it into a pre-Hilbert space $\mathfrak{H}$: If $A, B\in\mathfrak{M}$, let $(A, B)=\mathrm{Tr}\,(B^*A)$, so that $(A, A)^{1/2}=[\![A]\!]$, the Hilbert space or metric norm of A. If A is in $\mathfrak{M}$, then A acting

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by left multiplication is a bounded operator on $\mathfrak{H}$, so it can be extended to the Hilbert space closure $\mathscr{H}$. If $\mathfrak{A}$ is the weak closure of $\mathfrak{M}$, then it is well known that $\mathfrak{A}$ is a hyperfinite factor [2].

2. ***M*-semiregular subalgebras.** The following general construction leads to a large variety of maximal abelian subalgebras of $\mathfrak{A}$.

DEFINITIONS 2.1. Let $\{U_t : t=1, 2, \ldots\}$ be a set of selfadjoint unitaries such that: (1) $U_t \in \mathfrak{M}_t$; (2) U_t is zero except for 2 by 2 blocks along the main diagonal. Let $Y_t = U_1 U_2 \cdots U_t$, and for $A \in \mathfrak{A}$, define $A^{(t)} = Y_t A Y_t^*$ and $A^{[t]} = Y_t^* A Y_t$. For fixed t, the mappings $A \to A^{(t)}$ and $A \to A^{[t]}$ are *-automorphisms of $\mathfrak{A}$ and inverses of each other. Because of the form of U_t, the matrix unit ${}^pE_{jj}$ commutes with U_t for all $t>p$. Thus if A is a diagonal matrix in $\mathfrak{M}_p$, $p \leqq t$, then $A^{(t)} = A^{(t+1)}$, and so $\lim_{t\to\infty} A^{(t)} = A^{(\infty)}$ exists in $\mathfrak{M}$, hence in $\mathfrak{A}$.

In general, for $A \in \mathfrak{A}$, the limit $A^{(\infty)}$ does not exist. The mapping $A \to A^{(\infty)}$ is thus an isomorphism of some proper subalgebra of $\mathfrak{A}$ into $\mathfrak{A}$. This subalgebra, the domain of the mapping, we call $\mathfrak{D}$. If $\boldsymbol{E}$ is the set of diagonal matrices, then $\boldsymbol{E} \subset \mathfrak{D}$, as seen above. The ring $(\boldsymbol{E}^{(\infty)})^-$ is the maximal abelian subalgebra $\boldsymbol{R}$ which we study in this paper. (Cf. [7, pp. 285–286], for the proof that $\boldsymbol{R}$ is maximal abelian.) In Lemma 2.2 we will show that $\boldsymbol{E}^- \subset \mathfrak{D}$, and that $(\boldsymbol{E}^-)^{(\infty)} = (\boldsymbol{E}^{(\infty)})^-$ or $\boldsymbol{R}$.

LEMMA 2.2. *If $\boldsymbol{F} = \boldsymbol{E}^-$, then $\boldsymbol{F} \subset \mathfrak{D}$, and $\boldsymbol{F}^{(\infty)} = (\boldsymbol{E}^{(\infty)})^- = \boldsymbol{R}$.*

Proof. Suppose $A \in \boldsymbol{F}$. Then there is a sequence $A_n \in \boldsymbol{E} \cap \mathfrak{M}_n$, $A_n \to A$, with $A_n^{(\infty)} \in \mathfrak{M}$. Let $\varepsilon > 0$ be given, and choose n such that $[\![A_n - A]\!] < \varepsilon/2$. Consider

$$\begin{aligned}[\![A^{(s)} - A^{(t)}]\!] &= [\![Y_s A Y_s^* - Y_t A Y_t^*]\!] \\ &\leqq [\![Y_s A Y_s^* - Y_s A_n Y_s^*]\!] + [\![Y_s A_n Y_s^* - Y_t A_n Y_t^*]\!] + [\![Y_t A_n Y_t^* - Y_t A Y_t^*]\!].\end{aligned}$$

Choose s, t such that both are greater than or equal to n. Then $Y_s A_n Y_s^* = A_n^{(s)} = A_n^{(n)}$ and $Y_t A_n Y_t^* = A_n^{(t)} = A_n^{(n)}$. Hence $[\![Y_s A_n Y_s^* - Y_t A_n Y_t^*]\!] = 0$ if $s, t \geqq n$. Since Y_s and Y_t are unitary, the other two norms equal $[\![A - A_n]\!]$, and so the sum is less than ε. Therefore $A^{(t)}$ is Cauchy in the metric topology.

Now $A \in \mathfrak{A}$ and so $\|A\| < \infty$. Since $\|A^{(t)}\| = \|A\|$, $A^{(t)}$ is a bounded sequence. By [5, p. 723], $A^{(t)}$ is then Cauchy in the strong topology also, so its limit exists in $\mathfrak{A}$. Therefore $\boldsymbol{F} \subset \mathfrak{D}$.

We next show that $\boldsymbol{F}^{(\infty)} = (\boldsymbol{E}^{(\infty)})^-$ or $\boldsymbol{R}$. Let $A \in \boldsymbol{F}$, $A_n \in \boldsymbol{E} \cap \mathfrak{M}_n$, $A_n \to A$. Let $\varepsilon > 0$ be given, and choose n so that both $[\![A_n - A]\!] < \varepsilon/2$ and $[\![A^{(n)} - A^{(\infty)}]\!] < \varepsilon/2$. Then

$$\begin{aligned}[\![A_n^{(\infty)} - A^{(\infty)}]\!] &\leqq [\![A_n^{(\infty)} - A_n^{(n)}]\!] + [\![A_n^{(n)} - A^{(n)}]\!] + [\![A^{(n)} - A^{(\infty)}]\!] \\ &= [\![A_n^{(n)} - A_n^{(n)}]\!] + [\![A_n - A]\!] + [\![A^{(n)} - A^{(\infty)}]\!] \\ &\leqq 0 + \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

Therefore $A^{(\infty)} \in (\boldsymbol{E}^{(\infty)})^-$, and so $\boldsymbol{F}^{(\infty)} \subset \boldsymbol{R}$.

On the other hand, if $G \in \boldsymbol{R}$, there is a sequence $A_n \in \boldsymbol{E} \cap \mathfrak{M}_n$, $A_n^{(\infty)} \to G$, with $\|A_n^{(\infty)}\| = \|A_n^{(n)}\| = \|A_n\| \leqq \|G\|$ [4]. Since $A_n^{(\infty)}$ is metrically Cauchy, so is A_n, which is also strongly Cauchy because of the bound on the norm. Hence A_n has a limit $A \in \boldsymbol{F}$. By another standard argument, if $\varepsilon > 0$ be given, there exists N such that $[\![Y_t A Y_t^* - G]\!] < \varepsilon$ when $t \geqq N$. Therefore $G = \lim_{t\to\infty} Y_t A Y_t^*$ and so $G \in \boldsymbol{F}^{(\infty)}$. Hence $\boldsymbol{R} \subset \boldsymbol{F}^{(\infty)}$, and $\boldsymbol{F}^{(\infty)} = \boldsymbol{R}$.

REMARK. The normalizer of $\boldsymbol{R}$ in $\mathfrak{A}$ results in a similar way from the mapping $A \to A^{(\infty)}$. The subalgebra $r(\mathscr{C}_0^D)$, to be defined later, has the property that $\boldsymbol{E} \subset r(\mathscr{C}_0^D) \subset \mathfrak{D}$, and $r(\mathscr{C}_0^D)^{(\infty)} = N(\boldsymbol{R})$. (It appears that $r(\mathscr{C}_0^D) = \mathfrak{D}$, but we do not need this fact and have not proved it.)

By means of various choices of the sequence $\{U_t\}$, in §3 we construct a maximal abelian subalgebra $\boldsymbol{R}_n$ for each $n = 1, 2, 3, \ldots$, where $\boldsymbol{R}_n$ is M-semiregular, $M = n+1$. The chain $\boldsymbol{R}_n \subsetneqq N(\boldsymbol{R}) = \boldsymbol{P}_n \subsetneqq N^2(\boldsymbol{R}_n) \subsetneqq \cdots \subsetneqq N^{n+1}(\boldsymbol{R}_n) = N^M(\boldsymbol{R}_N) = \mathfrak{A}$ is such that $N^k(\boldsymbol{R}_n)$ is not a factor for $k = 1, 2, \ldots, n < M$, while $N^M(\boldsymbol{R}_n)$ is the factor $\mathfrak{A}$.

Furthermore, the subalgebras $\boldsymbol{R}_n$ are not conjugate under any *-automorphism of $\mathfrak{A}$. The integer n determines the number of normalizers between $\boldsymbol{R}_n$ and $\mathfrak{A}$ in the chain, and this is an automorphism invariant (cf. [7, pp. 282 and 305]).

Note. For convenience of notation, we often work with $N^k(\boldsymbol{P}_n) = N^{k+1}(\boldsymbol{R}_n)$, $k = 0, 1, \ldots, n$.

3. **Detailed construction of M-semiregular subalgebras.** In the construction of M-semiregular subalgebras, we use the following notations and definitions.

DEFINITIONS 3.1. We regard $n = 1, 2, 3, \ldots$ as fixed, and let

$$\Gamma = \{p : p = (3c+1)n,\ c = 0, 1, 2, \ldots\},$$

an infinite set of positive integers. We define $\mathscr{C}_n = \{{}^pE_{rs} : p \in \Gamma\}$. In the following paragraphs, we define a decomposition of $\mathscr{C}_n$ into 2^n disjoint subsets K_γ $(0 \leqq \gamma \leqq 2^n - 1)$, so that $\mathscr{C}_n = \bigcup_\gamma K_\gamma$.

Let $\mathfrak{G}_n$ be the set of all n-tuples $(a_1, a_2, \ldots, a_n)$, where $a_k = 0$ or 1. This is a commutative group under the operation of coordinate-wise addition modulo 2. If $\gamma = 0, 1, \ldots, 2^n - 1$ and $\gamma = \sum_{j=1}^n a_j 2^{n-j}$, we identify it with its binary expansion $(a_1, a_2, \ldots, a_n)$, so that we can consider γ as an element of $\mathfrak{G}_n$. The sum $\gamma_1 + \gamma_2$ is then defined by addition in $\mathfrak{G}_n$.

We determine the set K_γ in which ${}^pE_{rs}$ is contained as follows: For any index r $(0 \leqq r \leqq 2^{(3c+1)n} - 1)$, let $r = \sum_{k=0, k\not\equiv 2}^{3c} r_k 2^{kn}$. (Congruence is modulo 3 in this and in the following summations.) For $k \equiv 0$, we have $0 \leqq r_k < 2^n$, and so $r_k = \sum_{j=1}^n k_j 2^{n-j}$ with $(k_1, \ldots, k_n) \in \mathfrak{G}_n$. Designate this element of $\mathfrak{G}_n$ by $\psi(r_k)$. For $k \equiv 1$, $0 \leqq r_k < 2^{2n}$, and we let $\sigma(r_k) = 2(r_k \bmod 2^{n-1})$, so that $\psi(\sigma(r_k))$ is defined. Let

$$\Delta(r) = \sum_{k\equiv 0; k=0}^{3c} \psi(r_k) + \sum_{k\equiv 1; k=1}^{3c-2} \psi(\sigma(r_k)),$$

where the addition is coordinate-wise (mod 2), so that $\Delta(r) \in \mathfrak{G}_n$. Then $K_\gamma = \{{}^pE_{rs} : \Delta(r) + \Delta(s) = \gamma\}$ and we say that $K_\gamma = K({}^pE_{rs})$. Since this is independent of p, we sometimes write $K_\gamma = K(r, s)$.

DEFINITIONS 3.2. We also define the following sets of matrix units, again subsets of $\mathscr{C}_n$: $\mathscr{C}_0=\mathscr{N}_0=K_0$. For $j=1,2,\ldots,n$, $\mathscr{C}_j=\bigcup_\gamma \{K_\gamma : \gamma=(a_1,\ldots,a_j,0,0,\ldots,0)\}$ and $\mathscr{N}_j=\mathscr{C}_j\sim\mathscr{C}_{j-1}$. If we let

$$\mathfrak{M}^D = {}^1E_{00}\mathfrak{M}{}^1E_{00}+{}^1E_{11}\mathfrak{M}{}^1E_{11},$$

then we define $\mathscr{C}_j^D=\mathscr{C}_j\cap\mathfrak{M}^D$ and $\mathscr{N}_j^D=\mathscr{N}_j\cap\mathfrak{M}^D$, while $\mathscr{C}_j'=\mathscr{C}_j\sim\mathscr{C}_j^D$ and $\mathscr{N}_j'=\mathscr{N}_j\sim\mathscr{N}_j^D$. We let $r(\mathscr{C}_j^D)$ be the ring generated by the matrix units in $\mathscr{C}_j^D$, while $R(\mathscr{C}_j^D)$ is the ring generated by $\{F : F=({}^pE_{rs})^{(p)}$ with ${}^pE_{rs}\in\mathscr{C}_j^D\}$.

LEMMA 3.3. *Suppose* $p\in\Gamma$, ${}^{p+3n}E_{rs}\in K_\gamma$. *Let* $r=r'2^{3n}+r_12^n+r_0$ *and* $s=s'2^{3n}+s_12^n+s_0$ $(0\leqq r_1,s_1<2^{2n},\ 0\leqq r_0,s_0<2^n)$. *Then*

$$\gamma = \Delta(r)+\Delta(s) = (\Delta(r')+\Delta(s'))+\sigma+(\Delta(r_0)+\Delta(s_0)),$$

where $\sigma=\psi(\sigma(r_1))+\psi(\sigma(s_1))$.

Proof. This follows by computation from Definitions 3.1, since $\Delta(r)$ can be written as $\Delta(r')+\psi(\sigma(r_1))+\psi(r_0)$, and the same for $\Delta(s)$.

Construction 3.4. In constructing the maximal abelian subalgebra $\boldsymbol{R}_n$ according to §2.1, the sequence $\{U_t : t=1,2,3,\ldots\}$ is to be as follows: Let

$$B_1 = \begin{bmatrix} 2^{-1/2} & 2^{-1/2} \\ 2^{-1/2} & -2^{-1/2} \end{bmatrix}.$$

Let B_t be in $\mathfrak{M}_t$, with all entries zero except for 2 by 2 blocks like B_1 along the main diagonal.

For $n>1$, $U_t=I$ if $t<n$. If $p\in\Gamma$ and if $\Delta(r)=(a_1,a_2,\ldots,a_n)$, define:

$$\begin{aligned}
{}^pE_{rr}U_{p+1} &= {}^pE_{rr} && \text{if } a_n = 0,\\
&= {}^pE_{rr}B_{p+1} && \text{if } a_n = 1,\\
&\ \vdots\\
{}^pE_{rr}U_{p+n-j+1} &= {}^pE_{rr} && \text{if } a_j = 0,\\
&= {}^pE_{rr}B_{p+n-j+1} && \text{if } a_j = 1,\\
&\ \vdots\\
U_{p+n+1} &= I.\\
{}^pE_{rr}U_{p+n+2} &= {}^pE_{rr} && \text{if } a_2 = 0,\\
&= {}^pE_{rr}B_{p+n+2} && \text{if } a_2 = 1,\\
&\ \vdots\\
{}^pE_{rr}U_{p+n+j} &= {}^pE_{rr} && \text{if } a_j = 0,\\
&= {}^pE_{rr}B_{p+n+j} && \text{if } a_j = 1,\\
&\ \vdots\\
{}^pE_{rr}U_{p+2n} &= {}^pE_{rr} && \text{if } a_n = 0,\\
&= {}^pE_{rr}B_{p+2n} && \text{if } a_n = 1,\\
U_{p+2n+1} &= \cdots = U_{p+3n-2} = I,\\
{}^1E_{00}U_{p+3n-1} &= {}^1E_{00},\\
{}^1E_{11}U_{p+3n-1} &= {}^1E_{11}B_{p+3n-1},\\
U_{p+3n} &= I.
\end{aligned}$$

For $n=1$, $p \in \Gamma$, and if $\Delta(r) = (a_1)$, define:

$$\begin{aligned} {}^pE_{rr}U_{p+1} &= {}^pE_{rr} && \text{if } a_1 = 0, \\ &= {}^pE_{rr}B_{p+1} && \text{if } a_1 = 1, \\ {}^1E_{00}U_{p+2} &= {}^1E_{00}, \\ {}^1E_{11}U_{p+2} &= {}^1E_{11}B_{p+2}, \\ U_{p+3} &= I. \end{aligned}$$

REMARK. With this construction we aim to show that $N^{j+1}(\boldsymbol{R}) = N^j(\boldsymbol{P}) = R(\mathscr{C}_j^D)$ for $j=0, 1, \ldots, n-1$, and that none of these is a factor. However,

$$N^n(\boldsymbol{P}) = \mathfrak{A} = R(\mathscr{C}_n^D \cup \mathscr{C}_n').$$

(For $n=1$, the following three propositions hold with slight adaptations. Then nothing else is needed until Theorems 3.14 and 3.15.)

THEOREM 3.5. $N(\boldsymbol{R}) = \boldsymbol{P} = R(\mathscr{C}_0^D)$.

Proof. If $p \in \Gamma$, ${}^pE_{rs} \in \mathscr{C}_0^D$, then $\Delta(r)+\Delta(s)=(0, 0, \ldots, 0)$. So computation with the definitions of §3.4 shows that

$$U_{p+3n} \cdots U_{p+1}{}^pE_{rs}U_{p+1} \cdots U_{p+3n} = {}^pE_{rs}.$$

If $q \in \Gamma$, $q>p$, then $q=p+3hn$ for some integer h. Since ${}^pE_{rs}$ is a sum $\sum_v {}^qE_{r_v s_v}$, with all terms of the sum in $\mathscr{C}_0^D$, we have

$$U_q \cdots U_{p+1}{}^pE_{rs}U_{p+1} \cdots U_q = {}^pE_{rs} \in \mathscr{C}_0^D.$$

But if ${}^pE_{rs} \in \mathscr{N}_j$ $(j \geqq 1)$, then

$$U_{p+n-j+1} \cdots {}^pE_{rs} \cdots U_{p+n-j+1} = {}^pE_{rs}B_{p+n-j+1}.$$

Also, if ${}^pE_{rs} \in \mathscr{C}'_0$,

$$U_{p+3n-1} \cdots {}^pE_{rs} \cdots U_{p+3n-1} = {}^pE_{rs}B_{p+3n-1}.$$

Hence our construction satisfies the conditions of [7, §4.1], with $\mathscr{C}_0^D$ taking the place of K_0. Also, $d \leqq 3n-1$ is surely sufficient. Thus we can apply [7, Lemma 4.3] in order to conclude that any unitary V leaving $\boldsymbol{R}$ invariant is the metric limit of a sequence V_m in $\mathfrak{M}$ such that if $V_m \in \mathfrak{M}_p$ $(p \in \Gamma)$, then $V_m^{[p]} = \sum \alpha_{cd}{}^pE_{cd}$ with ${}^pE_{cd} \in \mathscr{C}_0^D$. So if $V \in N(\boldsymbol{R})$, then $V \in R(\mathscr{C}_0^D)$, and we have $N(\boldsymbol{R}) \subset R(\mathscr{C}_0^D)$.

On the other hand, consider a unitary V in $\mathfrak{M}_p$ $(p \in \Gamma)$ such that $V^{[p]} = \sum \pm {}^pE_{rs}$ with ${}^pE_{rs} \in \mathscr{C}_0^D$ and signs arbitrary. It is straightforward to show that V leaves $\boldsymbol{R}$ invariant. Since the collection of all unitaries of this type is sufficient to generate $R(\mathscr{C}_0^D)$, we have $R(\mathscr{C}_0^D) \subset N(\boldsymbol{R})$.

Therefore $N(\boldsymbol{R}) = \boldsymbol{P} = R(\mathscr{C}_0^D)$.

REMARK. The preceding proof also implies that $r(\mathscr{C}_0^D)$ is in $\mathfrak{D}$ and that $R(\mathscr{C}_0^D) = r(\mathscr{C}_0^D)^{(\infty)}$. For if $F = \sum \alpha_{rs}{}^pE_{rs}$ with ${}^pE_{rs} \in \mathscr{C}_0^D$, then $F^{(p)} = F^{(p+h)}$ for any $h>0$. Hence $\lim_{p\to\infty} F^{(p)} = F^{(\infty)}$ exists and $F \in \mathfrak{D}$. Using this information about $F \in r(\mathscr{C}_0^D) \cap \mathfrak{M}$, Lemma 2.2 and its proof can be rephrased to show that $r(\mathscr{C}_0^D) \subset \mathfrak{D}$, and that $R(\mathscr{C}_0^D)$, which is defined as the closure of $[r(\mathscr{C}_0^D) \cap \mathfrak{M}]^{(\infty)}$, can also be regarded simply as $r(\mathscr{C}_0^D)^{(\infty)}$.

LEMMA 3.6. *Let $p \in \Gamma$, $A^{[p]} = \sum \alpha_{cd}{}^pE_{cd}$ with ${}^pE_{cd}$ in $\mathscr{N}_j^D$ $(0 \leqq j \leqq n)$, $\mathscr{C}'_{n-1}$, or $\mathscr{N}'_n$. Then if $q \in \Gamma$, $q > p$, $A^{[q]} = \sum \beta_{rs}{}^qE_{rs}$ with ${}^qE_{rs}$ also in $\mathscr{N}_j^D$, $\mathscr{C}'_{n-1}$, or $\mathscr{N}'_n$ respectively.*

Proof. The case $\mathscr{N}_0^D$ has already been dealt with, since $\mathscr{N}_0^D = \mathscr{C}_0^D$.

We first consider $q = p + 3n$. Then $A^{[q]} = U_{p+3n} \cdots A^{[p]} \cdots U_{p+3n}$, and because of linearity it is sufficient to consider one term of $A^{[p]}$, say ${}^pE_{cd}$.

If $1 \leqq j \leqq n$ and ${}^pE_{cd} \in \mathscr{N}_j{}^D$, then Definition 3.4 shows that

$$U_{p+3n} \cdots {}^pE_{cd} \cdots U_{p+3n} = \sum \delta_{rs}{}^{p+3n}E_{rs}$$

is in $\mathfrak{M}_{p+n+j}$. Consider one term ${}^{p+3n}E_{rs}$. With $r = c \cdot 2^{3n} + r_1 2^n + r_0$ and $s = d \cdot 2^{3n} + s_1 2^n + s_0$, we thus have $r_0 = s_0$ and $r_1 \equiv s_1 \pmod{2^{n-j}}$. So $\sigma(r_1) = \sigma(s_1) \pmod{2^{n-j+1}}$; and therefore $\psi(\sigma(r_1)) + \psi(\sigma(s_1)) = (a_1, \ldots, a_{j-1}, 0, 0, \ldots)$, while $\psi(r_0) + \psi(s_0) = (0, 0, 0, \ldots)$. Hence, applying Lemma 3.3, ${}^{p+3n}E_{rs} \in \mathscr{N}_j$ as was ${}^pE_{cd}$. Now the action of the unitaries U_t surely preserves $\mathscr{C}_n^D$, and therefore ${}^{p+3n}E_{rs}$ is in $\mathscr{N}_j^D$.

Next suppose ${}^pE_{cd} \in \mathscr{C}'_{n-1}$ and consider $U_{p+3n} \cdots {}^pE_{cd} \cdots U_{p+3n}$. The product is in $\mathfrak{M}_{p+3n-1}$, by Definition 3.4, so one term ${}^{p+3n}E_{rs}$ has $r = c \cdot 2^{3n} + r_1 2^n + r_0$, $s = d \cdot 2^{3n} + s_1 2^n + s_0$ with $r_0 \equiv s_0 \pmod 2$. Thus $\psi(r_0) + \psi(s_0) = (\ldots, a_{n-1}, 0)$, and we can have ${}^{p+3n}E_{rs} \in \mathscr{N}_n$ if and only if $\psi(\sigma(r_1)) + \psi(\sigma(s_1)) = (\ldots, a_{n-1}, 1)$. But by definition, $\sigma(r_1) \equiv 0 \pmod 2$, so this cannot happen. As before, the action of the U_t's preserves $\mathscr{C}'_n$. Therefore ${}^{p+3n}E_{rs}$ is in $\mathscr{C}'_{n-1}$.

If ${}^pE_{cd} \in \mathscr{N}'_n$, then this time the computations of the preceding paragraph lead to the conclusion that the terms of $U_{p+3n} \cdots {}^pE_{cd} \cdots U_{p+3n}$ are in $\mathscr{N}'_n$. (Here $\psi(\sigma(r_1)) + \psi(\sigma(s_1)) = (\ldots, a_{n-1}, 0)$.)

If $q \in \Gamma$, $q > p$, then $q = p + 3hn$ for some integer h, and the desired result follows by induction.

LEMMA 3.7. *For $j = 1, 2, \ldots, n$, $R(\mathscr{C}_{j-1}^D) \subsetneqq R(\mathscr{C}_j^D) \subsetneqq R(\mathscr{C}_n)$.*

Proof. The inclusions are trivial and we need only show that they are proper inclusions.

Let F be a matrix unit in $\mathscr{N}_j^D$ (resp. $\mathscr{C}'_n$), so that $F \in \mathfrak{M}_p$ $(p \in \Gamma)$ and $F^{[p]} = {}^pE_{ab}$ in $\mathscr{N}_j^D$ $(\mathscr{C}'_n)$. Suppose that F is also in $R(\mathscr{C}_{j-1}^D)$ $(R(\mathscr{C}_n^D))$. Then there is a sequence $F_m \in \mathfrak{M}$ converging strongly to F, such that if $F_m \in \mathfrak{M}_q$ $(q \in \Gamma)$, $F_m^{[q]} = \sum \beta_{cd}{}^qE_{cd}$ with ${}^qE_{cd} \in \mathscr{C}_{j-1}^D(\mathscr{C}_n^D)$. Choose F_m such that $[\![F_m - F]\!] < 1/2^p$ and choose $q \in \Gamma$ such that $F_m, F \in \mathfrak{M}_q$. Then by Lemma 3.6, $F^{[q]} = \sum \alpha_{ab}{}^qE_{ab}$ with ${}^qE_{ab} \in \mathscr{N}_j^D$ $(\mathscr{C}'_n)$.

Case 1. $F \in \mathscr{N}_j^D$. Since $(F_m^{[q]}, F^{[q]}) = (F_m, F) = 0$, we have $1/2^{2p} > [\![F_m - F]\!]^2 = [\![F_m]\!]^2 + [\![F]\!]^2 > 1/2^p$, a contradiction. Therefore $F \notin R(\mathscr{C}_{j-1}^D)$.

Case 2. $F \in \mathscr{C}'_n$. Here

$$\begin{aligned}(F_m^{[q]}, F^{[q]}) &= ({}^1E_{ii}\, F_m^{[q]}\, {}^1E_{ii} + {}^1E_{jj}\, F_m^{[q]}\, {}^1E_{jj},\ {}^1E_{ii}\, F^{[q]}\, {}^1E_{jj}) \\ &= 0 \quad \text{where } i, j = 0 \text{ or } 1,\ i \neq j.\end{aligned}$$

So again $1/2^{2p} > [\![F_m - F]\!]^2 > 1/2^p$, a contradiction, and therefore $F \notin R(\mathscr{C}_n^D)$.

DEFINITION 3.8. We define the following projections in $\mathscr{C}_0^D$: For $k=2,\dots,n$ and $s=0,1,\dots,2^p-1$, let $P_k(s)$ be the operator such that $P_k(s)^{[p+3n]}=\sum_h {}^{p+3n}E_{s''+h,s''+h}$, where $s''=2^{3n}s$, $h\equiv 0 \pmod{2^{2n-k+1}}$ and $0\leqq h\leqq 2^{3n}-1$. Let $P'(s)$ be the operator such that $P'(s)^{[p+3n]}=\sum_h {}^{p+3n}E_{s''+h,s''+h}$, where $s''=2^{3n}s$, $h\equiv 0 \pmod{2^2}$ and $0\leqq h\leqq 2^{3n}-1$.

LEMMA 3.9. *Suppose* $W\in\mathfrak{M}_p$ $(p\in\Gamma)$ *is such that* $W^{[p]}=V^{[p]}+X^{[p]}$, *with* $V^{[p]}=\sum \beta_{rs}{}^pE_{rs}({}^pE_{rs}\in\mathscr{C}_n^D)$ *and* $X^{[p]}=\sum \alpha_{rs}{}^pE_{rs}({}^pE_{rs}\in\mathscr{C}_n')$. *Let* ${}^pE_{rt}$ *be a fixed matrix unit in* $\mathscr{C}_n^D$ *with* $K(r,t)=K_\gamma$. *Then*

$$(**)\qquad \begin{aligned} &{}^pE_{rr}[U_{p+3n}\cdots W^{[p]}\cdots U_{p+3n}]\sum_{s=0}^{2^p-1}P'(s)^{[p+3n]}[U_{p+3n}\cdots W^{*[p]}\cdots U_{p+3n}]{}^pE_{tt}\\ &\qquad = A(r,t)^{[p+3n]}+Q(r,t)^{[p+3n]},\end{aligned}$$

where $(A,Q)=0$ *and*

$$Q^{[p+3n]}=\sum_{s=0}^{2^p-1}\alpha_{rs}\bar{\alpha}_{ts}C(\gamma)\,{}^{p+3n}E_{ab}$$

with ${}^{p+3n}E_{ab}$ *in* $\mathscr{N}_{n-1}^D$ *or* $\mathscr{N}_n^D$, $C(\gamma)$ *a nonzero integer.*

Proof. The following statements are verified by calculations similar to those of [7, pp. 295–301].

Suppose $K({}^pE_{rs})=K_\alpha$ and $K({}^pE_{st})=K_\beta$, with both matrix units in $\mathscr{C}_n'$, $\alpha+\beta=\gamma$. If $\alpha=(a_1,a_2,\dots,a_n)$ and $\beta=(b_1,b_2,\dots,b_n)$, define $\omega_1=\omega(\alpha)=2(\sum_{i=2}^n a_i)+a_1+1$, $\omega_2=\omega(\beta)$, and $\mu(\alpha,\beta)=2(\sum_{i=2}^n a_ib_i)+a_1b_1$. Then the nonzero entries of the product $U_{p+3n}\cdots{}^pE_{rs}\cdots U_{p+3n}$ have numerical value $\pm(2^{-1/2})^{\omega_1}$, and similarly for ${}^pE_{st}$. Let $r_0=2^{3n-2}r$, $s_0=2^{3n-2}s$, $t_0=2^{3n-2}t$. Then 2^μ is the number of distinct δ's such that

$${}^{p+3n-2}E_{r_0r_0}[U_{p+3n}\cdots{}^pE_{rs}\cdots U_{p+3n}]\,{}^{p+3n-2}E_{s_0+\delta,s_0+\delta}$$

and

$${}^{p+3n-2}E_{s_0s_0}[U_{p+3n}\cdots{}^pE_{rs}\cdots U_{p+3n}]\,{}^{p+3n-2}E_{t_0+\delta,t_0+\delta}$$

are both nonzero.

Using the preceding, a matrix calculation shows that

$$(*)\qquad [U_{p+3n}\cdots{}^pE_{rs}\cdots U_{p+3n}]P'(s)^{[p+3n]}[U_{p+3n}\cdots{}^pE_{rs}\cdots U_{p+3n}]$$

has a term of the form $C(\gamma)\,{}^{p+3n}E_{r'',t''+2}$ and a term of the form $C(\gamma)\,{}^{p+3n}E_{r''+2,t''+2}$, where $r''=2^{3n}r$, $t''=2^{3n}t$, and $C(\gamma)=2^\mu(2^{-1/2})^{\omega_1+\omega_2}$. It is straightforward to show that $C(\gamma)$ depends only on γ and on the fact that ${}^pE_{rs}$ and ${}^pE_{st}$ are in $\mathscr{C}_n'$. By Lemma 3.3, if $K_\gamma\subset\mathscr{C}_{n-2}$, then $K(r'',t''+2)$ is in $\mathscr{N}_{n-1}$. If $K_\gamma\subset\mathscr{N}_{n-1}$ or $\mathscr{N}_n$, then so is $K(r''+2,t''+2)$. Also, since ${}^pE_{rt}\in\mathscr{C}_n^D$, so are these matrix units.

Now the product $(**)$ of the lemma equals

$$\sum_{s=0}^{2^p-1}[U_{p+3n}\cdots\delta_{rs}{}^pE_{rs}\cdots U_{p+3n}]P'(s)^{[p+3n]}[U_{p+3n}\cdots\delta_{ts}{}^pE_{st}\cdots U_{p+3n}].$$

Suppose $K_\gamma \subset \mathscr{C}_{n-2}$ and s such that ${}^pE_{rs}$ and ${}^pE_{st}$ are both in $\mathscr{C}'_n$. The summand corresponding to this s includes the term $\alpha_{rs}\bar{\alpha}_{ts}C(\gamma)\,{}^{p+3n}E_{r'',t''+2}$, which is in $\mathscr{N}_{n-1}$. Considering the summands corresponding to other s, we could not have one matrix unit in $\mathscr{C}'_n$, the other in $\mathscr{C}^D_n$, since ${}^pE_{rt} \in \mathscr{C}^D_n$. But if both are in $\mathscr{C}^D_n$, then the product is in $\mathfrak{M}_{p+2n}$, so there is no element in position $(r'', t''+2)$.

Suppose $K_\gamma \subset \mathscr{N}_n$ or $\mathscr{N}_{n-1}$. If s is such that ${}^pE_{rs}$ and ${}^pE_{st}$ are both in $\mathscr{C}'_n$, then the summand includes the term $\alpha_{rs}\bar{\alpha}_{ts}C(\gamma)\,{}^{p+3n}E_{r''+2,t''+2}$, which is in the same class as K_γ. Again, if s is such that both matrix units are in $\mathscr{C}^D_n$ there is no element in position $(r''+2, t''+2)$.

So if we let Q be as stated in the lemma, with $(a, b)=(r'', t''+2)$ or $(r''+2, t''+2)$ according to K_γ, then $(A, Q)=0$ and ${}^{p+3n}E_{ab} \in \mathscr{N}^D_{n-1}$ or $\mathscr{N}^D_n$.

LEMMA 3.10. *Suppose $W \in \mathfrak{M}_p$ $(p \in \Gamma)$ is such that $W^{[p]}=V^{[p]}+X^{[p]}$, with $V^{[p]}=\sum \beta_{rs}{}^pE_{rs}$ $({}^pE_{rs} \in \mathscr{C}^D_{k-1})$ and $X^{[p]}=\sum \alpha_{rs}{}^pE_{rs}$ $({}^pE_{rs} \in \mathscr{N}_k{}^D)$. Let ${}^pE_{rt}$ be a fixed matrix unit in $\mathscr{C}^D_{k-1}$ with $K(r, t)=K_\gamma$. Then*

$$(**)\qquad \begin{aligned} &{}^pE_{rr}[U_{p+3n}\cdots W^{[p]}\cdots U_{p+3n}]\sum_{s=0}^{2^p-1} P_k(s)^{[p+3n]}[U_{p+3n}\cdots W^{*[p]}\cdots U_{p+3n}]^pE_{tt}\\ &\qquad = A(r, t)^{[p+3n]}+Q(r, t)^{[p+3n]},\end{aligned}$$

where

$$Q^{[p+3n]} = \sum_{s=0}^{2^p-1} \alpha_{rs}\bar{\alpha}_{ts}D_k(\gamma)\,{}^{p+3n}E_{ab}$$

with ${}^{p+3n}E_{ab}$ in $\mathscr{N}^D_{k-1}$, $D_k(\gamma)$ a nonzero integer.

Proof. The proof is like that of the preceding lemma, with the following changes: $\omega_1=\omega(\alpha)=2(\sum_{i=2}^n a_i)+a_1$ (and a similar change in ω_2), $\mu(\alpha, \beta)=2(\sum_{i=2}^{k-1} a_ib_i)+a_kb_k+a_1b_1$, $r_0=2^{n+k-1}r$, $s_0=2^{n+k-1}s$, $t_0=2^{n+k-1}t$. Then 2^μ is the number of distinct δ's such that

$${}^{p+n+k-1}E_{r_0r_0}[U_{p+3n}\cdots {}^pE_{rs}\cdots U_{p+3n}]\,{}^{p+n+k-1}E_{s_0+\delta,s_0+\delta}$$

and

$${}^{p+n+k-1}E_{s_0s_0}[U_{p+3n}\cdots {}^pE_{st}\cdots U_{p+3n}]\,{}^{p+n+k-1}E_{t_0+\delta,t_0+\delta}$$

are both nonzero. The expression

$$(*)\qquad [U_{p+3n}\cdots {}^pE_{rs}\cdots U_{p+3n}]P_k(s)^{[p+3n]}[U_{p+3n}\cdots {}^pE_{st}\cdots U_{p+3n}]$$

has a term of the form $D_k(\gamma)\,{}^{p+3n}E_{r'',t''+\pi}$ and a term of the form $D_k(\gamma)\,{}^{p+3n}E_{r''+\pi,t''+\pi}$ where $r''=2^{3n}r$, $t''=2^{3n}t$, $\pi=2^{2n-k}$, and $D_k(\gamma)=2^\mu(2^{-1/2})^{\omega_1+\omega_2}$. Here $D_k(\gamma)$ depends only on γ and on k. By Lemma 3.3, if $K_\gamma \subset \mathscr{C}_{k-2}$, then $K(r'', t''+\pi)$ is in $\mathscr{N}_{k-1}$; if $K_\gamma \subset \mathscr{N}_{k-1}$, then $K(r''+\pi, t''+\pi)$ is in $\mathscr{N}_{k-1}$.

It can be verified, as in the preceding lemma, that $(A, Q)=0$ if we take Q as stated, with $(a, b)=(r'', t''+\pi)$ or $(r''+\pi, t''+\pi)$ according to K_γ.

LEMMA 3.11. *If the results of Lemmas* 3.9 *and* 3.10 *hold for* $q=p+3n$, *then they hold for any* $q=p+3hn$ (*i.e.*, $q\in\Gamma$). *Also,*

$$[\![Q]\!]^2 \geqq \left|\sum_{s=0}^{2^p-1}\alpha_{rs}\bar{\alpha}_{ts}\right|^2\Big/2^{p+5n}.$$

Proof. We first obtain bounds for $C(\gamma)$ and $D_k(\gamma)$. In both cases, we have $\mu\geqq 0$ and $\omega_1+\omega_2\leqq 2(2n-1)+2=4n$. Hence $C(\gamma)$ or $D_k(\gamma)=2^{\mu}(2^{-1/2})^{\omega_1+\omega_2}\geqq(2^{-1/2})^{4n}=1/2^{2n}$.

$$\begin{aligned}[\![Q^{[p+3n]}]\!]^2 &\geqq |C(\gamma)|^2\left|\sum\alpha_{rs}\bar{\alpha}_{ts}\right|^2/2^{p+3n}\\ &\geqq \left|\sum\alpha_{rs}\bar{\alpha}_{ts}\right|^2/2^{p+5n},\end{aligned}$$

and similarly in the case of $D_k(\gamma)$.

Now the unitaries $U_{p+3n+1},\ldots,U_{p+3hn}$ preserve the orthogonality of A and Q and the norm of Q. Also, by Lemma 3.6, matrix units in $\mathscr{N}_j^D$ ($j=1,2,\ldots,n$) are left in that class under the action of the unitaries U_t.

LEMMA 3.12. *For* $j=1,2,\ldots,n-1$, *let* $\mathscr{V}_j=\{V:V[R(\mathscr{C}_{j-1}^D)]V^*=R(\mathscr{C}_{j-1}^D),\ V$ *unitary*, $V\in\mathfrak{A}\}$. *If* $V\in\mathscr{V}_j$, *then there is a sequence* $V_m\in\mathfrak{M}$ *converging metrically to* V *such that if* $V_m\in\mathfrak{M}_p$ ($p\in\Gamma$), $V_m^{[p]}=\sum\beta_{rs}{}^pE_{rs}$ *with* ${}^pE_{rs}$ *in* $\mathscr{C}_j^D$. *Thus*, $N(R(\mathscr{C}_{j-1}^D))\subset R(\mathscr{C}_j^D)$.

Proof. (i) Since $V\in\mathfrak{A}$, $\|V\|\leqq 1$, there is a sequence $W_m\in\mathfrak{M}$, $\|W_m\|\leqq 1$, converging strongly and metrically to V [4]. If $W_m\in\mathfrak{M}_p$, let $W_m^{[p]}=V_m^{[p]}+X_m^{[p]}$, where $V_m^{[p]}=\sum\beta_{rs}{}^pE_{rs}$ (${}^pE_{rs}\in\mathscr{C}_n^D$) and $X_m^{[p]}=\sum\alpha_{rs}{}^pE_{rs}$ (${}^pE_{rs}\in\mathscr{C}_n'$). Because of the orthogonality of V_m and X_m, X_m itself is Cauchy in the metric topology. Now $\|W_m\|\leqq 1$ implies $\|V_m\|\leqq 1$ because of the definition of $\mathscr{C}_n^D$. Since $X_m=W_m-V_m$, we have $\|X_m\|\leqq 2$, and so X_m is also Cauchy in the strong topology [5, p. 723]. Let $X_m\to X\in\mathfrak{A}$. Suppose $\lim_m[\![X_m]\!]\neq 0$; then $\lim_m[\![X_mX_m^*]\!]\neq 0$ also. Hence $[\![X_mX_m^*]\!]^2>2^{5n}\varepsilon^2$ for all m and some $\varepsilon>0$. (Recall that n is fixed and related only to $\boldsymbol{R}=\boldsymbol{R}_n$.)

Choose W_m so that $[\![W_m-V]\!]<\varepsilon/4$. Suppose $W_m\in\mathfrak{M}_p$. Then

$$[\![X_m^{[p]}X_m^{[p]*}]\!]^2=(1/2^p)\sum\left|\sum_{s=0}^{2^p-1}\alpha_{rs}\bar{\alpha}_{ts}\right|^2>2^{5n}\varepsilon^2.$$

(The outer summation is over pairs (r,t) such that ${}^pE_{rt}\in\mathscr{C}_n^D$, since ${}^pE_{rs},\ {}^pE_{st}\in\mathscr{C}_n'$.) Fix p from here on.

Consider $\sum_{s=0}^{2^p-1}P'(s)^{[p+3n]}$, which has its matrix units in $\mathscr{C}_0^D$. Then $\sum_s P'(s)$ is in $R(\mathscr{C}_{j-1}^D)$ for any $j\geqq 1$, and if $V\in\mathscr{V}_j$, $V(\sum_s P'(s))V^*=T\in R(\mathscr{C}_{j-1}^D)$. So there exists a sequence $T_\nu\in\mathfrak{M}$, $[\![T_\nu-T]\!]\to 0$, and $T_\nu\in\mathfrak{M}_q$ ($q\in\Gamma$) implies $T_\nu^{[q]}=\sum\eta_{ih}{}^qE_{ih}$ with ${}^qE_{ih}$ in $\mathscr{C}_{j-1}^D$. Choose T_ν such that $[\![V(\sum_s P'(s))V^*-T_\nu]\!]<\varepsilon/2$. Since $\sum_s P'(s)$ is a projection, of norm at most one,

$$\left[\!\!\left[W_m\Big(\sum_s P'(s)\Big)W_m^*-V\Big(\sum_s P'(s)\Big)V^*\right]\!\!\right]<\varepsilon/2,$$

and thus it follows that

$$\left[\!\left[W_m\left(\sum_s P'(s)\right) W_m^* - T_v \right]\!\right] < \varepsilon.$$

On the other hand, we can apply Lemmas 3.9 and 3.11 with W_m replacing W. Take q to be such that $q \in \Gamma$, $q \geqq p+3n$, and $T_v \in \mathfrak{M}_q$. Since $Q^{[q]} = \sum \lambda_{cd}{}^qE_{cd}$ (${}^qE_{cd} \in \mathscr{N}^D_{n-1}$ or $\mathscr{N}^D_n$) and $T_v^{[q]} = \sum \eta_{th}{}^qE_{th}$ (${}^qE_{th} \in \mathscr{C}^D_{j-1}$, where $j-1<n-1$), we have $(T_v^{[q]}, Q^{[q]})=0$ also. Therefore

$$\begin{aligned} &[\![{}^pE_{rr}W_m^{[q]} \textstyle\sum P'(s)^{[q]} W_m^{*[q]}\,{}^pE_{tt} - {}^pE_{rr}T_v^{[q]}\,{}^pE_{tt}]\!]^2 \\ &\qquad = [\![A(r,t)^{[q]} + Q(r,t)^{[q]} - {}^pE_{rr}T_v^{[q]}\,{}^pE_{tt}]\!]^2 \\ &\qquad \geqq [\![Q(r,t)^{[q]}]\!]^2 \geqq \left|\sum_s \alpha_{rs}\bar{\alpha}_{ts}\right|^2 / 2^{p+5n}. \end{aligned}$$

Finally, we have:

$$\begin{aligned} \varepsilon^2 &\geqq \sum_{(r,t)} \left[\!\left[{}^pE_{rr}\left(W_m^{[q]} \sum_s P'(s) W_m^{*[q]} - T_v^{[q]}\right){}^pE_{tt} \right]\!\right]^2, \qquad {}^pE_{rt} \in \mathscr{C}^D_n \\ &\geqq \sum_{(r,t)} \left|\sum_s \alpha_{rs}\bar{\alpha}_{ts}\right|^2 / 2^{p+5n} > \varepsilon^2, \end{aligned}$$

which is a contradiction.

Therefore $\lim_l [\![X_l]\!]=0$ and so $\lim_l [\![V_l - V]\!]=0$, where $\|V_l\| \leqq 1$ and $V_l \in \mathfrak{M}_z$ ($z \in \Gamma$) implies $V_l^{[z]} = \sum \beta_{rs}{}^zE_{rs}$ with ${}^zE_{rs} \in \mathscr{C}^D_n$.

(ii) To show: Suppose $j<k\leqq n$ and suppose there exists $W_m \in \mathfrak{M}$ such that $\|W_m\| \leqq 1$, $\lim_m [\![W_m - V]\!]=0$, and $W_m \in \mathfrak{M}_p$ implies $W_m^{[p]} = \sum \delta_{rs}{}^pE_{rs}$ with ${}^pE_{rs}$ in $\mathscr{C}^D_k$. Then there exists V_m with the same properties except that $V_m^{[p]} = \sum \beta_{rs}{}^pE_{rs}$ with ${}^pE_{rs}$ in $\mathscr{C}^D_{k-1}$.

We let the assumed $W_m^{[p]} = V_m^{[p]} + X_m^{[p]}$, where the matrix units of the two summands are in $\mathscr{C}^D_{k-1}$ and $\mathscr{N}^D_k$ respectively. The argument proceeds much as in part (i), with $\sum_s P_k(s)$ replacing $\sum_s P'(s)$, so that Lemmas 3.10 and 3.11 apply. Since $V(\sum P_k(s))V^* = T$ in $R(\mathscr{C}^D_{j-1})$ and since $j-1<k-1$, the desired orthogonality holds between Q (in $\mathscr{N}^D_{k-1}$) and T_v (the sequence of matrices converging to T). We are led to conclude that $\lim_m [\![X_m]\!]=0$, and that V is the metric limit of V_m.

Since we can extend this as far as $k=j+1$ by a finite induction process, the lemma is proved.

THEOREM 3.13. *For $j=1, 2, \ldots, n-1$, if $R(\mathscr{V}_j)$ is the ring generated by $\mathscr{V}_j$ as defined in Lemma* 3.12, *then $R(\mathscr{V}_j)=R(\mathscr{C}^D_j)$. Thus, $N(R(\mathscr{C}^D_{j-1}))=R(\mathscr{C}^D_j)$.*

Proof. By Lemma 3.12, $R(\mathscr{V}_j) \subset R(\mathscr{C}^D_j)$.

For the reverse inclusion, take $T \in R(\mathscr{C}^D_{j-1})$. Let $V_1^{[p]} = \sum \pm{}^pE_{rs}$ with ${}^pE_{rs}$ in $\mathscr{C}^D_{j-1}$ and signs arbitrary. Then $V_1TV_1^*$ is in $R(\mathscr{C}^D_{j-1})$ since all three operators are.

Next let $V_2^{[p]} = \sum \pm {}^pE_{rs}$ with ${}^pE_{rs}$ in $\mathscr{N}_j^D$. Take a sequence $T_m \in \mathfrak{M}$, $T_m \to T$, and if $T \in \mathfrak{M}_q$, $T_m^{[q]} = \sum \beta_{cd}{}^qE_{cd}$ with ${}^qE_{cd}$ in $\mathscr{C}_{j-1}^D$. If $z = \max[p, q]$, then

$$V_2^{[q]}T_m^{[q]}V_2^{*[q]} = \left[\sum \delta_{rs}{}^zE_{rs}\right]\left[\sum \beta'_{cd}{}^zE_{cd}\right]\left[\sum \delta_{rs}{}^zE_{rs}\right],$$

where the matrix units of the first sum are in $\mathscr{N}_j^D$, those of the second in $\mathscr{C}_{j-1}^D$, and those of the third in $\mathscr{N}_j^D$, by Lemma 3.6. Calculating by means of §3.1, we see that each matrix unit of this product is in $\mathscr{C}_{j-1}^D$. Hence $V_2T_mV_2^*$ is in $R(\mathscr{C}_{j-1}^D)$, and so is its strong limit $V_2TV_2^*$.

But all unitaries of the form V_1 or V_2 are sufficient to generate $R(\mathscr{C}_j^D)$. Therefore $R(\mathscr{C}_j^D) \subset R(\mathscr{V}_j)$, and hence $R(\mathscr{V}_j) = R(\mathscr{C}_j^D)$.

THEOREM 3.14. *If* $\mathscr{V}_n = \{V : V[R(\mathscr{C}_{n-1}^D)]V^* = R(\mathscr{C}_{n-1}^D),\ V \text{ unitary},\ V \in \mathfrak{A}\}$, *then* $R(\mathscr{V}_n) = R(\mathscr{C}_n) = \mathfrak{A}$. *Thus*, $N(R(\mathscr{C}_{n-1}^D)) = \mathfrak{A}$.

Proof. Obviously $R(\mathscr{V}_n) \subset R(\mathscr{C}_n)$.

For the reverse inclusion, let T be in $R(\mathscr{C}_{n-1}^D)$. Consider in turn four types of unitaries $V_i^{[p]} = \sum \pm {}^pE_{rs}$ ($i = 1, 2, 3, 4$ and signs arbitrary). For $i = 1$, the matrix units are to be in $\mathscr{C}_{n-1}^D$; for $i = 2$, in $\mathscr{N}_n^D$; for $i = 3$, in $\mathscr{C}'_{n-1}$; for $i = 4$, in $\mathscr{N}'_n$. By Lemma 3.6, these classes are preserved under the unitaries U_t. So calculations like those in the proof of Theorem 3.13 show that $V_iTV_i^*$ is in $R(\mathscr{C}_{n-1}^D)$ for $i = 1, 2, 3, 4$.

But all unitaries of these types are sufficient to generate $R(\mathscr{C}_n)$, or $\mathfrak{A}$. Therefore $R(\mathscr{C}_n) \subset R(\mathscr{V}_n)$, and $R(\mathscr{V}_n) = \mathfrak{A}$.

REMARK. Theorems 3.13 and 3.14, together with Theorem 3.5 and Lemma 3.7, show that for each $\boldsymbol{R}_n$, $n = 1, 2, 3, \ldots$, we have $\boldsymbol{R}_n \subsetneq N(\boldsymbol{R}_n) \subsetneq \cdots \subsetneq N^{n+1}(\boldsymbol{R}_n) = \mathfrak{A}$. In order to prove that $\boldsymbol{R}_n$ is M-semiregular ($n+1 = M$), we need only show that $N(\boldsymbol{R}_n), N^2(\boldsymbol{R}_n), \ldots, N^n(\boldsymbol{R}_n)$ are not factors. ($N^{n+1}(\boldsymbol{R}_n) = N^M(\boldsymbol{R}_n)$ is the factor $\mathfrak{A}$.)

THEOREM 3.15. *For* $k = 1, 2, \ldots, n$, $N^k(\boldsymbol{R}_n)$ *is not a factor.*

Proof. If $k \neq n$, $N^k(\boldsymbol{R}_n) = N^{k-1}(\boldsymbol{P}_n) = R(\mathscr{V}_{k-1}) = R(\mathscr{C}_{k-1}^D)$. Consider the projection ${}^1E_{00} = {}^1E_{00}^{(\infty)} \in \boldsymbol{R}_n \subset N^k(\boldsymbol{R}_n)$. If A is any operator in $N^k(\boldsymbol{R}_n)$, there is a sequence $A_m \to A$ such that if $A_m \in \mathfrak{M}_p$, $A_m^{[p]} = \sum \alpha_{rs}{}^pE_{rs}$ with ${}^pE_{rs} \in \mathscr{C}_{k-1}^D$. Then

$$\begin{aligned}({}^1E_{00}A_m{}^1E_{00})^{[p]} &= {}^1E_{00}A_m^{[p]}\,{}^1E_{00} = \sum \alpha_{rs}{}^1E_{00}{}^pE_{rs}{}^1E_{00} \\ &= \sum \alpha_{rs}{}^pE_{rs} \qquad \text{(by definition of } \mathscr{C}_{k-1}^D) \\ &= A_m^{[p]}.\end{aligned}$$

Thus ${}^1E_{00}A_m{}^1E_{00} = A_m$, and taking strong limits, ${}^1E_{00}A\,{}^1E_{00} = A$.

Therefore ${}^1E_{00}$ commutes with $N^k(\boldsymbol{R}_n)$, ${}^1E_{00} \neq \alpha I$, ${}^1E_{00} \in N^k(\boldsymbol{R}_n)$, and so $N^k(\boldsymbol{R}_n)$ is not a factor.

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