THE NUMERICAL SOLUTION OF THE EIGENVALUE PROBLEM FOR COMPACT INTEGRAL OPERATORS

BY

KENDALL E. ATKINSON

1. **Introduction.** With linear integral equations of the second kind, an important method for their numerical solution [1]–[4], [6], [7], [9], [10], is to replace the integral operator with an approximating numerical integration operator. The resulting equation is equivalent to a finite linear system, and linear systems are relatively easy to treat. In [1], [2], the method is lifted into a functional analysis framework, and in this setting a complete error analysis is given. More precisely, the following are the basic assumptions of [1] and also of this paper.

A1. K and K_n, $n \geqq 1$, are linear operators on X into X, where X is a Banach space.

A2. $K_n x \to Kx$ as $n \to \infty$, for all $x \in X$.

A3. The sequence $\{K_n\}$ is compact in aggregate, i.e., the set

$$\{K_n x \mid n \geqq 1 \quad \text{and} \quad \|x\| \leqq 1, \quad \text{for } x \in X\}$$

has compact closure in X.

The method described in the first two sentences above is also used to find the eigenvalues and eigenvectors of an integral operator K [3], [5], [6], [7], [14], and the main purpose of this paper is to present a general theorem showing the convergence of the eigenvalues and eigenvectors of K_n, $n \geqq 1$, to those of K. Such results have been published before, but they have been limited to the cases of K selfadjoint or normal [5], [6], [14]. Also, these papers dealt with the only continuous kernels and generally limited themselves to the problem of the convergence of the eigenvalues. Papers that should be especially noted are those of Wielandt [14] and Brakhage [5].

The presentation given here is based completely on the hypotheses A1–A3 and there is no explicit reference to integral equations. From the hypotheses, it follows easily that (i) K is compact, (ii) the sequence of $\{K_n\}$ is uniformly bounded, say by B, and (iii) $\|(K-K_n)K\|$ and $\|(K-K_n)K_n\|$ tend to zero as $n \to \infty$. From the compactness of K, it follows that the spectrum of K, $\sigma(K)$, is a countable set of eigenvalues with zero as a possible noneigenvalue and the only possible point of accumulation. For the complex numbers λ of the resolvent set $\rho(K)$, $(\lambda-K)^{-1}$ is a bounded operator from X onto X. Letting λ_0 be a nonzero point of $\sigma(K)$, the projection operator $E(\lambda_0, K)$ is defined by

$$E(\lambda_0, K) = \frac{1}{2\pi i} \int_{|\lambda-\lambda_0|=\varepsilon} (\lambda-K)^{-1}\, d\lambda$$

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where ε is less than the distance from λ_0 to the remainder of $\sigma(K)$. Then

$$\text{Range } E(\lambda_0, K) \equiv E(\lambda_0, K)X$$

is the set of all simple and generalized eigenvectors of K corresponding to the eigenvalue λ_0. For a general review of these results, see [8, pp. 566–580] and [12, Chapter 5].

Using the above λ_0 and ε, let σ_n denote the part of $\sigma(K_n)$ which is within ε of λ_0. In Theorem 3, it is shown that $E(\sigma_n, K_n)$ can be defined for all sufficiently large n, and that $E(\sigma_n, K_n)x \to x$ as $n \to \infty$, for all $x \in E(\lambda_0, K)X$. The element $E(\sigma_n, K_n)x$ is a linear combination of eigenvectors of K_n with respect to the eigenvalues contained in σ_n. Theorems 2 and 3 together give a complete convergence theorem; Theorem 4 introduces a "somewhat computable" error bound.

2. **Convergence theorems.** The error analysis of [1], [2] is based on the hypotheses A1–A3 and on the following theorem [1], [2], [4].

THEOREM 1. *Let X be a Banach space, S and $(\lambda - S)^{-1}$ continuous linear operators on X into X, $\lambda \neq 0$, and T a compact linear operator on X into X. Furthermore, assume*

$$\|(T-S)T\| < |\lambda|/\|(\lambda-S)^{-1}\|. \tag{1}$$

Then $(\lambda-T)^{-1}$ exists with the bound

$$\|(\lambda-T)^{-1}\| \leqq \frac{1+\|T\|\;\|(\lambda-S)^{-1}\|}{|\lambda|-\|(\lambda-S)^{-1}\|\;\|(T-S)T\|}.$$

Also, for $x \in X$,

$$\|(\lambda-S)^{-1}x-(\lambda-T)^{-1}x\| \leqq \|(\lambda-S)^{-1}\| \frac{\|(T-S)x\|+\|(\lambda-S)^{-1}x\|\;\|(T-S)T\|}{|\lambda|-\|(\lambda-S)^{-1}\|\;\|(T-S)T\|}.$$

REMARKS. With $S=K$ and $T=K_n$, the inequality (1) will be satisfied for all sufficiently large n because $\|(K-K_n)K_n\| \to 0$ as $n \to \infty$; thus Theorem 1 implies the existence of $(\lambda-K_n)^{-1}$ for all sufficiently large n and the convergence of $(\lambda-K_n)^{-1}x$ to $(\lambda-K)^{-1}x$ for all $x \in X$.

For $0 \leqq t \leqq 1$, define

$$K_{n,t} = (1-t)K+tK_n, \tag{2}$$

and note that $K_{n,0}=K$, $K_{n,1}=K_n$, and $\|K_{n,t}\| \leqq B$. For each t, the operators K and $K_{n,t}$, $n \geqq 1$, can be shown to satisfy A1–A3, but this fact is not necessary for the present development. The introduction of these operators was suggested by Turner [13]. They provide a very neat way for proving Theorem 3, part of which the author has not been able to prove in any other way.

LEMMA. *Let F be a compact subset of $\rho(K)$ with $0 \notin F$. Then there is a real number $M > 0$ and an integer $N \geqq 1$ for which the following statements are true:*

(a) *For all $\lambda \in F$, $\|(\lambda - K)^{-1}\| \leqq M$.*

(b) *For $0 \leqq t \leqq 1$ and $n \geqq N$, $F \subset \rho(K_{n,t})$ and*

$$\|(\lambda - K_{n,t})^{-1}\| \leqq M_1 = 2(1 + MB)/c_0,$$

where c_0 is the distance from F to the origin.

(c) *For $0 \leqq t \leqq 1$, $x \in X$, $\lambda \in F$, and $n \geqq N$,*

$$\|(\lambda - K)^{-1}x - (\lambda - K_{n,t})^{-1}x\| \leqq \frac{2M}{c_0}[\|(K - K_{n,t})x\| + M\|x\| \, \|(K - K_{n,t})K_{n,t}\|].$$

(d) *For $0 \leqq t, s \leqq 1$, $|s - t| \leqq 1/4BM_1$, $n \geqq N$, and $\lambda \in F$,*

$$\|(\lambda - K_{n,s})^{-1} - (\lambda - K_{n,t})^{-1}\| \leqq 4BM_1^2|t - s|.$$

REMARKS. Although the details may seem overwhelming, statements (a)–(c) merely say that for each t, the remarks following Theorem 1 hold for $\{K_{n,t}\}$, uniformly with respect to $\lambda \in F$. Statement (d) shows the uniform continuity in t of $(\lambda - K_{n,t})^{-1}$, uniformly with respect to $\lambda \in F$ and $n \geqq N$.

Proof. (i) Since $(\lambda - K)^{-1}$ is a continuous function of λ on $\rho(K)$ and since F is a compact subset of $\rho(K)$, the quantity $(\lambda - K)^{-1}$ is bounded on F, say by M.

(ii) Theorem 1 with $S = K$ and $T = K_{n,t}$ will be used to establish parts (b) and (c), and as a first step the inequality

$$\|(K - K_{n,t})K_{n,t}\| < |\lambda|/\|(\lambda - K)^{-1}\| \tag{3}$$

will have to be examined. Since

$$\|(K - K_{n,t})K_{n,t}\| \leqq [\|(K - K_n)K_n\| + \tfrac{1}{4}\|(K - K_n)K\|]$$

for $0 \leqq t \leqq 1$, pick N so large that for $n \geqq N$,

$$\|(K - K_n)K_n\| + \tfrac{1}{4}\|(K - K_n)K\| \leqq c_0/2M.$$

Then

$$\begin{aligned} 0 < c_0/2 &\leqq c_0 - M[\|(K - K_n)K_n\| + \tfrac{1}{4}\|(K - K_n)K\|] \\ &\leqq |\lambda| - \|(\lambda - K)^{-1}\| \, \|(K - K_{n,t})K_{n,t}\| \end{aligned} \tag{4}$$

for $\lambda \in F$, $n \geqq N$, and $0 \leqq t \leqq 1$. Inequality (3) is satisfied, and Theorem 1 can be applied in a straightforward manner to obtain (b) and (c).

(iii) To show the continuity in t of $(\lambda - K_{n,t})^{-1}$, a well-known theorem based on Neumann series expansion will be used [8, p. 584] and [12, p. 164]. For $0 \leqq s, t \leqq 1$, $n \geqq N$, and $\lambda \in F$,

$$\|(\lambda - K_{n,t}) - (\lambda - K_{n,s})\| = |t - s| \, \|K - K_n\| \leqq 2B|t - s|.$$

For $|t-s| \leqq 1/4BM_1$,

$$\begin{aligned}\|(\lambda-K_{n,t})-(\lambda-K_{n,s})\| &\leqq 1/2M_1 \leqq 1/2\|(\lambda-K_{n,t})^{-1}\| \\ &< 1/\|(\lambda-K_{n,t})^{-1}\|.\end{aligned}$$

With this inequality, the theorem cited above can be used to give

$$\begin{aligned}\|(\lambda-K_{n,t})^{-1}-(\lambda-K_{n,s})^{-1}\| &\leqq \frac{\|(\lambda-K_{n,t})^{-1}\|^2\|K_{n,t}-K_{n,s}\|}{1-\|(\lambda-K_{n,t})^{-1}\|\ \|K_{n,t}-K_{n,s}\|} \\ &\leqq \frac{2BM_1^2|t-s|}{\frac{1}{2}} = 4BM_1^2|t-s|. \qquad \text{Q.E.D.}\end{aligned}$$

The following theorem says that no extraneous convergent sequence of eigenvalues is produced by the sequence of operators $\{K_n\}$. Taken with Theorem 3, it will imply that for each $\lambda \in \sigma(K)$, $\lambda \neq 0$, there is a sequence of sets σ_n of eigenvalues of K_n with the sets σ_n converging to λ as $n \to \infty$.

THEOREM 2. *Let R and ε be arbitrary small positive numbers. Then there is an N such that for $n \geqq N$, any eigenvalue λ of K_n satisfying $|\lambda| \geqq R$ is within ε of an eigenvalue λ_0 of K with $|\lambda_0| \geqq R$.*

Proof. Since B is a bound on K and K_n, $n \geqq 1$, all eigenvalues of K and K_n are within a circle of radius B about the origin. To avoid triviality, assume $R < B$. Let Δ be the annulus about the origin defined by $R \leqq |\lambda| \leqq B$, and then modify Δ as follows. For all $\lambda_0 \in \sigma(K) \cap \Delta$, remove from Δ all points λ for which $|\lambda-\lambda_0| < \varepsilon$. Call the resulting set Δ'. Since K is compact, $\sigma(K) \cap \Delta$ is finite, and therefore only a finite number of open sets are removed from Δ to form Δ'. This makes Δ' a compact set, and in addition the construction implies $\Delta' \subset \rho(K)$. By the Lemma with $t=1$ and $F=\Delta'$, there is an N such that $\Delta' \subset \rho(K_n)$ for all $n \geqq N$. Thus if $\lambda \in \sigma(K_n) \cap \Delta$ and $n \geqq N$, λ must be within ε of some element of $\sigma(K) \cap \Delta$ since $\sigma(K_n) \cap \Delta' = \varnothing$. Q.E.D.

Let $\lambda_0 \in \sigma(K)$, $\lambda_0 \neq 0$, and let $\varepsilon > 0$ be less than both $|\lambda_0|$ and the distance to the remainder of $\sigma(K)$; denote the circumference $|\lambda-\lambda_0| = \varepsilon$ by C_1. Denote by C_2 some simple closed rectifiable curve which is away from C_1 and which contains both the origin and the set $\sigma(K) \sim \{\lambda_0\}$ in its interior. Denote by F the set of points which are (i) on C_1 or C_2, or (ii) outside both C_1 and C_2 as well as satisfying $|\lambda| \leqq 3B/2$; F is a compact set in $\rho(K)$. Recall that $\sigma(K_{n,t})$ is contained inside the disk $|\lambda| \leqq 3B/2$ for all $n \geqq 1$ and $0 \leqq t \leqq 1$.

Now apply the Lemma to the set F. For all sufficiently large $n \geqq N$ and for $0 \leqq t \leqq 1$, $F \subset \rho(K_{n,t})$; thus $\sigma(K_{n,t})$ is contained in the interiors of C_1 and C_2. Denote by $\sigma_{n,t}$ the part of $\sigma(K_{n,t})$ contained in the interior of C_1; $\sigma_{n,1} \equiv \sigma_n$ and $\sigma_{n,0} = \{\lambda_0\}$. The projection operator

$$E(\sigma_{n,t}, K_{n,t}) = \frac{1}{2\pi i}\int_{C_1} (\lambda-K_{n,t})^{-1}\, d\lambda$$

is well defined for $n \geqq N$ and $0 \leqq t \leqq 1$. The additional results of applying the Lemma to the set F will be used in proving the following main theorem.

THEOREM 3. *With respect to the preceding remarks, the following statements are true for* $n \geqq N$

(a) *Dimension* $E(\sigma_n, K_n)X = E(\lambda_0, K)X$, *thereby showing* σ_n *to be nonempty and of the correct multiplicity.*

(b) *For every* $x \in E(\lambda_0, K)X$,

$$\|x - E(\sigma_n, K_n)x\| \leqq \frac{2M\varepsilon}{c_0} [\|(K-K_n)x\| + M\|x\| \, \|(K-K_n)K_n\|]; \tag{5}$$

this proves that $E(\sigma_n, K_n)x \to x$ *as* $n \to \infty$.

Proof. (i) First it will be shown that $E(\sigma_{n,t}, K_{n,t})$ is a continuous function of t, $0 \leqq t \leqq 1$. Recall the application of the Lemma to the set F (which includes C_1), and use part (d) which shows the continuity in t of $(\lambda - K_{n,t})^{-1}$. For $|t-s| \leqq 1/4BM_1$, $n \geqq N$, and $\lambda \in C_1$,

$$\begin{aligned} \|E(\sigma_{n,t}, K_{n,t}) - E(\sigma_{n,s}, K_{n,s})\| &\leqq \frac{1}{2\pi} \int_{C_1} \|(\lambda - K_{n,t})^{-1} - (\lambda - K_{n,s})^{-1}\| \, |d\lambda| \\ &\leqq \frac{1}{2\pi} (4BM_1^2|t-s|)(2\pi\varepsilon) = 4BM_1^2\varepsilon|t-s|, \end{aligned}$$

thus showing the desired continuity.

A standard theorem of functional analysis [11, p. 268] states that if the difference of two projections is of norm less than 1, then their ranges are of equal dimension. With this theorem, use the finite dimension of $E(\lambda_0, K)X$ and the continuity in t of $E(\sigma_{n,t}, K_{n,t})$ to deduce the result (a).

(ii) Inequality (5) follows from using part (c) of the Lemma with $t=1$. For $x \in E(\lambda_0, K)X$,

$$\begin{aligned} \|x - E(\sigma_n, K_n)x\| &= \|E(\lambda_0, K)x - E(\sigma_n, K_n)x\| \\ &= \left\| \frac{1}{2\pi i} \int_{C_1} [(\lambda - K)^{-1}x - (\lambda - K_n)^{-1}x] \, d\lambda \right\| \\ &\leqq \frac{2M\varepsilon}{c_0} [\|(K-K_n)x\| + M\|x\| \, \|(K-K_n)K_n\|]. \end{aligned}$$

Q.E.D.

The interpretation of this theorem with respect to the eigenvalues and eigenvectors of K_n should be fairly evident. However, the theorem says nothing about separating simple eigenvectors from generalized ones. More precisely, if x is a generalized eigenvector of K, are the approximating vectors $E(\sigma_n, K_n)x$ made up of simple or generalized eigenvectors of the K_n?

3. **A somewhat computable error bound.** Suppose that in solving $(\lambda - K_n)x = 0$ for several different values of n, we think a certain group of eigenvalues of the K_n

corresponds to some eigenvalue of the original operator K. For a particular n, let $\sigma_n=\{\lambda_1,\ldots,\lambda_q\}$ be the eigenvalues which are thought to correspond to some eigenvalue of the original operator K. In their listing in σ_n, let them be repeated according to the number of corresponding linearly independent eigenvectors they possess, and then let these eigenvectors be $\phi_1,\ldots,\phi_q$ with each having norm 1. Do not repeat a listing of an eigenvalue for its generalized eigenvectors.

In a somewhat arbitrary manner, define

$$\lambda_0 = \frac{1}{q}\sum_{i=1}^{q}\lambda_i \quad\text{and}\quad \eta = \operatorname*{Max}_{i=1,\ldots,q}|\lambda_0-\lambda_i|.$$

A test will be constructed to determine if there are eigenvalues of K within ε of λ_0, where $\varepsilon>0$ is a given small number satisfying the conditions (i) $\varepsilon>\eta$, (ii) the only eigenvalues of K_n which are within ε of λ_0 are the members of σ_n, and (iii) $(\lambda-K)^{-1}$ and $(\lambda-K_n)^{-1}$ exist on the circumference $|\lambda_0-\lambda|=\varepsilon$. Define

$$R(\varepsilon,n) = \frac{\|(K-K_n)K_n\|}{(1-\eta/\varepsilon)\left(\operatorname*{Min}_i|\lambda_i|\right)}\left[\operatorname*{Max}_{|\lambda_0-\lambda|=\varepsilon}\|(\lambda-K)^{-1}\|\right].$$

Also define $\sigma=\sigma(K)\cap\{\lambda : |\lambda-\lambda_0|<\varepsilon\}$; this is the set of eigenvalues of K which are within ε of λ_0.

THEOREM 4. *With respect to the preceding remarks, the following are true statements.*

(a) *If* $R(\varepsilon,n)<1$, *then* $\sigma\neq\varnothing$.

(b) *Let* $R(\varepsilon,n)<1$. *Then for each eigenvector* ϕ_i *of* K_n, *there is an element* ψ_i *from* $E(\sigma,K)X$ *such that* $\|\psi_i\|=1$ *and* $\|\phi_i-\psi_i\|\leqq 2R(\varepsilon,n)$. Note: *This does not say* $\psi_1,\ldots,\psi_q$ *are linearly independent.*

Proof. (i) In the remarks preceding the theorem, it was assumed that ε was chosen in such a way that $|\lambda-\lambda_0|=\varepsilon$ implies $(\lambda-K)^{-1}$ and $(\lambda-K_n)^{-1}$ exist. Let $|\lambda-\lambda_0|=\varepsilon$. Then for $i=1,\ldots,q$,

$$(\lambda-K_n)\phi_i = (\lambda-\lambda_i)\phi_i+(\lambda_i-K_n)\phi_i = (\lambda-\lambda_i)\phi_i,$$

and

$$(6)\qquad (\lambda-K_n)^{-1}\phi_i = \phi_i/(\lambda-\lambda_i).$$

Also,

$$\begin{aligned}(\lambda-K)\phi_i &= (\lambda-K)\phi_i-(\lambda_i-K_n)\phi_i\\ &= (\lambda-\lambda_i)\phi_i+(K_n-K)\phi_i,\end{aligned}$$

and

$$\frac{1}{\lambda-\lambda_i}\phi_i = (\lambda-K)^{-1}\phi_i+\frac{1}{\lambda-\lambda_i}(\lambda-K)^{-1}(K_n-K)\phi_i.$$

Using (6),

$$
(7)\qquad
\begin{aligned}
(\lambda-K_n)^{-1}\phi_i-(\lambda-K)^{-1}\phi_i &= \frac{1}{\lambda-\lambda_i}(\lambda-K)^{-1}(K_n-K)\phi_i \\
&= \frac{1}{\lambda_i(\lambda-\lambda_i)}(\lambda-K)^{-1}(K_n-K)K_n\phi_i.
\end{aligned}
$$

Using the usual definitions of projections, define $E(\sigma, K)$ and $E(\sigma_n, K_n)$. Then

$$
\begin{aligned}
\phi_j-E(\sigma, K)\phi_j &= E(\sigma_n, K_n)\phi_j-E(\sigma, K)\phi_j \\
&= \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\varepsilon}[(\lambda-K_n)^{-1}\phi_j-(\lambda-K)^{-1}\phi_j]\,d\lambda \\
&= \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\varepsilon}\frac{(\lambda-K)^{-1}(K_n-K)K_n\phi_j}{\lambda_j(\lambda-\lambda_j)}\,d\lambda,
\end{aligned}
$$

where (7) is used to obtain the last integral. Now take norms and recall that $\|\phi_j\|=1$. Then for $j=1,\ldots,q$,

$$
\begin{aligned}
\|\phi_j-E(\sigma, K)\phi_j\| &\leqq \frac{2\pi\varepsilon}{2\pi}\frac{1}{|\lambda_j|}\frac{\|(K-K_n)K_n\|}{\operatorname*{Min}_{|\lambda-\lambda_0|=\varepsilon}|\lambda-\lambda_j|}\operatorname*{Max}_{|\lambda-\lambda_0|=\varepsilon}\|(\lambda-K)^{-1}\| \\
&= \frac{\varepsilon}{\varepsilon-\eta}\frac{1}{|\lambda_j|}\|(K-K_n)K_n\|\operatorname*{Max}_{|\lambda-\lambda_0|=\varepsilon}\|(\lambda-K)^{-1}\| \\
&\leqq R(\varepsilon, n).
\end{aligned}
$$

$$
(8)\qquad \|\phi_j-E(\sigma, K)\phi_j\| \leqq R(\varepsilon, n), \qquad j = 1,\ldots,q.
$$

If $R(\varepsilon, n)<1$, then $\|\phi_j-E(\sigma, K)\phi_j\|<1$. Thus, from the assumption $\|\phi_j\|=1$, it follows that $E(\sigma, K)\phi_j\neq 0$, and therefore that $\sigma\neq\varnothing$.

(iii) To prove (b), define

$$
\psi_j = E(\sigma, K)\phi_j/\|E(\sigma, K)\phi_j\|.
$$

Then use (8) in the following.

$$
\begin{aligned}
\|\phi_j-\psi_j\| &\leqq \|\phi_j-E(\sigma, K)\phi_j\|+\|E(\sigma, K)\phi_j-\psi_j\| \\
&\leqq R(\varepsilon, n)+\|E(\sigma, K)[1-1/\|E(\sigma, K)\phi_j\|]\phi_j\| \\
&= R(\varepsilon, n)+|1-\|E(\sigma, K)\phi_j\|\,| \\
&\leqq R(\varepsilon, n)+\|\phi_j-E(\sigma, K)\phi_j\| \\
&\leqq 2R(\varepsilon, n).
\end{aligned}
$$

Q.E.D.

If $\sigma_n=\{\lambda_1\}$, then

$$
R(\varepsilon, n) = \frac{\|(K-K_n)K_n\|}{|\lambda_1|}\operatorname*{Max}_{|\lambda-\lambda_1|=\varepsilon}\|(\lambda-K)^{-1}\|.
$$

To compute the quantity $\mathrm{Max}_{|\lambda-\lambda_0|=\varepsilon} \|(\lambda-K)^{-1}\|$, the bound on $\|(\lambda-K)^{-1}\|$ given in [1] may be useful.

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UNIVERSITY OF WISCONSIN,
MADISON, WISCONSIN