ON A PIECE OF HYPERSURFACE IN A Riemannian Manifold with Mean Curvature Bounded Away from Zero

BY

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Introduction. In 1965, S. S. Chern proved the following theorem:

Theorem 0.1. Let $M$ be a compact piece of an oriented hypersurface of dimension $m$ with smooth boundary $\partial M$, which is immersed in an euclidean space $E$ of dimension $m + 1$. Suppose the mean curvature $H_1 \geq c > 0$. Let $a$ be a fixed unit vector which makes an angle $\leq \pi/2$ with the normals of $M$. Then

$$mcV_a \leq L_a,$$

where $V_a$ is the volume of the orthogonal projection of $M$ and $L_a$ that of $\partial M$ in the hyperplane perpendicular to $a$. If $M$ is defined by the equation

$$z = z(x_1, \ldots, x_m), \quad x_1^2 + \cdots + x_m^2 \leq R^2,$$

where $x_1, \ldots, x_m, z$ are rectangular coordinates in the space $E$ and the $z$-axis is chosen in the direction of $a$, then $cR \leq 1$ [1, p. 82](2).

This theorem is a generalization of Heinz's theorem for the $C^2$-surface defined by an equation $z = z(x, y)$ in euclidean three space [2].

The purpose of this paper is to extend this theorem to a hypersurface in a Riemann manifold.

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1. A certain generalization of Chern's theorem. We consider a Riemann manifold $R^{m+1}$ ($m + 1 \geq 3$) of class $C^\nu$ ($\nu \geq 3$) which admits a one-parameter group $G$ of transformations generated by an infinitesimal transformation:

$$(1.1) \quad x^i = x^i + \xi^i(x) \delta \tau$$

(where $x^i$ are local coordinates in $R^{m+1}$ and $\xi^i$ are the components of a contravariant vector $\xi$. If $\xi$ is a Killing vector, a homothetic Killing vector, a conformal Killing vector, etc. [3, p. 32], then the group $G$ is called isometric, homothetic, conformal, etc.

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447
In $R^{m+1}$, we consider a domain $D$. If the domain $D$ is simply covered by the paths of the transformations generated by $\xi$, and $\xi$ is everywhere of class $C^\tau$ and $\neq 0$ in $D$, then we call $D$ a regular domain with respect to the vector field $\xi$.

In a regular domain $D$, we choose a coordinate system such that the paths of the transformations generated by $\xi$ are new $x^1$-coordinate curves, that is, a coordinate system in which the vector $\xi$ has the components $\delta^i_1$ (where $\delta^i_j$ denotes the Kronecker delta). Then (1.1) becomes $\vec{x}^i = x^i + \delta^i_1 \tau$ and the domain $D$ admits the transformation given by

\[(1.1)' \quad \vec{x}^i = x^i + \delta^i_1 \tau.\]

Now we consider a compact piece $M$ of an oriented hypersurface of dimension $m$ with smooth boundary $\partial M$, which is immersed in a regular domain $D$ with respect to the vector field $\xi$. The immersion $x: M \to R^{m+1}$ is locally given by

\[(1.2) \quad x^i = x^i(u^\alpha), \quad i = 1, \ldots, m+1, \quad \alpha = 1, \ldots, m.\]

Here and henceforth, Latin indices run from 1 to $m+1$ and Greek indices from 1 to $m$. Let us consider a hypersurface which is orthogonal to the new $x^1$-coordinate curves and project $x(M)$ along the $x^1$-coordinate curve into the orthogonal hypersurface, and let $M'$ be the image of $M$. If $x'(p)$ is the image of $x(p)$, $p \in M$ under this projection, we have

\[(1.3) \quad x'^i = x^i(u^\alpha) - \delta^i_1 \tau(u^\alpha),\]

where $\tau$ is a function defined on $M$.

Let us consider a differential form of degree $m-1$ at a point $p$ of $M$, defined by

\[(1.4) \quad ((n, (\rho'/\rho)\rho^{-1}(\delta^i_1/\rho), dx, \ldots, dx)) \overset{\text{def.}}{=} g^{-\frac{1}{2}}(n, (\rho'/\rho)\rho^{-1}(\delta^i_1/\rho), dx, \ldots, dx),\]

where the symbol $(\quad)$ means a determinant of order $m+1$ whose columns are the components of respective vectors or vector-valued differential forms, $n$ is a unit normal vector at a point $p$ of $M$, $dx$ is a displacement along $M$, $g$ is the determinant of the metric tensor $g_{ij}$ of $R^{m+1}$, and the functions $\rho$ and $\rho'$ are defined by

\[
\begin{align*}
\rho(p) &= \text{length of the vector } \delta^i_1 \text{ at } x(p) = (g_{11}(x(p)))^{1/2}, \\
\rho'(p) &= \text{length of the vector } \delta^i_1 \text{ at } x'(p) = (g_{11}(x'(p)))^{1/2}.
\end{align*}
\]

Then the exterior differential of the differential form (1.4) divided by $m!$ becomes

\[(1.5) \quad \frac{1}{m!} d((n, (\rho'/\rho)\rho^{-1}\delta^i_1/\rho, dx, \ldots, dx)) = \frac{1}{m!} \{(Dn, (\rho'/\rho)\rho^{-1}\delta^i_1/\rho, dx, \ldots, dx) + (((n, D(\rho'm^{-1}/\rho_m \delta^i_1), dx, \ldots, dx)),
\]

where the symbol $D$ means the covariant differential.
The first term of the right-hand member of the above expression becomes

\[(1.6) \quad \frac{1}{m!} \left( (Dn, (\rho^{m-1}/\rho^m) \delta_1, dx, \ldots, dx) \right) = \left( -\frac{1}{m!} H^1_1 (\rho^{m-1}/\rho^m) n_i \delta^i_1 \, dV , \right) \]

where \( H^1_1 \) is the first mean curvature of \( M \) and \( dV \) is the volume element of \( M \). On the other hand, we easily obtain the following relation:

\[(1.7) \quad n_i \delta^i_1 \, dV = \left( -\frac{1}{m!} (\delta_1, dx, \ldots, dx) \right), \]

and from (1.3) we have

\[(1.8) \quad \frac{\left( -\frac{1}{m!} (\delta_1, dx, \ldots, dx) \right)}{g^{1/2}} = \frac{\left( \delta_1, dx', \ldots, dx' \right)}{g'^{1/2}}, \]

where \( n' \) and \( dV'_z \) mean the unit normal vector and the volume element at the corresponding point \( x'(\rho) \) of \( M' \) respectively and \( g' = g(x') \). Consequently from (1.7) and (1.8), we have

\[(1.9) \quad \frac{\left( -\frac{1}{m!} (\delta_1, dx, \ldots, dx) \right)}{g^{1/2}} = \frac{\left( \delta_1, dx', \ldots, dx' \right)}{g'^{1/2}}, \]

because \( \delta_1/\rho' \) is nothing but the unit normal vector of \( M' \).

Now we suppose that the transformation group \( G \) is conformal. Then \( R^{m+1} \) admits the infinitesimal transformation given by (1.1) with the additional condition

\[\xi_{ij} + \xi_{ij} = 2\phi(x)g_{ij}, \]

where the symbol ";' means the covariant derivative. In the new coordinate system of the domain \( D \), the additional condition becomes

\[\frac{\partial g_{ij}}{\partial x^1} = 2\phi(x)g_{ij}, \]

Therefore the metric tensor \( g_{ij} \) with respect to the new coordinate system has the form

\[(1.10) \quad g_{ij} = \exp \left( 2 \int \phi(x) \, dx^1 \right) f_{ij}(x^2, \ldots, x^{m+1}). \]

From this, we can easily deduce the following result:

\[(1.11) \quad \frac{g^{1/2}}{(g'^{1/2})} \left( \frac{\rho'}{\rho} \right)^m = \exp \left( \int_{x^1}^{x^1} \phi(x) \, dx^1 \right) \]
since $\rho = (g_{11})^{1/2}$ and $\rho' = (g'_{11})^{1/2}$. Therefore from (1.9) and (1.11), (1.6) becomes

\begin{equation}
\frac{1}{m!}((Dn, (\rho'^{m-1}/\rho^m)\delta_1, dx, \ldots, dx)) = (-1)^n H_1 \exp \left( \int_{x^{-1}} \phi \, dx^1 \right) dV_i.
\end{equation}

Put

\begin{equation}
v_{\mathcal{I}} = \frac{\rho'^{m-1}}{\rho^m} \delta_1^i g_{ij}.
\end{equation}

Then the second term of the right-hand member of (1.5) may be written as follows:

\begin{equation}
\frac{1}{m!}((n, D((\rho'^{m-1}/\rho^m)\delta_1), dx, \ldots, dx)) = \frac{(-1)^m}{2m} L_{\nu} g_{ij} g^{*ij} dV,
\end{equation}

where $L_{\nu} g_{ij}$ is the Lie derivative of the tensor $g_{ij}$ with respect to $\nu$, $g^{*ij}$ means $(\partial x^i/\partial u^\alpha)(\partial x^j/\partial u^\beta)g^{\alpha\beta}$, and $g_{\alpha\beta}$ is the metric tensor of $M$.

Accordingly from (1.12) and (1.14), we can rewrite (1.5) as follows:

\begin{equation}
\frac{(-1)^m}{m!} d((n, (\rho'/\rho)^{m-1}\delta_1/p, dx, \ldots, dx)) = \exp \left( \int_{x^{-1}} \phi \, dx^1 \right) H_1 \, dV_i + \frac{1}{2m} L_{\nu} g_{ij} g^{*ij} dV.
\end{equation}

Integrating both members of (1.15) over $M$ and applying Stokes' theorem we have

\begin{equation}
\frac{(-1)^m}{m!} \int_{\partial M} ((n, (\rho'/\rho)^{m-1}\delta_1/p, dx, \ldots, dx)) = \int_{M} \exp \left( \int_{x^{-1}} \phi \, dx^1 \right) H_1 \, dV_i + \frac{1}{2m} \int_{M} L_{\nu} g_{ij} g^{*ij} dV.
\end{equation}

On the other hand, from (1.3) and (1.11) we can see that

\begin{equation}
\frac{(-1)^m}{m!} ((n, (\rho'/\rho)^{m-1}\delta_1/p, dx, \ldots, dx)) = \frac{(-1)^m g^{1/2}}{m!} \left( \frac{\rho'}{\rho} \right)^m (g')^{1/2} (n, \delta_1/p', dx', \ldots, dx')
\end{equation}

\begin{equation}
= \frac{(-1)^m}{m!} \exp \left( \int_{x^{-1}} \phi \, dx^1 \right) (g')^{1/2} (n, \delta_1/p', dx', \ldots, dx').
\end{equation}

Now for each point $p \in M$, let us take the small piece $\bar{M}_p$ of the hypersurface passing through the corresponding point $x'(p)$ of $M'$, given by the following expression

\begin{equation}
x^i = x^i(u^\alpha) - \delta_1^i \tau_p, \quad \tau_p = \tau(p) = \text{const},
\end{equation}

where $\tau_p$ means $x^i(p) - x'^i(p)$. Then we have

\begin{equation}
\frac{\partial \bar{x}^i}{\partial u^\alpha} = \frac{\partial x^i}{\partial u^\alpha}.
\end{equation}
and we can obtain the following relation:

\[ n_i = \left[ \frac{\tilde{g}_{a\bar{b}}}{\sqrt{\det g_{a\bar{b}}}} \right]^{1/2} \frac{1}{\sqrt{\det (\tilde{g})^{1/2}}} \tilde{n}_i, \]

where \( \tilde{g}_{a\bar{b}} \) is the metric tensor of \( \tilde{M}_p \) and \( \tilde{n} \) is the unit normal vector of \( \tilde{M}_p \) at \( x'(p) \), \( \det g_{a\bar{b}} \) and \( \det \tilde{g}_{a\bar{b}} \) mean the determinants of \( g_{a\bar{b}} \) and \( \tilde{g}_{a\bar{b}} \), respectively. Since

\[ g_{a\bar{b}} = g_{ij}(x) \frac{\partial x^i}{\partial u^a} \frac{\partial x^j}{\partial u^\bar{b}}, \quad \tilde{g}_{a\bar{b}} = \tilde{g}_{ij}(x') \frac{\partial \tilde{x}^i}{\partial u^a} \frac{\partial \tilde{x}^j}{\partial u^\bar{b}}, \]

making use of (1.10) we have

\[ \det g_{a\bar{b}} = \exp \left( 2m \int_{x'^1}^{x^1} \phi \, dx^1 \right) \left| \det \tilde{g}_{a\bar{b}} \right|. \]

Therefore we have the following relation:

\[ n_i = \exp \left( \int_{x'^1}^{x^1} \phi \, dx^1 \right) \tilde{n}_i. \]

Since

\[ g_{ij}(x) = \exp \left( 2 \int_{x'^1}^{x^1} \phi \, dx^1 \right) g_{ij}(x'), \]

it follows that

\[ g^{ij}(x) = \exp \left( -2 \int_{x'^1}^{x^1} \phi \, dx^1 \right) g^{ij}(x'). \]

Thus we obtain

\[ n' = \exp \left( -\int_{x'^1}^{x^1} \phi \, dx^1 \right) \tilde{n}'. \]

Substituting the above result in (1.17), we get

\[ (1.18) \quad \frac{(-1)^m}{m!} ((n, (p^1/\rho)^{m-1} \delta_1/\rho, dx, \ldots, dx)) = \frac{(-1)^m}{m!} (g')^{1/2}(\tilde{n}, \delta_1/\rho', dx', \ldots, dx'). \]

Let \( e_A, A = 1, \ldots, m-1, \) be an oriented basis for the tangent plane of \( \partial M' \) at a point \( x'(p) \) on the image \( \partial M' \) of \( \partial M \), and let \( n'' \) be the unit vector of the tangent plane of \( M' \) at the same point \( x'(p) \) which is orthogonal to \( e_A, A = 1, \ldots, m-1, \) \( n' \) and such that

\[ ((n'', n', e_1, \ldots, e_{m-1})) > 0. \]

If \( dx' \) is a displacement along \( \partial M' \), then

\[ \frac{(-1)^m}{(m-1)!} ((n'', n', dx', \ldots, dx')) \]

is the volume element of \( \partial M' \). The unit normal vector \( \tilde{n} \) of \( \tilde{M}_p \) at \( x'(p) \) can be expressed as follows:

\[ \tilde{n} = \omega n'' + \beta n' + \sum_{A=1}^{m-1} \gamma^A e_A, \]
where $\alpha$, $\beta$, $\gamma^\mu$ satisfy
\[ \alpha^2 + \beta^2 + g_{ij}(x')e_i^\mu e_j^\nu \gamma^\mu \gamma^\nu = 1. \]

Therefore we have $|\alpha| \leq 1$. Since $\delta_1/\rho'$ is the unit normal vector $n'$, it follows that
\[ (-1)^m (m-1)! ((\delta_1/\rho', dx', \ldots, dx')) = (-1)^m \alpha((n', dx', \ldots, dx')). \]

By making use of a notation $dL'$ for the volume element of $\partial M'$, from (1.18) and (1.19), (1.17) becomes
\[ \frac{(-1)^m}{m!} ((n, (\rho'/\rho)^{m-1}\delta_1/\rho, dx, \ldots, dx)) = \frac{1}{m} \alpha dL', \quad |\alpha| \leq 1. \]

Substituting (1.20) in (1.16), we have
\[ \frac{1}{m} \int_{\partial M} \alpha dL' = \int_{M} \exp \left( \int_{x^1} \phi \, dx^1 \right) H_1 \, dV' + \frac{1}{2m} \int_{M} L_v g_{ij} g^{*ij} \, dV. \]

Consequently it follows that
\[ \frac{1}{m} \int_{\partial M} dL' \geq \int_{M} \exp \left( \int_{x^1} \phi \, dx^1 \right) H_1 \, dV' + \frac{1}{2m} \int_{M} L_v g_{ij} g^{*ij} \, dV. \]

From the above inequality, we can obtain the following

**Theorem 1.1 (Main Theorem).** Let $M$ be a compact piece of an oriented hypersurface of dimension $m$ with smooth boundary $\partial M$, which is immersed in a regular domain of a Riemann manifold of dimension $m+1$ admitting a conformal Killing vector field $\xi$ (i.e., $\xi_{ij} + \xi_{ij} = 2\phi g_{ij}$). Suppose the mean curvature $H_1 \geq c > 0$ ($c =$ const).

Let us suppose that the $\xi(x)$ makes an angle $\leq \pi/2$ with the normals of $M$, and let

(I) \[ \int_{x^1} \phi \, dx^1 \geq 0 \quad \text{everywhere on } M, \]

(II) \[ \int_{M} L_v g_{ij} g^{*ij} \, dV \geq 0. \]

Then
\[ mcV' \leq L', \]

where $V'_\varepsilon$ and $L'_\varepsilon$ are the volumes of the images $M'$ of $M$ and $\partial M'$ of $\partial M$, by the projection along the paths of the vector field $\xi$ into a hypersurface perpendicular to the paths of $\xi$.

2. **Some examples of the main theorem.** In this section, we shall give some examples which satisfy the hypotheses (I) and (II) in the main theorem.

The hypothesis (I) means that either $\xi$ is a Killing vector or $\rho \geq \rho'$ everywhere on $M$. 

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As an example in the case satisfying the hypothesis (II), we have the following

**Theorem 2.1.** Let $R^{n+1}$ admit a conformal Killing vector field (i.e., $\xi_{i;j} + \xi_{j;i} = 2\phi_{ij}$) and if the paths of the transformations generated by $\xi$ are geodesics (i.e., the new $x^1$-coordinate curves are geodesics), and $\phi \geq 0$ everywhere on $M$, then it follows that

$$L_\xi g_{ij} \geq 0$$

everywhere on $M$.

**Proof.** Let us choose a coordinate system such that the family of hypersurfaces perpendicular to the new $x^1$-coordinate curves are new $x^2, \ldots, x^{n+1}$-coordinate hypersurfaces. Then $g_{ij} = 0$ for $j = 2, \ldots, m+1$.

Since the $x^1$-coordinate curve is a geodesic, substituting

$$\frac{dx^1}{ds} = \frac{1}{\sqrt{g_{11}}}, \frac{dx^2}{ds} = \cdots = \frac{dx^{n+1}}{ds} = 0$$

in the differential equation of a geodesic:

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0, \quad i = 1, 2, \ldots, m+1,$$

we have

$$-\frac{1}{2} \frac{\partial g_{11}}{\partial x^j} + \Gamma^j_{11} = 0, \quad \Gamma^j_{11} = 0, \quad j = 2, \ldots, m+1.$$  

Substituting $\frac{\partial g_{ik}}{\partial x^1} = 2\phi g_{ik}$ in $\Gamma^j_{11}$ ($j = 2, \ldots, m+1$), we obtain

$$\Gamma^j_{11} = \frac{1}{2} g^{ik} \frac{\partial g_{11}}{\partial x^k}, \quad j = 2, \ldots, m+1.$$  

And also from $g^{11} = 0$, $j = 2, \ldots, m+1$, the conditions $\Gamma^j_{11} = 0$, $j = 2, \ldots, m+1$, become

$$\sum_{k=2}^{m+1} g^{ik} \frac{\partial g_{11}}{\partial x^k} = 0, \quad j = 2, \ldots, m+1.$$  

Since the determinant of $g^{ik}$ ($j, k = 2, \ldots, m+1$) is not equal to zero, the conditions $\Gamma^j_{11} = 0$ ($j = 2, \ldots, m+1$) are rewritten as follows:

$$\frac{\partial g_{11}}{\partial x^k} = 0, \quad k = 2, \ldots, m+1.$$  

And the condition

$$-\frac{1}{2} \frac{\partial g_{11}}{\partial x^1} + \Gamma^1_{11} = 0$$

follows clearly from $g_{1j} = 0$, $j = 2, \ldots, m+1$. Therefore in this case, a necessary and sufficient condition that the $x^1$-coordinate curve be a geodesic is that $\frac{\partial g_{11}}{\partial x^k} = 0$, $k = 2, \ldots, m+1$. 

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To calculate $L_v g_{ij} e^{*il}$, we go back to the definition (1.13):

$$v_i = \frac{\rho^\prime}{\rho^m} \delta^i_1 g_{ij},$$

where $\rho = (g_{11}(x))^{1/2}$ and $\rho^\prime = (g_{11}(x'))^{1/2}$. $\rho^\prime$ is constant on the $x^i$-coordinate curve. Then we have

$$v_{ij} = \frac{1}{2} \frac{\partial g_{ij}}{\partial x^i} (g_{11})^{(m-1)/2} - \frac{m}{2} \frac{(g_{11})^{(m-1)/2}}{2} \frac{\partial g_{11}}{\partial x^i} \delta^i_1 g_{il}$$

$$+ \frac{m-1}{2} \frac{g_{11}^{(m-3)/2}}{2} \frac{\partial g_{11}^i}{\partial x^i} \delta^i_1 g_{ij}$$

$$L_v g_{ij} = v_{ij} + \frac{m-1}{2} \frac{g_{11}^{(m-3)/2}}{2} \frac{\partial g_{11}^i}{\partial x^i} (g_{11})^{(m-1)/2}$$

because $\partial g_{ij}/\partial x^i = 2g_{ij}$. From the fact that the $x^i$-coordinate curves are geodesics and $(g_{11})^{1/2}$ is constant on the $x^i$-coordinate curve, we have the following relations:

$$\frac{\partial g_{11}}{\partial x^j} = 0, \quad j = 2, \ldots, m+1,$$

$$\frac{\partial g_{11}^i}{\partial x^j} = 0, \quad \frac{\partial g_{11}^i}{\partial x^j} = 0, \quad j = 2, \ldots, m+1,$$

since $x^i = x^j, j = 2, \ldots, m+1$. By making use of the above relations, we get

$$L_v g_{ij} e^{*il} = 2m \phi \frac{(g_{11})^{(m-1)/2}}{(g_{11})^{m/2}} - m \frac{(g_{11})^{(m-1)/2}}{(g_{11})^{m+2/2}} \frac{\partial g_{11}}{\partial x^1} \frac{\partial x^1}{\partial x^j} g^{a\beta}$$

$$= 2m \phi \frac{(g_{11})^{(m-1)/2}}{(g_{11})^{m/2}} \left(1 - g^{a\beta} \frac{\partial x^1}{\partial x^j} \frac{\partial x^1}{\partial x^j} \right),$$

and also we can see that

$$g_{ik} \frac{\delta^i_1}{(g_{11})^{1/2}} \frac{\partial x^k}{\partial u^a} = (g_{11})^{1/2} \frac{\partial x^i}{\partial u^a},$$

since the unit vector $\delta^i_1/(g_{11})^{1/2}$ at a point on $M$ is expressed as follows:

$$\frac{\delta^i_1}{(g_{11})^{1/2}} = \lambda x^i + \mu^a \frac{\partial x^i}{\partial u^a},$$

where

$$\lambda + g_{ij} \frac{\partial x^i}{\partial u^a} \frac{\partial x^j}{\partial u^a} \mu^a \mu^\beta = \lambda^2 + g_{a\beta} \mu^a \mu^\beta = 1,$$
we have
\[ g_{ij} \frac{\partial x^i}{\partial u^a} \frac{\partial x^j}{\partial u^b} = g_{ij} \frac{\partial x^i}{\partial u^a} \frac{\partial x^j}{\partial u^b} \mu^a = g_{\alpha \beta} \mu^\alpha. \]

Therefore it follows that
\[ g_{11} \frac{\partial x^1}{\partial u^a} \frac{\partial x^1}{\partial u^b} g^{a\beta} = g_{\alpha \beta} \mu^\alpha \mu^\beta = g_{\alpha \beta} \mu^\alpha \mu^\beta \leq 1. \]

Consequently if \( \phi \geq 0 \) everywhere on \( M \), then it follows that
\[ L_v g_{ij} g^{*ij} = 2m \phi \left( \frac{g^{11})^{m - 1/2}}{g_{11}^{m/2}} \right) \left( 1 - g_{11} \frac{\partial x^1}{\partial u^a} \frac{\partial x^1}{\partial u^b} g^{a\beta} \right) \geq 0 \]
everywhere on \( M \).

From Theorem 1.1 and Theorem 2.1, we can see

**Theorem 2.2.** Let \( M \) be a compact piece of an oriented hypersurface of dimension \( m \) with smooth boundary \( \partial M \), which is immersed in a regular domain of a Riemann manifold of dimension \( m + 1 \) admitting a conformal Killing vector field \( \xi \) with \( \xi_{ij} + \xi_{ji} = 2 \phi g_{ij}, \phi \geq 0 \). Suppose the mean curvature \( H \geq c > 0 \) \( (c = \text{const}) \) and suppose that \( \xi(x) \) makes an angle \( \leq \pi/2 \) with the normal at each point of \( M \). If the paths of the transformations generated by \( \xi \) are geodesics, then
\[ m V'_t \leq L'_t, \]
where \( V'_t \) and \( L'_t \) are the volumes of the image \( M' \) of \( M \) and the image \( \partial M' \) of \( \partial M \), by the projection along the paths into a hypersurface perpendicular to the paths, which never intersects \( M \).

**Corollary 2.1.** Let \( M \) be a compact piece of an oriented hypersurface of dimension \( m \) with smooth boundary \( \partial M \), which is immersed in a regular domain of a Riemann manifold of dimension \( m + 1 \) admitting a homothetic Killing vector field \( \xi(x) \) with \( \xi_{ij} + \xi_{ji} = 2 c_0 g_{ij}, c_0 > 0 \), or a Killing vector field. Suppose the mean curvature \( H \geq c > 0 \) \( (c = \text{const}) \) and that \( \xi(x) \) makes an angle \( \leq \pi/2 \) with the normal at each point of \( M \). If the paths of the transformations generated by \( \xi \) are geodesics, then
\[ m V'_t \leq L'_t, \]
where \( V'_t \) and \( L'_t \) are the volumes of the image \( M' \) of \( M \) and the image \( \partial M' \) of \( \partial M \), by the projection along the paths into a hypersurface perpendicular to the paths, which never intersects \( M \).

Especially if our manifold \( \mathbb{R}^{m+1} \) is an euclidean space \( E^{m+1} \) and if \( \xi \) is the homothetic Killing vector field on \( E^{m+1} \) with components \( \xi^i = x^i \), \( x^i \) being rectangular coordinates in the space \( E^{m+1} \), then the paths of the transformations generated by
ξ are the lines through O and a hypersurface perpendicular to these paths is a hypersphere of radius, say $R_0$, around O. The functions $\rho$ and $\rho'$ are given by

$$\rho = \left( \sum_{i=1}^{m+1} (x^i)^2 \right)^{1/2} \quad \text{and} \quad \rho' = R_0.$$ 

Thus, as a special case of Corollary 2.1, we have the following

**Corollary 2.2.** Let $M$ be a compact piece of an oriented hypersurface of dimension $m$ with smooth boundary $\partial M$, which is immersed in an euclidean space of dimension $m+1$. Suppose that the position vector $x(u)$ makes an angle $\frac{\pi}{2}$ with the normal of $x(M)$ at $x(u)$ for every $u \in M$ and that the mean curvature $H_x$ satisfies $H_x \geq c > 0$ ($c =$ const). Then

$$mcV'_i \leq L'_i,$$

where $V'_i$ and $L'_i$ are the volumes of the image $M'$ of $M$ and the image $\partial M'$ of $\partial M$, by the projection from the origin $O$ into a hypersphere around $O$ not intersecting $M$.

From this result we have (under the same assumptions as Corollary 2.2)

**Corollary 2.3.** Let $\rho, \theta_1, \ldots, \theta_m$ be polar coordinates in $E^{m+1}$, i.e.,

$$x^1 = \rho \sin \theta_m \sin \theta_{m-1} \cdots \sin \theta_2 \sin \theta_1,$$

$$x^2 = \rho \sin \theta_m \sin \theta_{m-1} \cdots \sin \theta_2 \cos \theta_1,$$

$$x^3 = \rho \sin \theta_m \sin \theta_{m-1} \cdots \sin \theta_3 \cos \theta_2,$$

$$\vdots$$

$$x^m = \rho \sin \theta_m \cos \theta_{m-1},$$

$$x^{m+1} = \rho \cos \theta_m.$$ 

If $M$ is defined by the equation

$$\rho = \rho(\theta_1, \ldots, \theta_m), \quad 0 \leq \theta_m \leq \phi_0 \ (= \text{const}),$$

then

$$cC_0 \leq 1,$$

where $C_0$ is a constant which depends only on $R_0$ and $\phi_0$ ($R_0$ is the radius of the hypersphere into which $M$ is projected).

**Proof.** For simplicity, we shall prove in case $m=2$. Since $M$ is written as follows:

$$x^1 = \rho(\theta_1, \theta_2) \sin \theta_2 \sin \theta_1,$$

$$x^2 = \rho(\theta_1, \theta_2) \sin \theta_2 \cos \theta_1, \quad 0 \leq \theta_2 \leq \phi_0,$$

$$x^3 = \rho(\theta_1, \theta_2) \cos \theta_2,$$

in this case, $M'$ is defined by

$$x^1 = R_0 \sin \theta_2 \sin \theta_1,$$

$$x^2 = R_0 \sin \theta_2 \cos \theta_1, \quad 0 \leq \theta_2 \leq \phi_0,$$

$$x^3 = R_0 \cos \theta_2,$$
and $\partial M'$ is the circle on the sphere $S_0$ with the center $O$ and radius $R_0$, given by
\[ x^1 = R_0 \sin \phi_0 \sin \theta_1, \]
\[ x^2 = R_0 \sin \phi_0 \cos \theta_1, \]
\[ x^3 = R_0 \cos \phi_0. \]

Let $g^*$ be the determinant of the metric tensor $g^*|_S$ on $S_0$ with respect to the co-ordinates $\theta_1, \theta_2$. Then $g^* = R_0^4 \sin^2 \theta_2$. Therefore the area of $M'$ is given by
\[ V'_e = \int_0^{2\pi} \int_0^\pi R_0^2 \sin \theta_2 \sin \theta_1 \, d\theta_2 \, d\theta_1 = 2R_0^2\pi(1 - \cos \phi_0), \]
and the length of $\partial M'$, which is a circle of radius $R_0 \sin \phi_0$, is given by
\[ L'_e = 2\pi R_0 \sin \phi_0. \]

From the results above, we have the following relation:
\[ 2V'_e = 2R_0 \frac{1 - \cos \phi_0}{\sin \phi_0} L'_e. \]

Putting
\[ C'_0 = 2R_0 \frac{1 - \cos \phi_0}{\sin \phi_0}, \]
and making use of Corollary 2.2, we obtain
\[ cC_0 L'_e \leq L'_e. \]
Thus $cC_0 \leq 1$.

Finally we consider a Killing vector field $\xi$ on $E^{m+1}$ which generates translations so that the paths of the transformations generated by $\xi$ are parallel lines in the direction of $\xi$ and a hypersurface perpendicular to these paths is a hyperplane perpendicular to $\xi$. In this case, Corollary 2.1 is nothing but Theorem 0.1. Thus Chern's theorem is included as a special case in Corollary 2.1.

REFERENCES


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