IRREDUCIBLE JORDAN ALGEBRAS OF SELF-ADJOINT OPERATORS

BY
ERLING STØRMER

1. **Introduction.** In [8] we classified all JW-factors of type I, and in particular all irreducible JW-algebras of the same type. In the present paper we shall complete the characterization of irreducible JW-algebras by showing that they are all of type I. As a consequence we can generalize Kadison's result [6] on the algebraic irreducibility of irreducible C^*-algebras to JC-algebras, and then use this for two applications, one practically as he did, to characterize pure states of JC-algebras in terms of their kernels, and the other to globalize the above mentioned type I result. The key topological result for these investigations is Theorem 2.1, which gives estimates on the norms on operators in certain real algebras. As another application of this theorem we shall in §3 study the relationship between Jordan ideals in a reversible JC- or JW-algebra and two-sided ideals in the enveloping C^*- or von Neumann algebra.

As this paper is a direct continuation of [8] we shall use the concepts of that paper freely, just recall that a JC-algebra (resp. JW-algebra) is a uniformly (resp. weakly) closed Jordan algebra of self-adjoint operators on a Hilbert space.

We are indebted to D. Topping for making available to us unpublished results on JC-algebras.

2. **Real algebras.** In this section we shall be concerned with real operator algebras and their enveloping C^*-algebras and von Neumann algebras. The core of all our later investigations is that a real algebra $\mathfrak{R}$ satisfying the extra reality condition $\mathfrak{R} \cap i\mathfrak{R} = \{0\}$ is in a sense "very real." If $\mathfrak{M}$ is a set of operators we denote by $\mathfrak{M}^+$ the set of positive operators in $\mathfrak{M}$.

THEOREM 2.1. *Let $\mathfrak{R}$ be a uniformly closed real self-adjoint operator algebra with identity acting on a Hilbert space. Then the following three conditions are equivalent.*

(i) $\mathfrak{R} \cap i\mathfrak{R} = \{0\}$,

(ii) $(i\mathfrak{R})^+ = \{0\}$,

(iii) *For all A, B in $\mathfrak{R}$, $\|A+iB\| \geqq \max\{\|A\|, \|B\|\}$,*

In particular if the above conditions obtain then $\mathfrak{R}+i\mathfrak{R}$ is a C^*-algebra.

Proof. (i) $\Rightarrow$ (ii). If $A \in \mathfrak{R}$ and $B = iA \geqq 0$ then $B^2 = A^*A$ is a positive operator in $\mathfrak{R}$. Since $\mathfrak{R}$ is self-adjoint and uniformly closed B, being the unique positive square root of B^2, belongs to $\mathfrak{R}$, hence to $\mathfrak{R} \cap i\mathfrak{R} = \{0\}$, $B = 0$.

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(ii) $\Rightarrow$ (i). If $\mathfrak{R} \cap i\mathfrak{R} \neq \{0\}$ it is a C^*-algebra. Since $(\mathfrak{R} \cap i\mathfrak{R})^+ \subset (i\mathfrak{R})^+ = \{0\}$, $\mathfrak{R} \cap i\mathfrak{R} = \{0\}$.

(ii) $\Rightarrow$ (iii). If $A + iB \geqq 0$, A, $B \in \mathfrak{R}$ then $A \geqq 0$. In fact, from the equivalence of (i) and (ii) A is self-adjoint, hence $A = A^+ - A^-$, A^+, $A^- \geqq 0$, $A^+A^- = 0$. Let f be any continuous real function which is zero on the positive reals. Then

$$f(A)Af(A) = -f(A)A^-f(A) \leqq 0.$$

Since $f(A)(A+iB)f(A) \geqq 0$, $f(A)iBf(A) \geqq 0$. By (ii) $f(A)Bf(A) = 0$, hence $f(A)Af(A) = 0$. Since this holds for all such f, $A^- = 0$, i.e., $A \geqq 0$. Let A, $B \in \mathfrak{R}$. In order to show $\|A+iB\| \geqq \max\{\|A\|, \|B\|\}$, we may assume $\|A+iB\| \leqq 1$. Then

$$0 \leqq I - (A+iB)^*(A+iB) = I - A^*A - B^*B - i(A^*B - B^*A).$$

By the above, $I \geqq A^*A + B^*B \geqq 0$, so $\|A\| \leqq 1$, $\|B\| \leqq 1$, and (iii) follows. Clearly (iii) $\Rightarrow$ (i).

If the above conditions obtain let $\{C_n = A_n + iB_n\}$ be a Cauchy sequence in $\mathfrak{R} + i\mathfrak{R}$. Then $\|A_n - A_m + i(B_n - B_m)\| \to 0$ as $n, m \to \infty$. By (iii) $\|A_n - A_m\| \to 0$, $\|B_n - B_m\| \to 0$ as $n, m \to \infty$. Thus $\{A_n\}$ and $\{B_n\}$ are Cauchy sequences in $\mathfrak{R}$, hence converge to A and B in $\mathfrak{R}$ respectively. Thus $C_n = A_n + iB_n \to A + iB$ in $\mathfrak{R} + i\mathfrak{R}$, which is thus uniformly closed, hence is a C^*-algebra.

As an immediate consequence of the proof of (ii) $\Rightarrow$ (iii) we have

COROLLARY 2.2. *Let $\mathfrak{R}$ be as in Theorem* 2.1 *and such that $\mathfrak{R} \cap i\mathfrak{R} = \{0\}$. If A, $B \in \mathfrak{R}$ are such that $A + iB \geqq 0$, then $A \geqq 0$.*

It is often desirable to have a stronger separation between $\mathfrak{R}$ and $i\mathfrak{R}$ than $\mathfrak{R} \cap i\mathfrak{R} = \{0\}$. The next lemma gives an answer to this problem.

LEMMA 2.3. *Let $\mathfrak{A}$ be a reversible JW-algebra such that $\mathfrak{R}(\mathfrak{A}) \cap i\mathfrak{R}(\mathfrak{A}) = \{0\}$. Then $\mathfrak{R}(\mathfrak{A})^- \cap i\mathfrak{R}(\mathfrak{A})^- = \{0\}$.*

Proof. Let $\mathfrak{J} = \mathfrak{R}(\mathfrak{A})^- \cap i\mathfrak{R}(\mathfrak{A})^-$. Then $\mathfrak{J}$ is a weakly closed two-sided ideal in $(\mathfrak{A})^-$ (see the argument in [7, Remark 2.2]). Thus there exists a central projection E in $(\mathfrak{A})^-$ such that $\mathfrak{J} = (\mathfrak{A})^-E$. Now $\mathfrak{J}_{SA} \subset \mathfrak{R}(\mathfrak{A})_{SA}^- \subset \mathfrak{A}$, since $\mathfrak{A}$ is reversible and weakly closed. Thus $E(\mathfrak{A})_{SA}^- \subset \mathfrak{A}$. By the structure theory for von Neumann algebras there exists a central projection F in $(\mathfrak{A})^-$ such that $F(\mathfrak{A})^-$ is of type I_1, i.e., is abelian, and $(I-F)(\mathfrak{A})^-$ has no type I_1 portion. Since $F(\mathfrak{A})^-$ is abelian, $F\mathfrak{R}(\mathfrak{A})^- = \mathfrak{R}(F\mathfrak{A})^- = F\mathfrak{A}$, hence $F\mathfrak{J} = \{0\}$, i.e., $FE = 0$. In order to show $\mathfrak{J} = \{0\}$ we may thus assume $(\mathfrak{A})^-$ has no type I_1 portion. Thus, as $E(\mathfrak{A})_{SA}^- \subset \mathfrak{A}$ [7, Theorem 2.16] shows that $E \neq 0$ implies

$$\{0\} \neq \mathfrak{R}((\mathfrak{A})_{SA}^- E) \cap i\mathfrak{R}((\mathfrak{A})_{SA}^- E) \subset \mathfrak{R}(\mathfrak{A}) \cap i\mathfrak{R}(\mathfrak{A}) = \{0\},$$

a contradiction. Thus $E = 0$, $\mathfrak{J} = \{0\}$.

For reversible JW-algebras the enveloping von Neumann algebra has a neat formulation.

THEOREM 2.4. *Let $\mathfrak{A}$ be a reversible JW-algebra. Then* $(\mathfrak{A})^- = \mathfrak{R}(\mathfrak{A})^- + i\mathfrak{R}(\mathfrak{A})^-$.

Proof. By [8, Lemma 6.1] there exist central projections E and F in $\mathfrak{A}$ such that $E\mathfrak{A}$ is the self-adjoint part of a von Neumann algebra, and $\mathfrak{R}(F\mathfrak{A}) \cap i\mathfrak{R}(F\mathfrak{A}) = \{0\}$, $E+F=I$. Since the theorem holds for self-adjoint parts of von Neumann algebras, we may assume $\mathfrak{R}(\mathfrak{A}) \cap i\mathfrak{R}(\mathfrak{A}) = \{0\}$, hence by Lemma 2.3 $\mathfrak{R}(\mathfrak{A})^- \cap i\mathfrak{R}(\mathfrak{A})^- = \{0\}$. Let $\mathfrak{B} = \mathfrak{R}(\mathfrak{A})^- + i\mathfrak{R}(\mathfrak{A})^-$. By Theorem 2.1 $\mathfrak{B}$ is a C^*-algebra. In order to show $\mathfrak{B} = (\mathfrak{A})^-$ it suffices to show $\mathfrak{B}$ is a von Neumann algebra. In order to accomplish this it suffices by a result of Kadison [5, Lemma 1] to show that the strong limit of each bounded monotone-increasing directed sequence of self-adjoint operators in $\mathfrak{B}$ lies in $\mathfrak{B}$. Let A_α, B_α be operators in $\mathfrak{R}(\mathfrak{A})^-$ such that $\{A_\alpha + iB_\alpha\}$ is a monotone-increasing directed sequence with least upper bound S. Then $A_\alpha + iB_\alpha \to S$ strongly [1, p. 331]. The directed sequence $\{A_\alpha\}$ is monotone-increasing. Indeed, if $\alpha \geqq \beta$ then $A_\alpha + iB_\alpha \geqq A_\beta + iB_\beta$, so that $A_\alpha - A_\beta + i(B_\alpha - B_\beta) \geqq 0$. By Corollary 2.2 $A_\alpha \geqq A_\beta$. Moreover, by Theorem 2.1 $\|A_\alpha\| \leqq \|S\|$. Let A be the least upper bound of the A_α. Then $A_\alpha \to A$ strongly [1, p. 331], and $A \in \mathfrak{R}(\mathfrak{A})^-$.

Let $x_1, \ldots, x_n$ be n given vectors, and let $\varepsilon > 0$. Since the directed sequences $\{A_\alpha\}$ and $\{A_\alpha + iB_\beta\}$ are monotone-increasing there exists α_0 such that if $\alpha \geqq \alpha_0$ then

$$\omega_{x_j}(A_\alpha) \leqq \omega_{x_j}(A) \leqq \omega_{x_j}(A_\alpha) + \varepsilon/2,$$

and there exists β_0 such that if $\beta \geqq \beta_0$ then

$$\omega_{x_j}(A_\beta + iB_\beta) \leqq \omega_{x_j}(S) \leqq \omega_{x_j}(A_\beta + iB_\beta) + \varepsilon/2$$

for $j = 1, \ldots, n$. Let $\gamma \geqq \alpha_0, \beta_0$. Then

$$\begin{aligned} |\omega_{x_j}(iB_\gamma - (S-A))| &= |\omega_{x_j}(iB_\gamma) - \omega_{x_j}(S) + \omega_{x_j}(A_\gamma + iB_\gamma) - \omega_{x_j}(A_\gamma + iB_\gamma) + \omega_{x_j}(A)| \\ &\leqq |-\omega_{x_j}(S) + \omega_{x_j}(A_\gamma + iB_\gamma)| + |\omega_{x_j}(A) - \omega_{x_j}(A_\gamma)| \\ &\leqq \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Thus $iB_\alpha \to S - A$ weakly, or, if $B = -i(S-A)$, then $B_\alpha \to B$ weakly. In particular $B \in \mathfrak{R}(\mathfrak{A})^-$. Since also

$$|\omega_{x_j}(A_\gamma + iB_\gamma - (A+iB))| \leqq |\omega_{x_j}(A_\gamma - A)| + |\omega_{x_j}(iB_\gamma - (S-A))| \leqq \varepsilon/2 + \varepsilon = 3\varepsilon/2,$$

for $j = 1, \ldots, n$, $A_\alpha + iB_\alpha \to A + iB$ weakly. The weak topology is Hausdorff, and $\{A_\alpha + iB_\alpha\}$ converges to both $A + iB$ and S weakly. Thus

$$S = A + iB \in \mathfrak{R}(\mathfrak{A})^- + i\mathfrak{R}(\mathfrak{A})^- = \mathfrak{B}.$$

The proof is complete.

3. **Ideals in *JC*-algebras.** In order to obtain information on reversible *JC*- or *JW*-algebras from their enveloping C^*- or von Neumann algebras, a knowledge of the relationship between their ideals is very helpful. The present section is devoted to this problem.

LEMMA 3.1. *Let $\mathfrak{A}$ be a reversible JC-algebra with $\mathfrak{R}(\mathfrak{A}) \cap i\mathfrak{R}(\mathfrak{A}) = \{0\}$. Let $\mathfrak{F}$ be a uniformly closed two-sided ideal in $(\mathfrak{A})$ such that $\mathfrak{F} \cap \mathfrak{A} = \{0\}$. Then there exist a uniformly closed Jordan ideal $\mathfrak{J}$ in $\mathfrak{A}$ and a central projection F in $(\mathfrak{A})^-$ such that*

$$\mathfrak{F}_{SA} = \mathfrak{J}F.$$

Moreover, there exists a C^*-isomorphism ψ of $\mathfrak{F}_{SA}$ onto $\mathfrak{J}$, ψ being the inverse mapping of the isomorphism $A \to AF$ of $\mathfrak{J}$ onto $\mathfrak{F}_{SA}$.

Proof. Let $\mathfrak{K} = \{A \in \mathfrak{R}(\mathfrak{A})$: there exists $B \in \mathfrak{R}(\mathfrak{A})$ with $A + iB \in \mathfrak{F}\}$. By Theorem 2.1 every operator in $\mathfrak{F}$ is of the form $A + iB$. Clearly $\mathfrak{K}$ is an ideal in $\mathfrak{R}(\mathfrak{A})$. Moreover, for each $A \in \mathfrak{K}$, B is unique, for if $A + iB$ and $A + iC$ both belong to $\mathfrak{F}$ then $i(B - C) \in \mathfrak{F}$, hence $(B - C)^*(B - C) \in \mathfrak{F} \cap \mathfrak{A} = \{0\}$, i.e., $B = C$. Let ρ be the map $\mathfrak{K} \to \mathfrak{K}$ carrying A onto B. Then ρ is a real linear isometry. In fact,

$$A^*A + iA^*B \in \mathfrak{F}, \tag{1}$$

and since $A^* - iB^* = (A + iB)^* \in \mathfrak{F}$, $A^*B - iB^*B \in \mathfrak{F}$. Thus

$$B^*B + iA^*B \in \mathfrak{F}. \tag{2}$$

By (1), (2) and uniqueness, $A^*A = B^*B$, hence in particular $\|A\| = \|B\|$; ρ is an isometry. If $C \in \mathfrak{R}(\mathfrak{A})$ and $A \in \mathfrak{K}$ then

$$\rho(AC) = \rho(A)C, \quad \rho(CA) = C\rho(A). \tag{3}$$

By [2, Proposition 1.7.2] there exists an increasing approximate unit $\{E_\nu\}$ for $\mathfrak{K}$ in $\mathfrak{K}$ (the result is stated for C^*-algebras, but can obviously be generalized to real algebras). By (3) $\rho(AE_\nu) = A\rho(E_\nu)$ for all $A \in \mathfrak{K}$. Since ρ is an isometry and $AE_\nu \to A$ uniformly, $A\rho(E_\nu) \to \rho(A)$ uniformly. The unit ball $\mathfrak{K}_1$ in $\mathfrak{K}^-$ is weakly compact. Let S be a weak limit point of $\{\rho(E_\nu)\}$ in $\mathfrak{K}_1$. Then AS is a weak limit point of $\{A\rho(E_\nu)\}$. Since $\rho(A)$ is a uniform limit of the same net, it is a priori its (unique) weak limit point. Thus

$$\rho(A) = AS = SA, \tag{4}$$

where the last equality follows by (3). Now $(A + iB)^* = A^* - iB^*$. Thus $\rho(A^*) = -B^* = -\rho(A)^*$. In particular, with A selfadjoint $\rho(A) = -\rho(A)^*$. This holds for the E_ν and therefore for their weak limit. Thus $S = -S^*$, and the operator $E = \frac{1}{2}(I + iS)$ is self-adjoint and commutes with $\mathfrak{K}$ by (4). Also $A \in \mathfrak{K}$ implies $AE \in \mathfrak{F}$. Moreover, since $S \in \mathfrak{K}^-$, $E \in (\mathfrak{A})^-$. Since $A \in \mathfrak{K}$ implies $A + i\rho(A) \in \mathfrak{F}$, $\rho(A) - iA \in \mathfrak{F}$, hence $\rho(\rho(A)) = -A$, and $A = -AS^2$. Thus

$$AE^2 = A\tfrac{1}{4}(I + iS)^2 = \tfrac{1}{4}A(I - S^2 + 2iS) = AE. \tag{5}$$

Now the projection $[\mathfrak{K}\mathfrak{X}] \in \mathfrak{A}' \cap (\mathfrak{A})^-$, since $\mathfrak{K}$ is an ideal in $\mathfrak{R}(\mathfrak{A})$, $\mathfrak{X}$ being the underlying Hilbert space. Let $F = E[\mathfrak{K}\mathfrak{X}]$. Then F is self-adjoint, being the product of two commuting selfadjoint operators, and $F^2 = E^2[\mathfrak{K}\mathfrak{X}]$. But if $A \in \mathfrak{K}$, $x \in \mathfrak{X}$, then $E^2Ax = EAx$ by (5), so $E^2[\mathfrak{K}\mathfrak{X}] = E[\mathfrak{K}\mathfrak{X}] = F$, and $F = F^2$ is a projection in

$(\mathfrak{A})^- \cap \mathfrak{K}'$. For $A \in \mathfrak{K}$, $A = A[\mathfrak{K}\mathfrak{X}]$, hence $AF = A[\mathfrak{K}\mathfrak{X}]E = AE \in \mathfrak{F}$. Let $\mathfrak{J} = \mathfrak{K}_{SA}$. Then $\mathfrak{J}$ is a uniformly closed Jordan ideal in $\mathfrak{A}$ with $\mathfrak{F}_{SA} = \mathfrak{J}F$. The map $\mathfrak{J} \to \mathfrak{F}_{SA}$ by $A \to AF$ is injective, for if $AF = 0$ then $AE = 0$, hence $A(I + iS) = 0$, i.e., $A = -iAS \in i\mathfrak{R}(\mathfrak{A}) \cap \mathfrak{R}(\mathfrak{A}) = \{0\}$, $A = 0$. Let ψ be the inverse mapping of $A \to AF$. Then ψ is a C^*-isomorphism of $\mathfrak{F}_{SA}$ onto $\mathfrak{J}$. The proof is complete.

LEMMA 3.2. *Let $\mathfrak{B}$ be a von Neumann algebra and $\mathfrak{A}$ a reversible JW-algebra such that $\mathfrak{R}(\mathfrak{A}) \cap i\mathfrak{R}(\mathfrak{A}) = \{0\}$, and $\mathfrak{B} = \mathfrak{R}(\mathfrak{A})^- + i\mathfrak{R}(\mathfrak{A})^-$. Let ϕ be the map $A + iB \to A^* + iB^*$ of $\mathfrak{B}$ into itself, $A, B \in \mathfrak{R}(\mathfrak{A})^-$. Then ϕ is a *-anti-automorphism of $\mathfrak{B}$, hence is in particular ultra-weakly continuous.*

Proof. Since $A + iB = \phi(A^* + iB^*)$, ϕ is surjective. By Lemma 2.3 $\mathfrak{R}(\mathfrak{A})^- \cap i\mathfrak{R}(\mathfrak{A})^- = \{0\}$, so ϕ is injective. If $A, B, C, D \in \mathfrak{R}(\mathfrak{A})^-$ then

$$\begin{aligned}
\phi((A+iB)(C+iD)) &= \phi(AC - BD + i(AD + BC)) \\
&= (AC - BD)^* + i(AD + BC)^* \\
&= (C^* + iD^*)(A^* + iB^*) \\
&= \phi(C + iD)\phi(A + iB). \\
\phi((A+iB)^*) &= \phi(A^* - iB^*) = A - iB = (A^* + iB^*)^* \\
&= \phi(A + iB)^*.
\end{aligned}$$

Clearly, ϕ is real linear. Hence, in order to show ϕ is a *-anti-automorphism of $\mathfrak{B}$, it remains to be shown that $\phi(i(A + iB)) = i\phi(A + iB)$. But $\phi(i(A + iB)) = \phi(-B + iA) = -B^* + iA^* = i(A^* + iB^*) = i\phi(A + iB)$, so the first part of the lemma is proved.

Since ϕ is a *-anti-automorphism of $\mathfrak{B}$ onto itself, ϕ is an order-isomorphism. Hence ϕ is positive and normal. By [1, Theorème 2, p. 56] ϕ is ultra-weakly continuous.

We remark that the above lemma gives a method for constructing *-anti-automorphisms of a von Neumann algebra $\mathfrak{B}$. It suffices by Theorem 2.4 to show the existence of a reversible JW-algebra $\mathfrak{A}$ such that $\mathfrak{R}(\mathfrak{A}) \cap i\mathfrak{R}(\mathfrak{A}) = \{0\}$ and $\mathfrak{A}' = \mathfrak{B}'$. If $\mathfrak{B}$ is a factor it suffices to find a reversible JW-algebra $\mathfrak{A}$ such that $\mathfrak{A}' = \mathfrak{B}'$ and $\mathfrak{A} \neq \mathfrak{B}_{SA}$ (cf. [8, Lemma 6.1]).

LEMMA 3.3. *Let $\mathfrak{B}$ and $\mathfrak{A}$ be as in Lemma 3.2. Then the map τ of $\mathfrak{B}$ onto $\mathfrak{R}(\mathfrak{A})^-$ defined by $\tau(A + iB) = A$, is ultra-weakly continuous.*

Proof. Let η denote the map $A + iB \to A - iB$. Then $\eta(S) = \phi(S^*)$, where ϕ is the anti-automorphism constructed in Lemma 3.2. Since the *-operation is ultra-weakly continuous, so is η. Let ι denote the identity map of $\mathfrak{B}$. Then $\tau = \frac{1}{2}(\iota + \eta)$, so is ultra-weakly continuous.

Notice that the map η in the above lemma is a conjugation of $\mathfrak{B}$, i.e., η satisfies all the axioms of an involution except that η is multiplicative instead of being an anti-automorphism.

We can now show the weakly closed analogue of Lemma 3.1.

LEMMA 3.4. *Let $\mathfrak{A}$ be a reversible JW-algebra such that $\mathfrak{R}(\mathfrak{A}) \cap i\mathfrak{R}(\mathfrak{A}) = \{0\}$. Let F be a central projection in $(\mathfrak{A})^-$ such that if $\mathfrak{F} = F(\mathfrak{A})^-$ then $\mathfrak{F} \cap \mathfrak{A} = \{0\}$. Then*

$$\mathfrak{F}_{SA} = \mathfrak{A}F.$$

Moreover, there exists a central projection $G \geqq F$ in $\mathfrak{A}$ and a C^*-isomorphism ψ of $\mathfrak{F}_{SA}$ onto $\mathfrak{A}G$, ψ being the inverse of the isomorphism $AG \to AF$ of $\mathfrak{A}G$ onto $\mathfrak{F}_{SA}$.

The proof is a modification of that of Lemma 3.1. Let $\mathfrak{K} = \{A \in \mathfrak{R}(\mathfrak{A})^-$: there exists $B \in \mathfrak{R}(\mathfrak{A})^-$ with $A + iB \in \mathfrak{F}\}$. Let ρ be the map of $\mathfrak{K}$ into itself defined by $A \to B$, where $A + iB \in \mathfrak{F}$. As in the proof of Lemma 3.1 ρ is an isometry. Let τ be as in Lemma 3.3. Then $\mathfrak{K} = \tau(\mathfrak{F})$. Now $\mathfrak{K}_{SA}$ is weakly closed. Indeed, let $A \in \mathfrak{K}_{SA}^-$. Multiplying by a scalar if necessary, we may assume $\|A\| \leqq 1$. Let $\mathfrak{K}_1$ denote the unit ball in $\mathfrak{K}_{SA}$. Since $\mathfrak{K}_{SA}$ is a Jordan algebra, the Kaplansky density theorem shows that $A \in \mathfrak{K}_1^-$. By Lemma 3.3 τ is weakly continuous on $\mathfrak{F}_2$—the ball of radius 2 around 0 in $\mathfrak{F}$. Since $\mathfrak{F}_2$ is weakly compact so is $\tau(\mathfrak{F}_2)$. Hence $\tau(\mathfrak{F}_2)$ is weakly closed. Since ρ is an isometry $\mathfrak{K}_1 \subset \tau(\mathfrak{F}_2)$. Since $\tau(\mathfrak{F}_2)$ is weakly closed $\mathfrak{K}_1^- \subset \tau(\mathfrak{F}_2)$, hence $A \in \tau(\mathfrak{F}_2)$. In particular $A \in \mathfrak{K}_{SA}$, so $\mathfrak{K}_{SA}$ is weakly closed as asserted. Let $\mathfrak{J} = \mathfrak{K}_{SA}$. Then $\mathfrak{J}$ is a weakly closed Jordan ideal in $\mathfrak{A}$, hence there exists a central projection G in $\mathfrak{A}$ such that $\mathfrak{J} = \mathfrak{A}G$, [9, Corollary 2 and Proposition 5]. Let $S = \rho(G)$. Let $E = \frac{1}{2}(I + iS)$. Let $F_1 = EG$. As in Lemma 3.1 $\mathfrak{F}_{SA} = \mathfrak{J}F_1 = \mathfrak{A}F_1$. Since the identity in $\mathfrak{F}$ is F, $F = F_1$. The proof is now completed as in Lemma 3.1.

We pause here to show an application of Lemma 3.4.

COROLLARY 3.5. *Let $\mathfrak{A}$ be a reversible JW-factor such that $(\mathfrak{A})^-$ is not a factor. Then there exist exactly two orthogonal nonzero projections E and F in the center of $(\mathfrak{A})^-$. Moreover, there exist C^*-isomorphisms of both $E(\mathfrak{A})_{SA}^-$ and $F(\mathfrak{A})_{SA}^-$ onto $\mathfrak{A}$, and $E(\mathfrak{A})_{SA}^- = E\mathfrak{A}$, $F(\mathfrak{A})_{SA}^- = F\mathfrak{A}$. In particular, $E(\mathfrak{A})_{SA}^-$ and $F(\mathfrak{A})_{SA}^-$ are C^*-isomorphic.*

Proof. From the hypothesis and [8, Lemma 6.1] $\mathfrak{R}(\mathfrak{A}) \cap i\mathfrak{R}(\mathfrak{A}) = \{0\}$.

If E and F are orthogonal central projections $\neq 0$, I in $(\mathfrak{A})^-$ then $\mathfrak{F} = F(\mathfrak{A})^-$ is a weakly closed two-sided ideal in $(\mathfrak{A})^-$ such that $\mathfrak{F} \cap \mathfrak{A} = \{0\}$. Since $\mathfrak{A}$ is a JW-factor the projection G in Lemma 3.4 must be the identity, hence there exists a C^*-isomorphism ψ of $F(\mathfrak{A})_{SA}^-$ onto $\mathfrak{A}$, and similarly from $E(\mathfrak{A})_{SA}^-$ onto $\mathfrak{A}$. Assume $E + F = I$, and that $F = F_1 + F_2$ with F_1 and F_2 central projections in $(\mathfrak{A})^-$. Then $\psi(F_1)$ and $\psi(F_2)$ are orthogonal central projections in $\mathfrak{A}$ with sum the identity. Hence one of them is 0 and the other I, and one of the F's is 0 and the other F. Thus F is a minimal nonzero central projection in $(\mathfrak{A})^-$ and similarly for E. The proof is complete.

Corollary 3.5 gives an alternative proof of [7, Theorem 5.2], in which all JW-factors of type I_n, $n \geqq 3$, were classified. Indeed, as in the original proof it suffices to consider case (4) in [8, Lemma 3.5], in which case there exists orthogonal nonzero central projections P and Q in $(\mathfrak{A})^-$ with $P + Q = I$. By [8, Theorem 8.2] $(\mathfrak{A})^-$ is of type I. By Corollary 3.5 $P(\mathfrak{A})^-$ is a factor of type I, hence is isomorphic

to all bounded operators $\mathfrak{B}(\mathfrak{K})$ on a Hilbert space $\mathfrak{K}$. Moreover, $\mathfrak{A}$ being the C^*-isomorphic image of $P(\mathfrak{A})_{SA}^-$ is the C^*-isomorphic image of $\mathfrak{B}(\mathfrak{K})_{SA}$. Even though this proof of [8, Theorem 5.2] is simpler than the old one the original proof has an advantage which will be apparent in the proof of Theorem 4.3 below.

We shall now consider cases where $\mathfrak{A}$ separates the ideals in $(\mathfrak{A})$ or $(\mathfrak{A})^-$, and consider the JW-algebras first.

LEMMA 3.6. *Let $\mathfrak{A}$ be a reversible JW-algebra. Let $\mathfrak{C}$ denote the center in $(\mathfrak{A})^-$. Assume $\mathfrak{A} \cap \mathfrak{C} = \mathfrak{C}_{SA}$. If $\mathfrak{F}$ is a uniformly closed two-sided ideal in $(\mathfrak{A})^-$ such that $\mathfrak{F} \cap \mathfrak{A} = \{0\}$, then $\mathfrak{F} = \{0\}$.*

Proof. Note that if $\mathfrak{F}$ is weakly closed then the lemma is immediate from Lemma 3.4. Because of our weaker assumptions a different proof is necessary. Assume $\mathfrak{F} \neq \{0\}$. By [8, Lemma 6.1] there exist central projections E and F in $\mathfrak{A}$ such that $E\mathfrak{A} = E(\mathfrak{A})_{SA}^-$, and $\mathfrak{R}(F\mathfrak{A}) \cap i\mathfrak{R}(F\mathfrak{A}) = \{0\}$, and $E + F = I$. Since $\mathfrak{F} \cap \mathfrak{A} = \{0\}$, $\mathfrak{F} \cap E(\mathfrak{A})^- = \{0\}$, so we may assume $\mathfrak{R}(\mathfrak{A}) \cap i\mathfrak{R}(\mathfrak{A}) = \{0\}$. Let E be a projection in $\mathfrak{F}$. By Theorem 2.4 there exist $A \in \mathfrak{A}$, $B \in \mathfrak{R}(\mathfrak{A})^-$ such that $E = A + iB$. Then $E = E^2 = A^2 - B^2 + i(AB + BA)$, hence $A = A^2 - B^2$, or $B^2 = A^2 - A \in \mathfrak{A}$. Let ϕ be a representation of $(\mathfrak{A})^-$ with kernel $\mathfrak{F}$. Then $0 = \phi(E) = \phi(A) + i\phi(B)$, hence $\phi(A) = -i\phi(B)$. Thus $\phi(A)^2 = -\phi(B)^2 = -\phi(A^2 - A)$, and $2A^2 - A \in \mathfrak{F}$. Therefore $A = 2A^2$, so by spectral theory $A = \frac{1}{2}F$ with F a projection in $\mathfrak{A}$. Furthermore $F \geqq E$, because $FA = A$, so $FB = B$, and $FE = E$. Since $E = \frac{1}{2}F + iB$, $B = i(\frac{1}{2}F - E) \in \mathfrak{R}(\mathfrak{A})$.

Since $\mathfrak{F}$ is a uniformly closed two-sided ideal in a von Neumann algebra there exists a monotone increasing net $\{E_\alpha\}$ of projections in $\mathfrak{F}$ such that $E_\alpha \to P$ strongly, where P is the identity projection in $\mathfrak{F}^-$, hence a central projection in $(\mathfrak{A})^-$, hence $P \in \mathfrak{A}$ by assumption. Moreover, by the above paragraph $E_\alpha = \frac{1}{2}F_\alpha + iB_\alpha$ with F_α a projection in $\mathfrak{A}$, $F_\alpha \geqq E_\alpha$, and $B_\alpha \in \mathfrak{R}(\mathfrak{A})$. If $E_\alpha \geqq E_\beta$ then

$$0 \leqq E_\alpha - E_\beta = \tfrac{1}{2}(F_\alpha - F_\beta) + i(B_\alpha - B_\beta),$$

so by Corollary 2.2 $F_\alpha - F_\beta \geqq 0$, hence $\{F_\alpha\}$ is a monotone increasing net of projections in $\mathfrak{A}$. Let F be their strong limit. Then $0 \leqq F \leqq I$ and $F \in \mathfrak{A}$. As in the proof of Theorem 2.4 $B_\alpha = i(\frac{1}{2}F_\alpha - E_\alpha) \to i(\frac{1}{2}F - P) \in i\mathfrak{R}(\mathfrak{A})^-$ weakly. Since $B_\alpha \in \mathfrak{R}(\mathfrak{A})$ and $B_\alpha \to i(\frac{1}{2}F - P)$ weakly, $i(\frac{1}{2}F - P) \in \mathfrak{R}(\mathfrak{A})^-$, hence in $\mathfrak{R}(\mathfrak{A})^- \cap i\mathfrak{R}(\mathfrak{A})^-$, which is zero by Lemma 2.3. Since $\frac{1}{2}F - P \neq 0$, this is a contradiction, $\mathfrak{F} = \{0\}$, the proof is complete.

An analogue of Lemma 3.6 can also be proved for JC-algebras in view of Lemma 3.1. We shall do it only in the irreducible case.

LEMMA 3.7. *Let $\mathfrak{A}$ be a reversible irreducible JC-algebra. If $\mathfrak{F}$ is a uniformly closed two-sided ideal in $(\mathfrak{A})$ such that $\mathfrak{F} \cap \mathfrak{A} = \{0\}$ then $\mathfrak{F} = \{0\}$.*

Proof. If $\mathfrak{R}(\mathfrak{A}) \cap i\mathfrak{R}(\mathfrak{A}) = \{0\}$ then there exist by Lemma 3.1 a uniformly closed Jordan ideal $\mathfrak{J}$ in $\mathfrak{A}$, and a central projection F in $(\mathfrak{A})^-$ such that $\mathfrak{F}_{SA} = \mathfrak{J}F$. But if

$\mathfrak{F} \neq \{0\}$ then $F = I$ hence $\mathfrak{F}_{SA} = \mathfrak{F} \subset \mathfrak{A}$, contrary to assumption. If $\mathfrak{L} = \mathfrak{R}(\mathfrak{A}) \cap i\mathfrak{R}(\mathfrak{A}) \neq \{0\}$ then $\mathfrak{L}$ is a uniformly closed two-sided ideal in $(\mathfrak{A})$ such that $\mathfrak{L}_{SA} \subset \mathfrak{A}$ [7, Remark 2.2]. Thus $\mathfrak{F} \cap \mathfrak{L} = \{0\}$. Since $(\mathfrak{A})$ is an irreducible C^*-algebra it has no ideal divisors of zero [2, Lemme 2.11.3]. Since $\mathfrak{L} \neq \{0\}$, $\mathfrak{F} = \{0\}$.

4. Irreducible *JW*-algebras. We shall now obtain our main result, which together with [8, Theorems 3.9 and 7.1], classifies all irreducible *JW*-algebras.

THEOREM 4.1. *Every irreducible JW-algebra is of type* I.

Proof. Let $\mathfrak{A}$ be an irreducible *JW*-algebra acting on a Hilbert space $\mathfrak{X}$. If $\mathfrak{A}$ is not reversible it is by [8, Corollary 6.5] totally nonreversible, hence by [8, Theorem 7.1] of type I_2. If $\mathfrak{A}$ is reversible and $\mathfrak{R}(\mathfrak{A}) \cap i\mathfrak{R}(\mathfrak{A}) \neq \{0\}$ then $\mathfrak{A} = \mathfrak{B}(\mathfrak{X})_{SA}$ by [8, Theorem 6.4], hence of type I. It remains to consider the case when $\mathfrak{A}$ is reversible and $\mathfrak{R}(\mathfrak{A}) \cap i\mathfrak{R}(\mathfrak{A}) = \{0\}$. Let $\mathfrak{C}(\mathfrak{X})$ denote the completely continuous operators on $\mathfrak{X}$. In order to show $\mathfrak{A}$ is of type I it suffices to show $\mathfrak{A} \cap \mathfrak{C}(\mathfrak{X}) \neq \{0\}$, but this is immediate from Lemma 3.6. The proof is complete.

We shall need a generalization to *JW*-algebras of a well-known result for von Neumann algebras.

LEMMA 4.2. *Let $\mathfrak{A}$ be a JC-algebra. Then the following are equivalent*:

(i) *$\mathfrak{A}$ is a JW-algebra,*

(ii) *$\mathfrak{A}$ is strongly closed,*

(iii) *$\mathfrak{A}$ is ultra-weakly closed,*

(iv) *$\mathfrak{A}$ is ultra-strongly closed.*

Proof. By [1, Théorème 1, p. 40] (i) $\Leftrightarrow$ (ii), (iii) $\Leftrightarrow$ (iv) and (i) $\Rightarrow$ (iii). It remains to be shown that (iv) $\Rightarrow$ (ii). Let $\mathfrak{A}$ be ultra-strongly closed. Let $\mathfrak{A}_1$ be the unit ball in $\mathfrak{A}$ and $\mathfrak{A}_1^-$ the unit ball in $\mathfrak{A}^-$. On $\mathfrak{A}_1$ and $\mathfrak{A}_1^-$ the strong and ultra-strong topologies coincide [1, p. 36]. Since $\mathfrak{A}$ is strongly dense in $\mathfrak{A}^-$, $\mathfrak{A}_1$ is strongly dense in $\mathfrak{A}_1^-$ by the Kaplansky density theorem. Since $\mathfrak{A}_1$ is ultra-strongly closed, it is strongly closed, hence $\mathfrak{A}_1 = \mathfrak{A}_1^-$, $\mathfrak{A} \supset \mathfrak{A}^-$, and they are equal.

A factor is of type I if and only if there exists a vector state which acts purely on it. The same is true for *JW*-factors.

THEOREM 4.3. *Let $\mathfrak{A}$ be a JW-factor. Then $\mathfrak{A}$ is of type* I *if and only if there exists a vector state which is pure on $\mathfrak{A}$.*

Proof. If $\mathfrak{A}$ is of type I let E be an abelian projection in $\mathfrak{A}$. If x is a unit vector in E then ω_x is pure on $\mathfrak{A}$. Conversely assume ω_x is pure on $\mathfrak{A}$. In order to show $\mathfrak{A}$ is of type I we may by [8, Theorem 7.1] assume $\mathfrak{A}$ is reversible. Let ω be a pure state extension of ω_x to $(\mathfrak{A})$. Then $\omega = \omega_y \phi$ with ϕ an irreducible representation of $(\mathfrak{A})$. Thus $\phi(\mathfrak{A})$ is an irreducible *JC*-algebra, and by Theorem 4.1 $\phi(\mathfrak{A})^-$ is of type I. By [8, Theorem 3.9] there exists an abelian projection F in $\phi(\mathfrak{A})^-$ containing y, as can easily be shown. By exactly the same argument as in the proof of [8, Theorem 5.1] ϕ is ultra-weakly continuous on $\mathfrak{A}$. Kernel $\phi|\mathfrak{A}$ is an ultra-weakly closed

Jordan ideal in $\mathfrak{A}$, hence weakly closed by Lemma 4.2. Thus there exists a central projection P in $\mathfrak{A}$ such that $P\mathfrak{A}=\text{kernel } \phi|\mathfrak{A}$. Since $\phi\neq 0$ and $\mathfrak{A}$ is a JW-factor, $P=0$, $\phi|\mathfrak{A}$ is an isomorphism, and ultra-weakly continuous. As for von Neumann algebras $\phi(\mathfrak{A})=\phi(\mathfrak{A})^-$. Let $E=(\phi|\mathfrak{A})^{-1}(F)$. Then E is an abelian projection in $\mathfrak{A}$, $\mathfrak{A}$ is of type I.

It might seem that an application of Corollary 3.5 would simplify the above proof considerably. However, if $\mathfrak{A}$ is reversible, ω_x is a pure state of $\mathfrak{A}$, and $(\mathfrak{A})^-$ is a factor, it is not a priori clear that ω_x is pure on $(\mathfrak{A})^-$, hence that $(\mathfrak{A})^-$ is of type I.

COROLLARY 4.4. *Let $\mathfrak{A}$ be a JW-factor, ω_x a vector state, and E the support of ω_x in $\mathfrak{A}$. Then the following are equivalent.*

(i) *ω_x is pure,*

(ii) *E is abelian,*

(iii) *the null space of $\omega_x=\{A\in\mathfrak{A} : EAE=0\}$.*

Proof. (i) $\Rightarrow$ (ii). Let ω_x be pure on $\mathfrak{A}$. In view of [8, Theorem 7.1] we may assume $\mathfrak{A}$ is reversible. In the notation of the proof of Theorem 4.3 E is abelian, and $\omega_x(E)=\omega_y\phi(E)=\omega_y(F)=1$, so $x\in E$. Since E is minimal it equals the support of ω_x.

(ii) $\Rightarrow$ (iii). If E is abelian $EAE=\omega_x(A)E$ [9, Corollary 24], so the null space of ω_x is as in (iii).

(iii) $\Rightarrow$ (i). If the null space $\mathfrak{N}$ of ω_x equals $\{A\in\mathfrak{A} : EAE=0\}$ then

$$\mathfrak{N} = \{A\in\mathfrak{A} : A = AF+FA-FAF\},$$

where $F=I-E$. Clearly $\omega_x(F)=0$, hence if $\omega_x=\frac{1}{2}(\omega_1+\omega_2)$ with ω_1 and ω_2 states of $\mathfrak{A}$, then $\omega_1(F)=\omega_2(F)=0$. By the Schwarz inequality $\omega_1(A)=\omega_2(A)=0$ for all $A\in\mathfrak{N}$. Thus $\omega_x=\omega_1=\omega_2$, ω_x is pure.

5. **Irreducible *JC*-algebras.** In this section we shall generalize Kadison's result [6, Theorem 1] that an irreducible C^*-algebra is algebraically irreducible. In order to reduce the argument to reversible JW-algebras the next result is important.

LEMMA 5.1. *In a JW-factor of type I_2 the ultra-strong and the uniform topologies coincide. In particular, if $\mathfrak{A}$ is a JC-algebra whose weak closure is a JW-factor of type I_2, then $\mathfrak{A}$ is itself weakly closed.*

Proof. Let $\mathfrak{B}$ be a JW-factor of type I_2, and let $A_\alpha\in\mathfrak{B}$, $A_\alpha\to A$ strongly on the unit ball. Then $(A_\alpha-A)^2\to 0$ strongly, as multiplication is strongly continuous on the unit ball. In the notation of the proof of [8, Theorem 7.1, (1) $\Rightarrow$ (2)] $\|A_\alpha-A\|_2^2\to 0$, hence $\|A_\alpha-A\|\leqq 2^{1/2}\|A_\alpha-A\|_2\to 0$, i.e., $A_\alpha\to A$ uniformly, and the ultra-strong topology is finer than the uniform. Since the converse is obvious the two topologies coincide.

If $\mathfrak{A}$ is a JC-algebra such that $\mathfrak{A}^-$ is a JW-factor of type I_2, then by the above $\mathfrak{A}$ is ultra-strongly closed, hence weakly closed by Lemma 4.2.

THEOREM 5.2. *Let $\mathfrak{A}$ be a JC-algebra acting irreducibly upon a Hilbert space $\mathfrak{X}$ and $\{x_j\}$, $\{y_j\}$ be two sets containing n vectors each, the first set consisting of linearly independent vectors. Assume there exists B in $\mathfrak{A}^-$ such that $Bx_j=y_j$, $j=1,\ldots,n$. Then there exists A in $\mathfrak{A}$ such that $Ax_j=y_j$.*

Sketch of proof. The proof is a slight modification of Kadison's proof. By Lemma 5.1 and [8, Theorem 7.1] we may assume $\mathfrak{A}^-$ is reversible. By Theorem 4.1 and [8, Theorem 3.9] there are three cases. In case (2), the complex case $\mathfrak{A}^-=\mathfrak{B}(\mathfrak{X})_{SA}$, and the theorem follows from Kadison's arguments. In case (1), the real case, there exists an orthonormal basis $(e_\alpha)_{\alpha\in J}$ for $\mathfrak{X}$ for which (Ae_α, e_β) is real for all α, $\beta\in J$ and all $A\in\mathfrak{A}^-$. Following Kadison's arguments we can prove the theorem when the coefficients of the x_j and y_j relative to the basis are all real. In the general case there exist unique vectors u_j, v_j, w_j, z_j in $\mathfrak{X}$ with real coefficients relative to the basis such that $x_j=u_j+iv_j$, $y_j=w_j+iz_j$. Since $Bx_j=y_j$ and B has real coefficients relative to the basis, $Bu_j=w_j$, $Bv_j=z_j$. We can now find A in $\mathfrak{A}$ such that $Au_j=w_j$, $Av_j=z_j$. Then $Ax_j=A(u_j+iv_j)=w_j+iz_j=y_j$, and the theorem is proved in the real case.

It remains to consider case (3), the quaternionian case. It is an easy consequence of Corollary 4.4 (iii) that every vector state ω_x is pure on $\mathfrak{A}^-$. Indeed, it suffices to show that no vector state is faithful, and by cutting down by a four dimensional projection we may even assume the Hilbert space to be four dimensional. We can then write x as a vector (q_1, q_2) in $\boldsymbol{Q}\times\boldsymbol{Q}$, and $\mathfrak{A}^-$ as 2×2 matrices over $\boldsymbol{Q}$. Exactly as in the complex case it follows that ω_x cannot be faithful on $\mathfrak{A}^-$. As a consequence of this it follows that if $x_1,\ldots,x_n$ are orthogonal vectors in $\mathfrak{X}$ and $\mathfrak{A}^-$ is given by case (3), and E_j, $j=1,\ldots,n$, are abelian projections containing the x_j, then either $E_jE_k=0$ or $E_j=E_k$ for $j\neq k$. In order to complete the proof of the theorem, we may assume the x_j form an orthonormal set. If E_j is the abelian projection containing x_j, then we may also assume all the E_j are orthogonal, hence that the x_j as vectors in $\boldsymbol{Q}^m$ are orthogonal when we have reduced to the finite dimensional case, as in Kadison's proof. From now on the argument is given by Kadison.

It is well known that an irreducible C^*-algebra $\mathfrak{A}$ on a Hilbert space $\mathfrak{X}$ either contains the completely continuous operators $\mathfrak{C}(\mathfrak{X})$ or $\mathfrak{A}\cap\mathfrak{C}(\mathfrak{X})=\{0\}$. This generalizes to JC-algebras.

LEMMA 5.3. *Let $\mathfrak{A}$ be an irreducible JC-algebra acting on a Hilbert space $\mathfrak{X}$. Then either $\mathfrak{A}\cap\mathfrak{C}(\mathfrak{X})=\mathfrak{A}^-\cap\mathfrak{C}(\mathfrak{X})$, or $\mathfrak{A}\cap\mathfrak{C}(\mathfrak{X})=\{0\}$.*

Proof. Let $\mathfrak{J}=\mathfrak{A}\cap\mathfrak{C}(\mathfrak{X})$. Assume $\mathfrak{J}\neq\{0\}$. Clearly $\mathfrak{J}^-=\mathfrak{A}^-$, since $\mathfrak{J}^-$ is an ideal in $\mathfrak{A}^-$. Let E be a finite dimensional projection in $\mathfrak{J}$. By Theorem 4.1 $\mathfrak{A}^-$ is of type I. If it is of type I_2 then $\mathfrak{A}^-\cap\mathfrak{C}(\mathfrak{X})=\{0\}$ unless $\mathfrak{X}$ is finite dimensional, in which case the lemma is trivial. We may thus assume $\mathfrak{A}^-$ is of type I_∞. There exists then an abelian projection F in $\mathfrak{A}^-$ orthogonal to E. Let E_1 be an abelian subprojection of E in $\mathfrak{A}^-$ and let S be a symmetry in $\mathfrak{A}^-$ such that $SE_1S=F$ [9, Corollary 26]. Let $W=E_1SF+FSE_1$. Then $W\in\mathfrak{A}^-$, W is zero on $E-E_1$, it is an isometry of

E_1 onto F and an isometry of F onto E_1. Since $\mathfrak{J}$ is an irreducible JC-algebra there exists by Theorem 5.2 V in $\mathfrak{J}$ with the same properties. Then $F=VEV\in\mathfrak{J}$. If G is any abelian projection in $(\mathfrak{A})^-$ then there exists a symmetry T in $\mathfrak{A}^-$ such that $TGT=F$. Then T is an isometry of F onto G and of G onto F. By Theorem 5.2 there exists U in $\mathfrak{J}$ with the same properties. Then $UFU=G\in\mathfrak{J}$. Thus every abelian projection in $\mathfrak{A}^-$ belongs to $\mathfrak{J}$, hence $\mathfrak{A}^-\cap\mathfrak{C}(\mathfrak{X})\subset\mathfrak{J}$. The proof is complete.

6. ***JW*-algebras of type I.** It is possible to generalize Theorem 4.1 to a general global theorem. In order to do this we shall use CCR-algebra techniques, especially those developed by Dixmier [3], and modify them for JC-algebras. If $\mathfrak{A}$ is a GCR-algebra we identify $\hat{\mathfrak{A}}$ and its structure space Z. If $\mathfrak{P}$ denotes the set of $A\in\mathfrak{A}^+$ for which the map $\pi\to\operatorname{Tr}\pi(A)$ is finite and continuous on $\hat{\mathfrak{A}}$, then $\mathfrak{P}$ is the positive part of a two-sided ideal $\mathfrak{M}$ in $\mathfrak{A}$. Let $J(\mathfrak{A})$ denote the uniform closure of $\mathfrak{M}$. Then $\mathfrak{A}$ is said to have *continuous trace* if $J(\mathfrak{A})=\mathfrak{A}$. We first modify [3, Lemma 10] to our purposes.

LEMMA 6.1. *Let $\mathfrak{A}$ be a reversible JC-algebra such that $(\mathfrak{A})$ is a CCR-algebra with continuous trace and structure space Z. Then there exist a positive operator B in $\mathfrak{A}$ and an open set U in Z such that $\pi(B)$ is an abelian projection in $\pi(\mathfrak{A})$ for all $\pi\in U$.*

Proof. There are two cases.

Case 1. For all $\pi\in Z$, $\pi(\mathfrak{A})^-$ has all its abelian projections of rank 2. Fix $\pi_0\in Z$. By Lemma 5.3 $\pi_0(\mathfrak{A})\cap\mathfrak{C}(\mathfrak{X})=\pi_0(\mathfrak{A})^-\cap\mathfrak{C}(\mathfrak{X})$. There exists therefore $B_1\in\mathfrak{A}^+$ such that $\pi_0(B_1)=F$ with F an abelian projection (of rank 2) in $\pi_0(\mathfrak{A})$, and such that the map $\pi\to\operatorname{Tr}\pi(B_1)$ is continuous. Let $B_2=f(B_1)$, where f is a real continuous function, $f(t)=t$ for $0\leqq t\leqq 1$, $f(t)=1$ for $t\geqq 1$. Then $\|B_2\|\leqq 1$, $\pi_0(B_2)=F$. In a neighborhood U of π_0, $\operatorname{Tr}\pi(B_2)\leqq 10/4$, and $\|\pi(B_2)\|\geqq 3/4$. Now $\pi(B_2)=\sum\lambda_jE_j$ with the E_j 2-dimensional orthogonal projections. Therefore $\operatorname{Tr}\pi(B_2)=2\sum\lambda_j$. Hence $\sum\lambda_j\leqq 10/4\cdot\frac{1}{2}=5/4$. Since the largest of the λ's is $\geqq 3/4$, the others are $\leqq\frac{1}{2}$. Let g be an increasing continuous real function which is 0 for $t\leqq\frac{1}{2}$ and 1 for $t\geqq 3/4$. Let $B=g(B_2)$. Then B satisfies the conditions of the lemma.

Case 2. There exists $\pi_0\in Z$ such that $\pi_0(\mathfrak{A})^-$ has abelian projections of rank 1. In this case the proof is almost a direct copy of that of [3, Lemma 10], and can be read out of the proof of case 1.

LEMMA 6.2. *Let $\mathfrak{A}$ be as in Lemma 6.1. Assume further that for each nonzero uniformly closed two-sided ideal $\mathfrak{F}$ in $(\mathfrak{A})$, $\mathfrak{F}\cap\mathfrak{A}\neq\{0\}$. Then there exists A in $\mathfrak{A}$ such that $A\mathfrak{A}A$ is abelian and nonzero.*

Proof. Let B and U be as in Lemma 6.1. Let $\mathfrak{F}$ be the ideal in $(\mathfrak{A})$ whose structure space is U. Let $\mathfrak{J}=\mathfrak{F}\cap\mathfrak{A}$. Then there exists A_1 in $\mathfrak{J}^+$ such that if $A=BA_1B$, then $A\neq 0$. By [4, Theorem 3] $A\in\mathfrak{J}^+$. If $\pi\in Z$ and $C\in\mathfrak{A}$ then

$$\pi(ACA) = \pi(B(A_1BCBA_1)B) = \pi(B)\pi(A_1BCBA_1)\pi(B) = \mu\pi(B)$$

with μ real. Thus $\pi(A\mathfrak{A}A)$ is abelian, and $A\mathfrak{A}A$ is abelian.

LEMMA 6.3. *Let $\mathfrak{A}$ be a reversible JW-algebra. Let $\mathfrak{C}$ denote the center of* $(\mathfrak{A})^-$. *Assume $\mathfrak{A} \cap \mathfrak{C} = \mathfrak{C}_{SA}$. Let $\mathfrak{F}$ be a nonzero GCR-ideal in* $(\mathfrak{A})^-$. *Then there exists a nonzero abelian projection in $\mathfrak{A}$.*

Proof. By [3, Proposition 9] $\mathfrak{F}$ contains a nonzero *CCR*-ideal with continuous trace. Replacing $\mathfrak{F}$ by this ideal we may assume $\mathfrak{F}$ has continuous trace. Let $\mathfrak{J} = \mathfrak{F} \cap \mathfrak{A}$. By Lemma 3.6 $\mathfrak{J} \neq \{0\}$. By [4, Theorem 2] $(\mathfrak{J})$ is an ideal in $\mathfrak{F}$. By [3, Proposition 10] $(\mathfrak{J})$ has continuous trace. Apply Lemma 6.2 to $\mathfrak{J}$. Then there exists A in $\mathfrak{J}^+$ such that $A\mathfrak{J}A$ is abelian and nonzero. Let E be any nonzero spectral projection of A. Then $E \in \mathfrak{J}$, since it is majorized by a scalar multiple of A, and $E\mathfrak{A}E$ is abelian, since equal to $E(E\mathfrak{A}E)E \subset E\mathfrak{J}E$, which is abelian. The proof is complete.

We can now prove the main result of this section.

THEOREM 6.4. *Let $\mathfrak{A}$ be a JW-algebra such that $\mathfrak{A}''$ is a von Neumann algebra of type* I. *Then $\mathfrak{A}$ is itself of type* I.

Proof. From [9, Theorem 5] there exists a central projection E in $\mathfrak{A}$ such that $(I-E)\mathfrak{A}$ is of type I and $E\mathfrak{A}$ has no type I portion. Confining our attention to $E\mathfrak{A}$ we may assume $\mathfrak{A}$ has no type I portion. By [8, Theorems 6.4 and 6.6] $\mathfrak{A}$ is then reversible and $\mathfrak{R}(\mathfrak{A}) \cap i\mathfrak{R}(\mathfrak{A}) = \{0\}$. We may also assume $\mathfrak{A}'' = (\mathfrak{A})^-$. Let $\mathfrak{C}$ denote the center of $\mathfrak{A}''$. Then $\mathfrak{C}_{SA} = \mathfrak{C} \cap \mathfrak{A}$. In fact, if not let E_1 be a central projection in $\mathfrak{A}''$ which is not in $\mathfrak{A}$. Let F_1 be the smallest central projection in $\mathfrak{A}$ such that $F_1 \geqq E_1$. Then $F_1 \neq E_1$. Now $E_1\mathfrak{A}''$ is an ideal in $\mathfrak{A}''$, hence $E_1\mathfrak{A}'' \cap \mathfrak{A}$ is a weakly closed Jordan ideal in $\mathfrak{A}$. By [9, Corollary 2 and Proposition 5] there exists a central projection F_2 in $\mathfrak{A}$ such that $E_1\mathfrak{A}'' \cap \mathfrak{A} = F_2\mathfrak{A}$. Clearly $F_2 \leqq E_1$, so $F_2 < E_1$. Let $F_3 = F_1 - F_2$. Then $F_3 \neq 0$ and is central in $\mathfrak{A}$. Consider $F_3\mathfrak{A}''$, which is of type I. Let $E_2 = E_1F_3 \neq 0$. Let $\mathfrak{F} = E_2\mathfrak{A}''$. As above there exists a central projection F_4 in $\mathfrak{A}$ such that $\mathfrak{F} \cap \mathfrak{A} = F_4\mathfrak{A}$. Clearly $F_4 \leqq E_2 \leqq F_3$. Since $\mathfrak{F} \subset E_1\mathfrak{A}''$, $\mathfrak{F} \cap \mathfrak{A} \subset F_2\mathfrak{A}$, so that $F_4 \leqq F_2$. But $F_3F_2 = 0$, hence $F_4 = 0$. Thus $\mathfrak{F} \cap \mathfrak{A} = \{0\}$. By Lemma 3.4 there exist a central projection G in $\mathfrak{A}$ and a C^*-isomorphism ψ of $\mathfrak{F}_{SA}$ onto $G\mathfrak{A}$. If F is an abelian projection in $\mathfrak{F}$ then $\psi(F)$ is abelian in $G\mathfrak{A}$, so $G\mathfrak{A}$ is of type I, contrary to assumption. Thus $\mathfrak{C}_{SA} = \mathfrak{C} \cap \mathfrak{A}$, as asserted. By [1, Proposition 2, p. 252] there exists a central projection P in $\mathfrak{A}''$ such that $\mathfrak{A}''P$ is homogeneous of type I_n. Replacing $\mathfrak{A}''$ by $\mathfrak{A}''P$ we may assume $\mathfrak{A}''$ is homogeneous of type I_n. Then $\mathfrak{A}''$ is spatially isomorphic to a von Neumann algebra of the form $\mathfrak{C} \otimes \mathfrak{B}(\mathfrak{X})$. We may thus assume $\mathfrak{A}'' = \mathfrak{C} \otimes \mathfrak{B}(\mathfrak{X})$. Let $\mathfrak{F} = \mathfrak{C} \otimes \mathfrak{C}(\mathfrak{X})$. Then $\mathfrak{F}$ is a *CCR*-ideal in $\mathfrak{A}''$. By Lemma 6.3 there exists a nonzero abelian projection in $\mathfrak{A}$, contrary to assumption. Thus the projection E from the beginning of the proof is zero, $\mathfrak{A}$ is of type I.

In [8, Theorem 8.2] it was shown that if $\mathfrak{A}$ is a reversible *JW*-algebra of type I then $\mathfrak{A}''$ is of type I. Combining this with Theorem 6.4 we obtain

COROLLARY 6.5. *Let $\mathfrak{A}$ be a reversible JW-algebra. Then $\mathfrak{A}$ is of type* I *if and only if $\mathfrak{A}''$ is a von Neumann algebra of type* I.

7. **Pure states.** In [6] Kadison characterized pure states of C^*-algebras in terms of their left kernels. Topping has modified the concept of left kernel to the following: if ω is a state of a JC-algebra $\mathfrak{A}$ then its *kernel* $\mathfrak{M}$ is the set of A in $\mathfrak{A}$ for which $\omega(A^2)=0$. Then $\mathfrak{M}$ is a quadratic ideal in the sense of Topping [9]. He conjectured that ω is pure if and only if $\mathfrak{M}$ is a maximal quadratic ideal. We shall prove this via a proof which is a slight modification of Kadison's original proof. By an argument similar to that in [6] Topping has shown that each uniformly closed quadratic ideal is the intersection of the kernels of all pure states annihilating it. In particular, there exists at least one pure state annihilating a maximal quadratic ideal. If $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are two sets of operators in a JC-algebra we denote by $\mathfrak{M}_1 \circ \mathfrak{M}_2$ the set of operators of the form $A \circ B$ $(=\frac{1}{2}(AB+BA))$ with $A \in \mathfrak{M}_1$, $B \in \mathfrak{M}_2$.

THEOREM 7.1. *Let $\mathfrak{A}$ be a JC-algebra, ω a state of $\mathfrak{A}$, $\mathfrak{M}$ its kernel, and $\mathfrak{N}$ its null space. Then the following three conditions are equivalent.*

(i) *ω is a pure state,*

(ii) $\mathfrak{N}=\mathfrak{M} \circ \mathfrak{A}+\mathfrak{M}$,

(iii) *$\mathfrak{M}$ is a maximal quadratic ideal.*

Proof. (i) $\Rightarrow$ (ii). Let $\bar{\omega}$ be a pure state extension of ω to $(\mathfrak{A})$. Let $\mathfrak{J}$ be the left kernel of $\bar{\omega}$, $\bar{\omega}=\omega_x\phi$, where ω_x is pure on the irreducible JC-algebra $\phi(\mathfrak{A})$, hence on the irreducible JW-algebra $\phi(\mathfrak{A})^-$. Thus, if E denotes the support projection of ω_x in $\phi(\mathfrak{A})^-$ then E is abelian by Corollary 4.4. Let $B \in \mathfrak{N}$. Then, with $F=I-E$, $\phi(B)=\phi(B)F+F\phi(B)-F\phi(B)F$ (Corollary 4.4). Since $F \in \mathfrak{J}$,

$$\begin{aligned} B+\mathfrak{J} &= \phi(B)(I+\mathfrak{J}) \\ &= (\phi(B)F+F\phi(B)-F\phi(B)F)(I+\mathfrak{J}) \\ &= F\phi(B)(I+\mathfrak{J}) \\ &= F(B+\mathfrak{J}). \end{aligned}$$

Since $F \in \mathfrak{J}$, $F(I+\mathfrak{J})=0$. Thus by Theorem 5.2 there exists A in $\mathfrak{A}$ such that $\phi(A)(I+\mathfrak{J})=0$, $\phi(A)(B+\mathfrak{J})=B+\mathfrak{J}$. Then $A \in \mathfrak{J}$, hence $A \in \mathfrak{M}=\mathfrak{J} \cap \mathfrak{A}$. Let $C=AB+BA-ABA$. Then $C \in \mathfrak{M} \circ \mathfrak{A}+\mathfrak{M}$, and $\phi(C-B)(I+\mathfrak{J})=\phi(C)(I+\mathfrak{J})$ $-\phi(B)(I+\mathfrak{J})=\phi(A)(B+\mathfrak{J})-(B+\mathfrak{J})=0$, hence $C-B \in \mathfrak{M}$. Then $B=C-D$ with $D \in \mathfrak{M}$. Since $\mathfrak{M}$ is linear $ABA+D \in \mathfrak{M}$, hence $B=C-D=2A \circ B-(ABA+D)$ $\in \mathfrak{M} \circ \mathfrak{A}+\mathfrak{M}$, i.e., $\mathfrak{N} \subset \mathfrak{M} \circ \mathfrak{A}+\mathfrak{M}$. Since the converse inclusion is trivial, they are equal.

(ii) $\Rightarrow$ (i). If $\omega=\frac{1}{2}(\omega_1+\omega_2)$ with ω_i states then $\omega_i(\mathfrak{M})=0$. Thus $\omega_i(\mathfrak{N})=0$, so $\omega_1=\omega_2=\omega$ on $\mathfrak{A}$, ω is pure.

(ii) $\Rightarrow$ (iii). Let $\mathfrak{M}'$ be a maximal quadratic ideal containing $\mathfrak{M}$, and $\mathfrak{N}=\mathfrak{M} \circ \mathfrak{A}$ $+\mathfrak{M}$. Let ω' be a pure state of $\mathfrak{A}$ whose kernel is $\mathfrak{M}'$. Let $\mathfrak{N}'$ be the null space of ω'. Then $\mathfrak{N}'=\mathfrak{M}' \circ \mathfrak{A}+\mathfrak{M}'$ by (i) $\Leftrightarrow$ (ii). Thus $\mathfrak{N}' \supset \mathfrak{M} \circ \mathfrak{A}+\mathfrak{M}=\mathfrak{N}$, so $\mathfrak{N}'=\mathfrak{N}$, $\omega'=\omega$. Thus $\mathfrak{M}'=\mathfrak{M}$, $\mathfrak{M}$ is maximal.

(iii) $\Rightarrow$ (ii). Let ω' be a pure state whose kernel is $\mathfrak{M}$. By (i) $\Leftrightarrow$ (ii) $\mathfrak{N}'$, the null space of ω', equals $\mathfrak{M} \circ \mathfrak{A} + \mathfrak{M} \subset \mathfrak{N}$. Thus $\mathfrak{N}' = \mathfrak{N}$, $\omega' = \omega$, hence ω is a pure state. The proof is complete.

Added in proof. A slightly stronger version of Lemma 5.1 has been proved independently by D. Topping, Corollary 2 in *An isomorphism invariant for spin factors*, J. Math. Mech. **15** (1966), 1055–1064.

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University of Oslo,
Oslo, Norway