INFINITE-PRODUCT MARKOV PROCESSES(1)

BY

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1. Introduction and summary. We discuss Markov random functions \( x(t) = (x_1(t), x_2(t), \ldots) \) where the component processes \( x_i(t), i = 1, 2, \ldots \) are themselves Markov processes independent of one another. As remarked in the final paragraph, this is one way to study the temporal development of random counting measures or point processes, if we want to identify two sequences \( x(t) \) and \( x'(t) \) when the coordinates of \( x'(t) \) are a permutation of the coordinates of \( x(t) \). However the present paper studies some properties of the product process \( x(t) \) when we do not make this identification.

If \( E_i \) is the state space of \( x_i(t) \) and \( E = E_1 \times E_2 \times \cdots \) is the state space of \( x(t) \), we shall consider stochastically closed subsets of \( E \). If \( H \subseteq E \) is stochastically closed and \( x', x'' \in H \), we give conditions under which \( H \) is a union of two disjoint stochastically closed sets, one containing \( x' \), the other \( x'' \). In some cases, every stochastically closed set has two points with this property.

A particular study is made of the case where \( x_i(t) \) is a temporally homogeneous additive process in \( R_1 \) (similar results hold in \( R_k \)). If \( x', x'' \), and \( H \) are as above and \( \sum (x'_i - x''_i)^2 = \infty \), then the above decomposition is possible. If \( t \) takes only non-negative integer values, an analogous result holds for a process of sums of independent random variables.

We also obtain an extension (the italicized words in the statement below) of a sufficient condition of Shepp [7] for distinguishability of a random sequence from its translates (but not of Shepp's necessary condition). Let \( Z_1, Z_2, \ldots \) be independent identically distributed random variables with the common distribution \( G \), and let \( y_1, y_2, \ldots \) be a sequence of numbers such that \( \sum (y_i - b)^2 = \infty \) for each real \( b \); it will be evident later why we assume this rather than merely \( \sum y_i^2 = \infty \). Let \( P_G \) and \( P_B \) be the product measures on \( R_1 \times R_1 \times \cdots \) corresponding respectively to \( Z_1, Z_2, \ldots \) and \( Z_1 + y_1, Z_2 + y_2, \ldots \). Then there is a measurable set \( \Gamma \) in \( R_1 \times R_1 \times \cdots \), which depends on the sequence \( y_1, y_2, \ldots \) but does not depend on \( G \), such that \( P_G(\Gamma) = 1 \), \( P_B(\Gamma) = 0 \).

2. Definition of the product Markov process. We use the terminology and notation of E. B. Dynkin [5], [6]. For each \( i = 1, 2, \ldots \) let

\[
X_i = (x_i(t), M_{it}, P_{1x_i})
\]

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be a nonterminating Markov process with state space \((E_i, \mathcal{B}_i)\) and stationary transition function \(P_i(t, x_i, \Gamma) = P_{i|\omega}(x_i(t) \in \Gamma), \quad \Gamma \in \mathcal{B}_i,\) where \(P_i(t, x_i, E_i) = 1, \quad t \geq 0.\) Here \(x_i(t) = x_i(t, \omega_i), \quad \omega_i \in \Omega_i.\) It is assumed that \(M_{it} \subseteq M_i^0, \quad t \geq 0,\) and that \(P_{i|\omega}\) is for each \(i = 1, 2, \ldots\) and each \(x_i \in E_i\) a probability measure on \(M_i^0.\) For the rest, each process \(X_i\) is assumed to satisfy conditions 3.1 A–G [6, Vol. 1, pp. 77–78].

The product process \(X = (x(t), M_t, P_x)\) will be defined from the following elements(2).

\[
\begin{align*}
    x &= (x_1, x_2, \ldots), \quad x(t) = (x_1(t), x_2(t), \ldots), \quad \omega = (\omega_1, \omega_2, \ldots), \\
    E &= \prod E_i, \quad \Omega = \prod \Omega_i, \quad \mathcal{B} = \prod \mathcal{B}_i, \\
    M_t &= \prod M_{it}, \quad M_t^0 = \prod M_i^0, \\
    P_x &= \prod P_{i|\omega} P(t, x, \cdot) = \prod P_i(t, x_i, \cdot).
\end{align*}
\]

It can be verified that the product process \(X\) satisfies each of the conditions 3.1 A–G [6, Vol. 1, pp. 77–78]. (The verification is routine and will not be included here.) Hence, the process \(X = (x(t), M_t, P_x)\) is Markov.

3. Properties of the product process. We will make the following additional assumptions in the rest of this paper. In each of A1–A3, \(i\) runs through the positive integers.

A1. \(E_i\) is a compact metric space (hence separable) with metric \(D_i,\) and \(\mathcal{B}_i\) is generated by the open sets. (In case \(E_i\) is locally compact separable metric, it is compactified as indicated later.)

A2. If \(f\) is continuous on \(E_i,\) then so is \(T_{i|\omega} f,\) where

\[
(3.1) \quad T_{i|\omega} f(x_i) = \mathcal{E}_{i|\omega} f(x_i(t)) = \int_{E_i} P_i(t, x_i, dy_i) f(y_i).
\]

(the "Feller" property).

A3. \(P_i\) is stochastically continuous; i.e., if \(U \subseteq E_i\) is open and \(x_i \in U,\) then \(\lim_{t \downarrow 0} P_i(t, x_i, U) = 1.\)

From A1, \(E\) is a compact topological space if we take the metric

\[
(3.2) \quad D(x, y) = \sum_{i=1}^{\infty} 2^{-i} \frac{D_i(x_i, y_i)}{1 + D_i(x_i, y_i)}
\]

and the measurable sets \(\mathcal{B}\) are generated by the open sets.

**Theorem 3.1.** If \(X_i\) satisfies A1–A3, \(i = 1, 2, \ldots,\) then the product process \(X\) has a stochastically continuous Feller transition function \(P(t, x, \Gamma), \quad \Gamma \in \mathcal{B}.\)

(2) The fields \(M_t\) and \(M_t^0\) will be enlarged below.
Proof. Let \( f \) be continuous (hence bounded and uniformly continuous) on \( E \) and let \( y \) be an arbitrary fixed point of \( E \). For each \( x \in E \) and \( n = 1, 2, \ldots \) put

\[
x^{(n)} = (x_1, x_2, \ldots, x_n, y_{n+1}, y_{n+2}, \ldots).
\]

Then \( D(x, x^{(n)}) \leq 2^{-n}, \ x \in E, \) and hence, given \( \varepsilon > 0, \) we can choose the positive integer \( n_0 \) so that \( n \geq n_0 \) implies \( |f(x) - f(x^{(n)})| < \varepsilon, \ x \in E. \)

Putting \( x^{(n)}(t) = (x_1(t), x_2(t), \ldots, x_n(t), y_{n+1}, y_{n+2}, \ldots), \ g_n(x_1, \ldots, x_n) = f(x^{(n)}), \) we obtain

\[
T_t f(x) = \mathcal{E}_x[f(x(t)) - f(x^{(n)}(t))] + \mathcal{E}_{x_1x_2\cdots x_n} g_n(x_1(t), \ldots, x_n(t)),
\]

where \( \mathcal{E}_{x_1x_2\cdots x_n} \) denotes expectation with respect to the finite product process \( (x_1(t), x_2(t), \ldots, x_n(t)). \) If \( n \geq n_0, \) the first term on the right side of (3.3) is bounded by \( \varepsilon. \) The second term is a continuous function of \( x, \) since, using the Stone-Weierstrass theorem, we can approximate \( g_n \) uniformly within \( \varepsilon \) on \( E_1 \times E_2 \times \cdots \times E_n \) by a finite sum of products of the form

\[
g_{n_1}(x_1)g_{n_2}(x_2)\cdots g_{n_n}(x_n) = h(x_1, \ldots, x_n),
\]

where \( g_{n_i} \) is continuous on \( E_i. \) Since each \( X_i \) is Feller,

\[
\mathcal{E}_x h(x_1(t), \ldots, x_n(t)) = \prod_{i=1}^n \mathcal{E}_{x_i} g_{n_i}(x_i(t))
\]

is continuous in \( x_i, \ i = 1, \ldots, n, \) and hence is continuous in \( x. \) Hence \( T_t f(x) \) is continuous in \( x. \)

If \( U \subset E \) is open and \( x \in U, \) we find an open set \( U' = U_1 \times U_2 \times \cdots \times U_k \times E_{k+1} \times \cdots, U_i \) open in \( E_i, \) such that \( x \in U' \subset U. \) Then \( P(t, x, U) \equiv \prod_{i=1}^k P_i(t, x, U_i), \) and the product approaches 1 as \( t \to 0. \)

4. Standard character of the product process. Since the transition function \( P(t, x, \Gamma) \) of \( X \) is a stochastically continuous Feller transition function on a compact metric space, \( P \) is the transition function of a standard Markov process which is equivalent to \( X. \) This follows from [6, Theorem 3.14, Vol. 1, p. 104]. We shall not repeat here the complete definition of "standard," but note that a standard process is strongly Markov and has right-continuous sample functions.

Actually, by virtue of assumptions A1–A3, we can and will take each of the processes \( X_i \) to be standard. Moreover, following the discussion in [5] and [6] leading to Theorem 3.14 of [6], we see that the process \( X \) is standard provided we replace \( M_t \) and \( M^0 \) by the larger fields \( \overline{M}_{t+0} \) and \( \overline{M}^0. \) We shall suppose that this has been done, and henceforth can and will assume that our process \( X \) is standard.

\(^{(3)}\) Cairoli, in [1] and an earlier note, has announced a number of results about products of two Markov processes; e.g., such properties as being strong, Hunt, standard, Feller, strongly Feller are productive. Since the entire proof for the Feller property of denumerable products is short, we give it here.
5. Decomposition of the state space.

**Definition 5.1.** A nonempty set \( F \subseteq E \) is **stochastically closed** if \( F \in \mathcal{B} \) and

\[
P_x(x(t) \in F, t \geq 0) = 1, \ x \in F.
\]

The fact that \( \{ \} \) in (5.1) is \( M^0 \)-measurable follows from [6, §4.5, Vol. 1, pp. 110–111]. A set \( F \in \mathcal{B} \) is **indecomposable** if it does not contain two disjoint stochastically closed subsets.

**Definition 5.2.** A stochastically closed set \( F \) is **perfectly decomposable** if \( F = F_1 + F_2 \), where \( F_1 \) and \( F_2 \) are stochastically closed.

**Definition 5.3.** The points \( x' \in F \) and \( x'' \in F \) are said to **divide** the stochastically closed set \( F \) if \( F = F_1 + F_2 \), \( x' \in F_1 \), \( x'' \in F_2 \), and \( F_1 \) and \( F_2 \) are stochastically closed.

In many types of Markov processes, the state space is a union of indecomposable stochastically closed sets plus perhaps certain other sets. This is not characteristic of the process \( X \) of §4. In fact, in important cases, we will see that every stochastically closed set is perfectly decomposable, and we will give a sufficient condition for recognizing when two points divide a stochastically closed set.

It is convenient to have a sufficient condition for perfect decomposability in terms of transition probabilities. Theorem 6.1 will provide this. The following preliminary lemma is an easy consequence of known results on excessive functions.

**Lemma 5.1.** Suppose \( E = E_1 + E_2 \), where

\[
P(t, x, E_1) = 1, \quad x \in E_1, \quad t \geq 0, \tag{5.2}
\]

\[
P(t, x, E_2) = 1, \quad x \in E_2, \quad t \geq 0.
\]

Then \( E_1 \) and \( E_2 \) are stochastically closed.

Note that the first equality in (5.2) would not by itself imply the stochastic closure of \( E_1 \).

**Proof of Lemma 5.1.** Define \( f \) on \( E \) by \( f(x) = 0 \), \( x \in E_1 \); \( f(x) = \infty \), \( x \in E_2 \). Then \( f \) is excessive [6, Vol. 2, p. 1]. That is, \( f(x) \leq E_x f(x(t)) \) and \( \lim_{t \to 0} E_x f(x(t)) = f(x) \), \( x \in E \). Since \( X \) is standard, \( E_2 \) cannot be reached from any point of \( E_1 \), i.e., \( E_1 \) is stochastically closed (ibid., p. 10, Theorem 12.5). A similar argument applies for \( E_2 \).

6. A criterion for division. We define an important subalgebra of \( \mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \cdots \).

**Definition 6.1.** Let \( \mathcal{B}_\infty = \bigcap_{n=1}^{\infty} (\mathcal{B}_n \times \mathcal{B}_{n+1} \times \cdots) \). We call the sets in \( \mathcal{B}_\infty \) "tail sets."

**Theorem 6.1.** Let \( H \in \mathcal{B} \) be stochastically closed. Suppose that \( x' \in H \), \( x'' \in H \), and that there is a set \( \Gamma \in \mathcal{B}_\infty \) such that \( P(t, x', \Gamma) = 1 \) for almost every (Lebesgue) \( t \) while \( P(t, x'', \Gamma) = 0 \) for a.e. \( t \). Then \( x' \) and \( x'' \) divide \( H \). (See Definitions 5.1 and 5.3.)

(*) We write \( F = F_1 + F_2 \) to mean that \( F \) is the union of the disjoint nonempty sets \( F_1 \) and \( F_2 \).
Note 1. For certain applications, the allowance by the hypotheses of an exceptional null set for $t$ may be useful. It helped in an earlier version (later modified) of the proof of Theorem 7.1.

Note 2. The result is not true for processes $X$ in arbitrary state spaces $E$ where a field analogous to $\mathcal{S}_X$ is not defined. (It is the 0-1 property of $3S_X$ that counts.) Consider for example a process with states 1, 2, and 3 where 1 and 2 are absorbing and $0 < P(t, 3, \{i\}) < 1$, $t > 0$, $i = 1, 2$. Take $H = E$, $x' = 1$, $x'' = 2$, $\Gamma = \{1\}$. Then the hypotheses of the theorem (except the meaningless one about $\mathcal{S}_X$) are true, but $E$ is not perfectly decomposable.

Proof of Theorem 6.1. First define

$$H_1 = \{x : x \in E, P(t, x, \Gamma) = 1 \text{ for a.e. } t\}.$$  

Then $x' \in H_1$, which is thus not empty. For fixed $\Gamma_i \in \mathcal{S}_i$, $i = 1, 2, \ldots$, the transition function $P_i(t, x_i, \Gamma_i)$ is measurable $\mathcal{B}_i \times \mathcal{S}_i$, where $\mathcal{B}_i$ is the ordinary Borel algebra of the nonnegative reals (see [6, Vol. 1, §3.17, pp. 98–99]). Since $\Gamma \in \mathcal{B}_n \times \mathcal{B}_{n+1} \times \cdots$, it follows that $P(t, x, \Gamma)$ is measurable $\mathcal{B}_{(0, \omega)} \times \mathcal{B}_n \times \mathcal{B}_{n+1} \times \cdots$ and hence measurable $\mathcal{B}_{(0, \omega)} \times \mathcal{B}_\omega$. Since $H_1$ is exactly the set of $x$ such that

$$\int_0^\infty (1 - P(t, x, \Gamma)) \, dt = 0,$$

we see that $H_1 \in \mathcal{B}_\omega$.

Next note that

$$P(t, x'', H_1) = 0, \quad \text{all } t \geq 0.$$

For, from the assumption of the theorem, letting $t \geq 0$ be fixed,

$$0 = P(s+t, x'', \Gamma) = \int P(t, x'', dy)P(s, y, \Gamma) \quad \text{for a.e. } s,$$

i.e., for a.e. $s$ we have: $P(s, y, \Gamma) = 0$ for a.e. $y(P(t, x'', \cdot)$ measure). Hence for a.e. $y(P(t, x'', \cdot)$ measure) we have: $P(s, y, \Gamma) = 0$ for a.e. $s$. That is, $P(t, x'', H_1) = 0$. This proves (6.2).

Next, letting $s \geq 0$ be fixed, note that if $x \in H_1$, then $P(s+t, x, \Gamma) = 1$ for a.e. $t$, implying by an argument like that above that for a.e. $y(P(s, x, \cdot)$-measure) we have $P(t, y, \Gamma) = 1$ for a.e. $t$, i.e., $y \in H_1$.

Hence,

$$P(s, x, H_1) = 1, \quad \text{all } s \geq 0, x \in H_1.$$

Suppose that for some $t_0 \geq 0$ and $x \in E$ we have $P(t_0, x, H_1) > 0$. Since $H_1$ is a tail event and $P(t_0, x, \cdot)$ is a product measure, $P(t_0, x, H_1)$ is 1 or 0, and hence is 1. Using this fact, the relation

$$P(t_0 + h, x, H_1) = \int_{H_1} P(t_0, x, dy)P(h, y, H_1), \quad h \geq 0,$$
and (6.4), we see that

\[(6.6) \quad P(t_0, x, H_1) > 0 \Rightarrow P(t, x, H_1) = 1, \quad \text{all } t \geq t_0.\]

Now define

\[(6.7) \quad H^* = \{x : x \in E, P(t, x, H_1) = 0, \text{all } t \geq 0\},
\]

\[H' = E - H^*.\]

Note that \(x'' \in H^*,\) from (6.2), and \(x' \in H_1 \subset H',\) from (6.1) and (6.4). Moreover, arguing as we did for \(H_1\) above, we see that \(H'' \in \mathcal{B}_\omega\) and hence \(H' \in \mathcal{B}_\omega.\)

Let \(f\) be the indicator function of \(H_1.\) From (6.6) and the definition of \(H'\) we see that

\[(6.8) \quad \mathcal{E}_x \int_0^\infty f(x(s)) \, ds = \int_0^\infty P(s, x, H_1) \, ds = \infty, \quad x \in H',\]

while \(\mathcal{E}_x \int_t^\infty f(x(s)) \, ds = 0, \ x \in H^*, \ t \geq 0.\) Hence

\[(6.9) \quad 0 = \mathcal{E}_x \int_t^\infty f(x(s)) \, ds = \mathcal{E}_x \mathcal{E}_x \int_0^\infty f(x(s)) \, ds, \quad x \in H^*, \ t \geq 0.\]

Because of (6.8), (6.9) implies \(P_x(x(t) \in H') = 0, \ x \in H^*, \ t \geq 0,\) or

\[(6.10) \quad P(t, x, H') = 1, \quad x \in H^*, \ t \geq 0.\]

Now suppose \(x \in H'.\) From (6.6) and the definition of \(H',\) we can find a positive number \(t\) such that

\[(6.11) \quad 1 = P(t+s, x, H_1) = \int_{H'} P(s, x, dy)P(t, y, H_1) + \int_{H^*} P(s, x, dy)P(t, y, H_1)
\]

\[= \int_{H'} P(s, x, dy)P(t, y, H_1), \quad s \geq 0,\]

and the last integral in (6.11) can be 1 only if \(P(s, x, H') = 1, \ s \geq 0.\) Hence

\[(6.12) \quad P(t, x, H') = 1, \quad x \in H', \ t \geq 0.\]

Lemma 5.1, (6.10), and (6.12) imply the stochastic closure of \(H'\) and \(H^*.\) Since \(H\) is stochastically closed by assumption, we have \(H = (H \cap H') + (H \cap H''),\) where \(H \cap H'\) and \(H \cap H''\) are stochastically closed and \(x' \in H \cap H', \ x'' \in H \cap H''.\) □

**Corollary to Theorem 6.1.** If \(E\) contains two disjoint stochastically closed \(\mathcal{B}_\omega^-\)-measurable sets \(K'\) and \(K'',\) then \(E\) is perfectly decomposable. (Take any \(x' \in K', \ x'' \in K'',\) and put \(\Gamma = K'.\))

### 7. The case of additive processes.

To get more specific results we now consider the case \(E_i = R_1 \cup \{\infty\}, \ i = 1, 2, \ldots,\) where \(R_1\) is the real line and \(\infty\) is a compactifying point. Let \(\mathcal{B}_i\) be the \(\sigma\)-algebra generated by adding \(\{\infty\}\) to the linear Borel sets. Assume that the part of \(X_i\) on \(R_1\) is a temporally homogeneous additive
process (same law for each $i$), while $P_x(t, \infty, \{\infty\}) = 1$ and $P_x(t, x_1, \{\infty\}) = 0$, $t \geq 0$, $x_1 \in R_1$. To avoid trivialities we assume that $x_i(t)$, for fixed $t > 0$, is not constant with probability 1.

It can be verified, and is essentially known, that each process $X_i$ satisfies A1–A3 of §3, and thus may be assumed standard. Hence we may apply the results of §§3–6.

**Definition 7.1.** Let $F = \{x : x \in E, x_i \in R_1, i = 1, 2, \ldots\}$. Put $\theta = (0, 0, \ldots) \in F$. Note that $F$ is stochastically closed.

**Theorem 7.1.** Suppose $H \subseteq F$ is stochastically closed, and $x', x'' \in H$ with $\sum_{i=1}^{\infty} (x'_i - x''_i)^2 = \infty$. Then $x'$ and $x''$ divide $H$ (Definition 5.3). (Here $\infty$ means "infinitely large," and is not the compactifying point.)

**Proof.** We will show how to construct a set $\Gamma$ satisfying the conditions of Theorem 6.1. Note that if $x_i(t)$ has a finite second moment for $i(0, \infty$, and if $\delta^2 x_i(t) = 0$, we can immediately define $\phi$

$$r = \frac{\sum_{i=1}^{n_k} (x'_i - x''_i)(x_i - x'_i)}{\sum_{j=1}^{n_k} (x'_j - x''_j)^2} = 0,$$

where the sequence $\{n_k\}$ will be defined in (7.4). Similar criteria are familiar and are suggested by considering the likelihood ratio in the Gaussian case.

The additivity of the processes $X_i$ enables us to take $x' = (0, 0, \ldots) = \theta$ without loss of generality. We then put $y_i = x''_i$, $i = 1, 2, \ldots$, and thus $\sum y_i^2 = \infty$.

We shall use the function $\phi$ defined on $R_1 \cup \{\infty\}$:

$$\phi(u) =
\begin{cases}
1 & \text{if } u > 0 \text{ or } u = \infty, \\
0 & \text{if } u = 0, \\
-1 & \text{if } u < 0.
\end{cases}
$$

Note that if the sequence $\{|y_i|\}$ is unbounded, the solution is particularly easy. Suppose $y_{n_1} < y_{n_2} < \ldots$.

Put

$$\Gamma_1 = \bigcup_{n=1}^{\infty} \left\{ x : \lim_{i \to \infty} \sup_{i=1}^{n} \frac{h_n(x_{n_1}) + \cdots + h_n(x_{n_i})}{i} > 0 \right\},$$

where $h_n$ is the indicator function of the interval $(-n, n)$. From the strong law of large numbers, $P(t, \theta, \Gamma_1) = 1$ and $P(t, y, \Gamma_2) = 0, t \geq 0$. Hence $\Gamma_1$ can be the set $\Gamma$ of Theorem 6.1. Note that the definition of $\Gamma_1$, while depending on $y$, does not depend on the distribution of the $x_i(t)$.

Thus, for the remainder of the proof we can and will assume

$$|y_i| \leq K, \quad i = 1, 2, \ldots,$$

$y$ being a fixed sequence such that $\sum y_i^2 = \infty$.

(5) In the fraction put $x'_i - x''_i = 0$, say, if $x_j = \infty$. 


(a) Assume first that if $x_i(0) \neq \infty$, then $x_i(t) - x_i(0)$ has for each $t \geq 0$ a symmetric distribution. Let $n_1 < n_2 < \cdots$ be positive integers such that

\[
\sum_{k=1}^{\infty} \left( \sum_{j=1}^{n_k} y_j^2 \right)^{-1} < \infty.
\]

Define $T_{k,n}(x)$ and $\psi_n(x)$, $x \in E$, $k, n = 1, 2, \ldots$, by

\[
T_{k,n}(x) = \frac{\sum_{i=1}^{n_k} y_i \int_{-n}^n \phi(x_i + s) \, ds}{\sum_{i=1}^{n_k} y_i^2},
\]

\[
\psi_n(x) = \lim sup_{k \to \infty} \left| T_{k,n}(x) \right|,
\]

agreeing that $x_i + s = \infty$ if $x_i = \infty$ in (7.5). Put

\[
\Gamma^{(n)} = \{ x : x \in E, \psi_n(x) = 0 \}, \quad \Gamma_2 = \bigcap_{n=1}^{\infty} \Gamma^{(n)}.
\]

Obviously $\Gamma^{(n)}$ and $\Gamma_2$ are $\mathcal{B}_\infty$-measurable. We shall see that $P(t, \theta, \Gamma^{(n)}) = 1$, $t > 0$, while $P(t, \theta, \Gamma^{(n)}) = 0$ for all sufficiently large $n$, so that $P(t, \theta, \Gamma_2) = 1$ and $P(t, \theta, \Gamma_2) = 0$.

Since $\phi$ is an odd function on $R$, and $x_i(t) - x_i(0)$ has a symmetric distribution, $\delta \phi(x_i(t) + s) + \delta \phi(x_i(t) - s) = 0$ and hence $\delta \phi T_{k,n}(x(t)) = 0$. Letting $\sigma_{k,n}^2(x)$ be the variance of $T_{k,n}(x(t))$ calculated for $x(0) = x$, we have

\[
\sigma_{k,n}^2(x) \leq \frac{\sum_{i=1}^{n_k} y_i^2 (2n)^2}{\left( \sum_{i=1}^{n_k} y_i^2 \right)^2} \leq 4n^2 \left( \sum_{i=1}^{n_k} y_i^2 \right)^{-1}, \quad x \in E,
\]

and $\sum_{k=1}^{\infty} \sigma_{k,n}^2(x) < \infty$ because of (7.4). Hence $P_{\theta}(\lim_{k \to \infty} T_{k,n}(x(t)) = 0) = 1$. That is,

\[
P(t, \theta, \Gamma^{(n)}) = 1, \quad n = 1, 2, \ldots, \quad t \geq 0.
\]

Because of the symmetry of the distribution of $x_i(t)$, we have for $b > 0$

\[
b \delta \phi(x_i(t) + b) = b(P_{\theta}(x_i(t) > b) + P_{\theta}(0 \leq x_i(t) \leq b)
\]

\[
+ P_{\theta}(-b < x_i(t) < 0) - P_{\theta}(x_i(t) < -b)),
\]

whence

\[
b \delta \phi(x_i(t) + b) \geq |b| P_{\theta}(0 \leq x_i(t) \leq |b|).
\]

A similar argument shows that (7.10) is still true for $b < 0$. 

Now recall (7.3) and consider, for \( n \geq K \geq y_i \geq 0 \) (we consider \( y_i < 0 \) below),

\[
y_i \int_{-\infty}^{\infty} \phi(x_i(t) + s) \, ds = y_i \int_{-\infty}^{n} \phi(x_i(t) + s) \, ds
\]

\[
y_i \int_{-n}^{-y_i} \phi(x_i(t) + s) \, ds + y_i \int_{n}^{n+y_i} \phi(x_i(t) + s) \, ds.
\]

(7.11)

Of the last two expectations in (7.11), the first vanishes for the same reason that \( \delta x_{T_{n,n}} = 0 \) above. As for the second, using (7.10), we find

\[
y_i \int_{-n}^{-y_i} \phi(x_i(t) + s) \, ds \geq y_i \int_{-n}^{n+y_i} P_{\theta}(0 \leq x_i(t) \leq s) \, ds,
\]

and hence, from (7.11) and (7.12),

\[
y_i \int_{-n}^{n} \phi(x_i(t) + s) \, ds \geq 2y_i^2 P_{\theta}(0 \leq x_i(t) \leq n-K).
\]

Almost the same argument shows that (7.13) is also true if \( n \geq K \) and \( y_i < 0 \), and hence it holds whenever \( n \geq K \). From (7.5) and (7.13),

\[
y_i \infty_{T_{k,n}}(x(t)) = 2P_{\theta}(0 \leq x_i(t) \leq n-K).
\]

(7.14)

Using (7.14) and (7.7) and noting from (7.4) that \( \lim_{k \to \infty} \sigma^2_{n,n}(y) = 0 \), we see that when \( n \) is large enough so that the right side of (7.14) is positive, \( P(t, y, \Gamma^{(n)}) = 0 \), \( t > 0 \). From this and (7.8) we see that \( \Gamma_2 \) as defined by (7.6) can be the set \( \Gamma \) of Theorem 6.1. Note that \( \Gamma_2 \) was defined without reference to the distribution of the \( x_i(t) \).

(b) Drop the assumption of symmetry. Suppose, however, that

\[
\sum_{i=1}^{\infty} (y_i - b)^2 = \infty \quad \text{for each } b, \quad -\infty < b < \infty.
\]

(7.15)

(The contrary case is simple, but is essentially different and is treated below.) It follows from (7.15) by routine arguments that will not be given here, that there exist distinct positive integers \( m_1, m_2, \ldots \) (the sequence \( \{m_i\} \) is not necessarily increasing) such that

\[
(y_{m_1} - y_{m_2})^2 + (y_{m_3} - y_{m_4})^2 + \cdots = \infty.
\]

(7.16)

Define the mapping \( \xi \) of \( E \) onto \( E \) by

\[
\xi(x) = (x_{m_1} - x_{m_2}, x_{m_3} - x_{m_4}, \ldots),
\]

where it is agreed that if either \( x_{m_i} \) or \( x_{m_i+1} = \infty \), then so is their difference. Let \( w(t) = (w_1(t), w_2(t), \ldots) = \xi(x(t)) \). If \( x(0) = \theta \), then the \( w_i(t) \) are symmetric independent random variables.

Put \( y' = \xi(y) \), and let \( n_1 < n_2 \cdots \) be a sequence such that (7.4) holds with each component of \( y \) replaced by the corresponding one of \( y' \) and \( \{n_k\} \) replaced by \( \{n_k'\} \). Let \( \Gamma_2' \) be constructed just as was \( \Gamma_2 \), except that \( y \) is replaced by \( y' \) and \( \{n_k\} \)
by \( \{n_k\} \). Then \( P_\theta(w(t) \in \Gamma_2') = 1 \) and \( P_\theta(w(t) \in \Gamma_2') = 0 \), from the argument of (a) above. Putting \( \Gamma_3 = \xi^{-1}(\Gamma_2') \), we see that \( P(t, \theta, \Gamma_3) = 1 \), \( P(t, y, \Gamma_3) = 0 \), \( t > 0 \), and hence \( \Gamma_3 \) can serve as \( \Gamma \) of Theorem 6.1. Also, \( \Gamma_3 \) was defined without reference to the distribution of the \( x_i(t) \).

(c) Finally, suppose that for some \( b \neq 0 \),

\[
\sum (y_i - b)^2 < \infty.
\]

The difference of this case from the others is evident if we suppose each \( x_i(t) \) has a finite variance and \( \mathbb{E}_{\phi}(x_i(t)) = \mathbb{E}_{\phi}(x_i(t) - t)^2 = t \). If \( x(0) = \theta \), then the components of \( x(2) \) have means 2 and variances 2, while if \( x(\theta) = (1, 1, \ldots) \), the components of \( x(1) \) have means 2 and variances 1. That is, distinguishing the starting point \( \theta \) from the starting point \((1, 1, \ldots)\) requires distinguishing not only translations but also changes of scale.

The canonical form of the logarithm of the characteristic function of \( x_i(t) \), if \( x_i(0) = 0 \), is

\[
tR(s) + itI(s), \quad -\infty < s < \infty,
\]

where \( R \) and \( I \) are finite functions whose precise form is immaterial. Pick \( \alpha \) so that

\[
0 < b\alpha < 2\pi \quad \text{and} \quad R(\alpha) \neq 0.
\]

This is possible since the nondegeneracy of \( x_i(t) \) implies that \( R \) cannot vanish identically in a neighborhood of 0.

Letting \( \text{Re} \) ( ) and \( \text{Im} \) ( ) denote real and imaginary parts and agreeing that \( e^{ia\infty} = 0 \), we put

\[
\Gamma_4 = \left\{ x : x \in E, \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} e^{iax_j} \text{ exists (} = U(x), \text{ say)} \right\}
\]

and

\[
\exp \left[ \frac{iI(\alpha)}{R(\alpha)} \text{Re} (\log U(x)) - i \text{Im} (\log U(x)) \right] = 1.
\]

It is easily verified that \( P_\theta(x(t) \in \Gamma_4) = 1 \), \( t > 0 \). On the other hand, we have, using (7.15),

\[
P_\theta \left( \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} e^{iax_j(t)} + iay \right) = \exp \left[ tR(\alpha) + itI(\alpha) + ib\alpha \right] = 1,
\]

which shows that \( P_\theta(y + x(t) \in \Gamma_4) = 0 \), \( t > 0 \). Hence (7.20) furnishes an appropriate \( \Gamma \). This completes the proof of Theorem 7.1.

An easy corollary of Theorem 7.1 is that every stochastically closed subset \( H \) of \( F \) is perfectly decomposable. For if \( x' \in H \), then

\[
P_{x'} \left\{ \sum_{t=1}^{\infty} (x_i(t) - x'_i)^2 < \infty \right\} = 0, \quad t > 0.
\]

Hence there must be an \( x'' \in H \) such that \( \sum (x''_i - x'_i)^2 = \infty \).
8. Extensions and relation to other results. Essentially the same methods apply if each \( x(t) \) is an additive process in a \( p \)-dimensional Euclidean space, \( x(t) = (x_1(t), \ldots, x_p(t)) \), with \( \|x(t)\|^2 = \sum_i (x_i(t))^2 \), assuming that each component \( x_i(t) \) is nondegenerate. Theorem 7.1 is true if we take \( F = \{ x : \text{all components of each } x_i \text{ are finite} \} \) and assume that \( \sum \|x_i' - x_i\|^2 = \infty \). The definition of \( T_{k,n}(x) \) in (7.5) is changed to

\[
\sum_{i=1}^{n_k} \sum_{j=1}^{p} y_{ij} \int_{-n}^{n} \phi(x_{ij} + s) \, ds
\]

If \( t \) is discrete, \( t = 0, 1, 2, \ldots \), then Theorem 6.1 still holds, if we replace "almost every \( t \)" in the hypotheses by "every \( t = 0, 1, \ldots \). In Theorem 7.1, we can then assume that \( x_i(t+1) - x_i(t), \ t = 0, 1, \ldots, i = 1, 2, \ldots, \) are arbitrary identically distributed independent random variables. The proof of Theorem 7.1 is unchanged, with one exception. In case (c), where \( \sum (y_i - b)^2 < \infty \) for some \( b \neq 0 \), we take \( a \) close enough to \( 0 \) so that \( \delta e^{ax_i^2(t)} \neq 0 \), besides requiring (7.19).

Shepp [7] considered the following problem. Let \( Z_1, Z_2, \ldots \) be independent random variables with a common distribution \( G \), let \( y = (y_1, y_2, \ldots) \) be a nonrandom sequence, and let \( P_a \) and \( P_b \) be the probability measures on \( R_1 \times R_1 \times \cdots \) generated respectively by \( Z_1, Z_2, \ldots \) and \( Z_1 + y_1, Z_2 + y_2, \ldots \). Shepp showed by another method(6) that if \( \sum y_i^2 = \infty \), there is a measurable \( \Gamma_0 \) in \( R_1 \times R_1 \times \cdots \) such that \( P_a(\Gamma_0) = 1 \) and \( P_b(\Gamma_0) = 0 \); this is part of his necessary and sufficient conditions. Since we have found a set \( \Gamma \) such that \( P(t, \theta, \Gamma) = 1 \) and \( P(t, y, \Gamma) = 0, \ t = 1, 2, \ldots \) in the discrete case, we obtain Shepp's sufficient condition by putting \( t = 1 \) and \( \Gamma_0 = \Gamma \cap F \) (see Definition 7.1). However, we can now state the following extension of Shepp's result, noting that except for the case (c) above where \( \sum (y_i - b)^2 < \infty \) for some \( b \), our set \( \Gamma_0 \) is constructed without reference to the distribution of the \( x_i(t) \).

**Theorem 8.1.** Let \( Z_1, Z_2, \ldots \) be independent identically distributed random variables and let \( y_1, y_2, \ldots \) be a nonrandom sequence such that \( \sum_{t=1}^{\infty} (y_i - b)^2 = \infty \) for each real \( b \). Then there is a measurable set \( \Gamma_0 \) in \( R_1 \times R_1 \times \cdots \), whose construction depends on \( y_1, y_2, \ldots \) but not on the distribution of the \( Z_i \), such that \( P_a(\Gamma_0) = 1 \) and \( P_b(\Gamma_0) = 0 \).

A similar result holds if \( Z_i \) are \( p \)-dimensional vectors.

9. Point processes or counting measures. For some purposes it is appropriate to identify the two sequences \( (x_1, x_2, \ldots) \) and \( (x'_1, x'_2, \ldots) \) if one is obtained by permuting the coordinates of the other. The equivalence classes so obtained are point

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(6) Use was made of a well-known theorem of Kakutani on product measures. The sufficient condition of Shepp also follows from recent results of Dudley [4], but without the distribution-free property of Theorem 8.1 that is crucial for our purposes.
processes or counting measures. Thus we have a way of treating Markov processes whose states are infinite counting measures. In particular, for the case where the $x_i(t)$ are additive processes, we have the kind of situation considered by Doob [3, p. 404], Dobrushin [2], and others. The author hopes to discuss such processes in a later paper.

REFERENCES


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