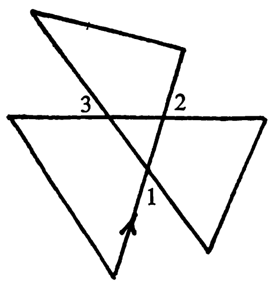
A CHARACTERIZATION OF THE DOUBLE POINT STRUCTURE OF THE PROJECTION OF A POLYGONAL KNOT IN REGULAR POSITION

BY
L. B. TREYBIG[1]

1. **Introduction.** In [2] Gauss considered the idea of associating with certain types of plane curves a finite sequence of integers. For an illustration of the idea see Figure 1. Gauss listed all possible types of these curves with up to four crossing points and also conjectured that in any given one of the finite integer sequences between each two occurrences of a symbol there were an even number, perhaps zero, of the other symbols.



1 2 3 1 2 3

FIGURE 1

In [5] Nagy proved Gauss' conjecture and also observed that such a point set could be colored alternately with two colors a and b such that as a point moved from one section through a crossing point into another section the colors of the sections changed. Nagy used the idea of colored sections to define some simple closed curves (made up of sections of the same color) which he called cycles. Nagy's cycles do bear some resemblance to the abstract cycles(W) (see §4) used later on in this paper, in that for every cycle of Nagy, there does correspond naturally a cycle(W), but the converse does not hold.

It now seems appropriate to lay an exact foundation before proceeding. Suppose that $A_n=\{a_1, a_2, \ldots, a_{2n}\}$ is a set of $2n$ points lying in (0, 1) such that $a_i<a_{i+1}$, $i=1, 2, \ldots, 2n-1$, and that W is a decomposition of A_n into two element sets. Suppose also that f is a mapping (continuous transformation) of the half open interval [0, 1) into the plane such that (1) $f(t)=f(t')$ for $t<t'$ if and only if

Presented to the Society, August 16, 1965; received by the editors April 15, 1966.

(1) This research was partially supported by NSF Grant GP3989.

$\{t, t'\} \in W$, (2) Im f can be expressed as the sum of a finite number of straight line intervals such that no point of $f(A_n)$ is an endpoint of one of the intervals and, (3) $f(t) \to f(0)$ as $t \to 1$. Note that Im f can be considered (see [1]) as the projection of a polygonal knot in regular position, where the set $f(A_n)$ is the set of double points of the projection. The main purpose of this paper is to give a condition on the decomposition W that is both necessary and sufficient for there to exist an f having the above properties.

The author first became interested in this problem through the work of D. E. Penney [6]. Suppose that g is a one to one mapping of $[0, 1)$ into E^3 such that (1) $\pi g = f$, where $\pi(x, y, z) = (x, y, 0)$, (2) Im g is the sum of a finite number of straight line intervals, and (3) $g(t) \to g(0)$ as $t \to 1$. Penney has been studying the idea of associating with g (or Im g) a "word" $f(a_1)^{e_1} f(a_2)^{e_2} \cdots f(a_{2n})^{e_{2n}}$, where if $f(a_i) = f(a_j)$, and the z coordinate of $g(a_i)$ is larger than the z coordinate of $g(a_j)$, then $e_i = 1$ (or is suppressed) and $e_j = -1$. Penney has been able to use his methods to show certain knots are isomorphic, but as yet has no results on showing knots are not isomorphic. The author dces feel that the results of this paper will shed some light on that problem.

In his papers [7], [8] C. J. Titus considers mappings very similar to the ones in this paper. Suppose g is a mapping from the boundary C of the set $D = \{z : |z| < 1\}$ into the plane such that g is differentiable, such that Im g has only a finite number of double points, and at each double point the tangents are linearly independent. Such a mapping g is called a normal mapping. In [8], Titus solves the following problem suggested by C. Loewner:

PROBLEM B. Given a normal representation g of a closed oriented curve find necessary and sufficient conditions that g be an interior boundary. (I.e., that g can be extended continuously to $\bar{D}$ so as to be interior and sense preserving on D.)

For another reference in this area see also Marx [3].

2. **Definitions.** If A_n is a set of $2n$ points in $(0, 1)$ let $N(A_n)$ denote the set of all continuous transformations f from $[0, 1)$ into the plane such that there is a decomposition W of A_n into two element sets such that f and W are as in paragraph one of §1. Such a W will be said to determine the double point structure of f, and it is clear that each such f in $N(A_n)$ determines a unique W. The set $Q(A_n)$ of abstract intervals of A_n is the set of ordered pairs $(a_1, a_2), (a_2, a_3), \ldots, (a_{2n}, a_1)$. The numbers which determine such a pair are called its endpoints, and the geometric realizations of these intervals are, respectively, $[a_1, a_2]$, $[a_2, a_3]$, $\ldots$, $[a_{2n}, 1) + [0, a_1]$. (Denote $[0, a_1] + [a_{2n}, 1)$ by $[a_{2n}, a_1]$, and call a_{2n} and a_1 the endpoints of this set.) The geometric realization of an interval I of $Q(A_n)$ will also be denoted by $G(I)$ and the set of all $G(I)$ for $I \in B \subset Q(A_n)$ will be denoted by $G(B)$. W will be said to have property P provided it is true that if $U, V \subset W$ and $U = W - V$, then there exists $\{a, b\} \in U$ and $\{c, d\} \in V$ such that $a < c < b < d$ or $c < a < d < b$. If the double point structure of the element f of $N(A_n)$ is determined

by a W that has property P, then f is said to have property P or be prime. W is said to have property Q provided that if $\{a, b\} \in W$ where $a<b$ then there exists $\{c, d\} \in W$ such that $a<c<b<d$ or $c<a<d<b$. As before, an f associated with a W with property Q will also have property Q. Figure 1 is associated with an f with property P. In Figure 2 the left-hand figure is determined by an f with property Q but not property P and the right-hand figure is determined by an f which does not have property Q (and hence not property P).

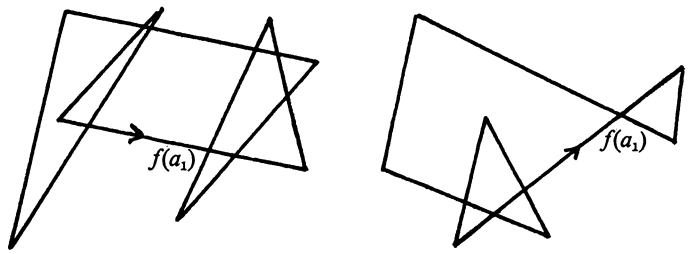


FIGURE 2

3. **Some preliminary topology theorems.** Through the remainder of this paper let $A_n=\{a_1, a_2, \ldots, a_{2n}\}$ denote a subset of (0, 1) such that $a_i<a_{i+1}$, $i=1, \ldots, 2n-1$.

THEOREM 1. *Suppose that $B_n=\{b_1, b_2, \ldots, b_{2n}\}$ is a subset of* (0, 1) *such that $b_i<b_{i+1}$, $i=1, \ldots, 2n-1$, and that W is a decomposition of A_n into two element sets. Let k be any cyclic permutation of* 1, 2, ..., 2n *and let W' be a decomposition of B_{2n} such that $\{b_{k(u)}, b_{k(v)}\} \in W'$ if and only if $\{a_u, a_v\} \in W$. Then, there is an f in $N(A_n)$ whose double point structure is determined by W if and only if there is a g in $N(B_n)$ whose double point structure is determined by W'.*

Proof. There is a one-to-one function h from [0, 1) onto [0, 1) such that (1) $h(a_u)=b_{k(u)}$, $u=1, \ldots, 2n$, and (2) (a) $h(t) \to h(0)$ as $t \to 1$ and h has exactly one point of discontinuity, or (b) h is continuous. Each of the functions f and g can be defined in terms of the other by use of the equation $f=gh$.

THEOREM 2. *If $f \in N(A_n)$ and $B \in G(A_n)$, then there exist two complementary domains U and V of* $\operatorname{Im} f$ *such that $f(B) \subset \bar{U} \cdot \bar{V}$ and no point of $f(B)-f(B)\cdot f(A_n)$ is a boundary point of any other complementary domain of* $\operatorname{Im} f$.

Proof. Suppose $B=[a_i, a_{i+1}]$. (The case where $B=[a_{2n}, a_1]$ involves the same ideas.) Let $y, s_1, s_2, \ldots, t_1, t_2, \ldots$ denote real numbers such that (1) $\{s_p\}$ and $\{t_p\}$ converge to a_i and a_{i+1}, respectively, and (2) $a_i<s_{p+1}<s_p<y<t_p<t_{p+1}<a_{i+1}$ for $p=1, 2, 3, \ldots$. In case $f(a_i)=f(a_{i+1})$, $f(B)$ is a simple closed curve and in case $f(a_i)\neq f(a_{i+1})$, $f(B)$ is an arc and there is also an arc from $f(a_i)$ to $f(a_{i+1})$ in $f([0, a_i]+[a_{i+1}, 1))$, so in either event there is a simple closed curve J such that (1) $J \subset \operatorname{Im} f$, (2) $f(a_i), f(a_{i+1}) \in J$, and (3) $f([s_p, t_p])$ is an arc lying in J for $p=1, 2, \ldots$.

By Theorem 10, p. 166 of [4] there is an arc $xf(y)z$ such that $x \in \operatorname{Int} J$, $y \in \operatorname{Ext} J$, and $xf(y)z \cdot J = f(y)$. Since $f([0,1)-(a_i, a_{i+1}))$ is closed and does not intersect $f((a_i, a_{i+1}))$ there is a subarc $x_1 f(y) z_1$ of $xf(y)z$ such that $x_1 \in \operatorname{Int} J$, $z_1 \in \operatorname{Ext} J$, and $x_1 f(y) z_1 \cdot \operatorname{Im} f = f(y)$. Since $\operatorname{seg} x_1 f(y)$ and $\operatorname{seg} f(y) z_1$ clearly lie in different complementary domains of $\operatorname{Im} f$, an application of Theorem 37, p. 185 of [4] yields that there exists two complementary domains U_p and V_p $(p=1,2,\ldots)$ of $\operatorname{Im} f$ such that (1) $U_p \subset \operatorname{Int} J$, $V_p \subset \operatorname{Ext} J$, (2) $f([s_p, t_p]) \subset \overline{U}_p \cdot \overline{V}_p$, and (3) no point of $f((s_p, t_p))$ is a boundary point of any other complementary domain of $\operatorname{Im} f$. Clearly $U_1 = U_2 = \cdots$ and $V_1 = V_2 = \cdots$, so U_1 and V_1 satisfy the conclusion of the theorem.

THEOREM 3. *If $f \in N(A_n)$ and $(a_r, a_s) \in Q(A_n)$ such that $f(a_r) = f(a_s)$, then* $\operatorname{Im} f$ *$-f(G(a_r, a_s))$ is a subset of one of the complementary domains of the simple closed curve $f(G(a_r, a_s))$.*

Proof. $\operatorname{Im} f - f(G(a_r, a_s))$ is connected by the continuity of f and (in one case) the fact that $f(t) \to f(0)$ as $t \to 1$.

THEOREM 4. *If $f \in N(A_n)$ and $a_i \in A_n$ then* (1) *there exist at least three but no more than four complementary domains of image f whose boundaries contain $f(a_i)$, and* (2) *if f also has property Q there are exactly four such domains.*

Proof. Suppose a_j is the other point of A_n such that $f(a_j) = f(a_i)$ and $i<j$. There exist numbers a, b, c, d such that $a < a_i < c$, $b < a_j < d$, and a circle J with center at $f(a_i)$ such that (1) $\operatorname{Im} f \cdot (J + \operatorname{Int} J)$ consists of two straight line intervals, $Af(a_i)C$ and $Bf(a_j)D$, where $f(a)=A$, $f(b)=B$, $f(c)=C$ and $f(d)=D$. (2) $f(A_n)\cdot(J+\operatorname{Int} J)$ $=f(a_i)$. Let P, Q, R, and S denote additional points on the circle J which occur in the circular order $APBQCRDSA$. Let U_1, U_2, U_3, and U_4 denote the complementary domains of $\operatorname{Im} f$ which contain seg APB, seg BQC, seg CRD, and seg DSA, respectively. Since the arcs APB and BQC abut (see [4, p. 180]) on $f(a_{j-1})Bf(a_j)$ from opposite sides it follows that $U_1 \neq U_2$. Similarly $U_2 \neq U_3$, $U_3 \neq U_4$, $U_4 \neq U_1$. Each point of $\operatorname{Int} J$ belongs to one of U_1, U_2, U_3, U_4, $Af(a_i)C$, $Bf(a_j)D$, so there are at most four complementary domains of $\operatorname{Im} f$ whose boundaries contain $f(a_i)$.

Suppose there are only two such domains. Then, $U_1 = U_3$ and $U_2 = U_4$. There exist arcs PTR and QUS which lie, respectively, in U_1 and U_2. Let P_1 (Q_1) denote the last point of APB (BQC) in the order from P to R (Q to S) on the arc PTR (QUS). Let R_1 (S_1) denote the first point of PTR (QUS) which follows P_1 (Q_1) and lies on CRD (DSA). But the points P_1 and R_1 separate the points S_1 and Q_1 on J and the subarcs P_1R_1 and Q_1S_1 of PTR and QUS, respectively, lie, except for their endpoints, in $\operatorname{Ext} J$. By Theorem 11, p. 147 of [4] these arcs intersect. This yields a contradiction.

Now suppose that f has property Q but that $U_2 = U_4$. There exist integers m and n such that $f(a_m) = f(a_n)$, $a_i < a_m < a_j$ and either $a_n < a_i$ or $a_j < a_n$. Suppose $a_j < a_n$. Since $f([a_m, b] + [d, a_n])$ is a locally connected compact continuum lying except for $f(b)$ and $f(d)$ in Ext J, it contains an arc BVD lying except for B and D in Ext J. If Q_1S_1 denotes an arc as above which lies in U_2, then Q_1S_1 and BVD must intersect in a point which lies in $\operatorname{Im} f$ and also in U_2. Again, this is a contradiction.

THEOREM 5. *If $f \in N(A_n)$ and a_i and a_j are two elements of A_n such that $f(a_i)=f(a_j)$ and K and L are elements of $G(A_n)$ containing a_i and a_j, respectively, then there is a complementary domain U of* $\operatorname{Im} f$ *such that $f(K)+f(L)\subset \operatorname{Bd} U$.*

Proof. The validity of Theorem 5 may be seen by an inspection of the proof of Theorem 4 and use of Theorem 2.

THEOREM 6. *If $f \in N(A_n)$ and f has property Q, then no point separates* $\operatorname{Im} f$.

Proof. If some point x separates $\operatorname{Im} f$ then x clearly must be of the form $f(a_i)=f(a_j)=x$ where $i<j$. However, there exist integers m and n such that $f(a_m)=f(a_n)$, $a_i<a_m<a_j$ and either $a_n<a_i$ or $a_j<a_n$. But this means that $f([0, a_i)+(a_i, a_j)+(a_j, 1))$ is connected, a contradiction.

THEOREM 7. *If $f \in N(A_n)$ and U is a complementary domain of* $\operatorname{Im} f$, *then there is a unique subcollection H of $G(A_n)$ (namely, $\{K : K \in G(A_n)$ and $f(K-K\cdot A_n)$ intersects* $\operatorname{Bd} U\}$) *such that* (1) $f(H^*)=\operatorname{Bd} U$ *and* (2) $f(H^*)$ *is the boundary of no other complementary domain of* $\operatorname{Im} f$.

Proof. By Theorem 42, p. 191 of [4] $\operatorname{Bd} U$ is a continuous curve, so each point of $\operatorname{Bd} U$ is a limit point of $\operatorname{Bd} U-f(A_n)\cdot \operatorname{Bd} U$. Let H denote the subset of $G(A_n)$ described above. By Theorem 2, each $K\in H$ has the property that $f(K)\subset \operatorname{Bd} U$. Since H is finite and each point of $\operatorname{Bd} U$ is a limit point of the closed set $f(H^*)$, it follows that $f(H^*)=\operatorname{Bd} U$. The uniqueness follows from the fact that f is one-to-one on $[0, 1)-A_n$.

If $f(H^*)$ were the boundary of some other complementary domain V of $\operatorname{Im} f$, then by Theorem 43, p. 193 of [4], $f(H^*)$ is a simple closed curve J and $\operatorname{Im} f$ could not contain a point not on J because that would mean that either U or V would have a boundary point not on J. Thus the existence of A_n is denied, a contradiction.

THEOREM 8 (NAGY [5]). *If $f \in N(A_n)$ and a_i and a_j are elements of A_n such that $f(a_i)=f(a_j)$ then $j-i$ is odd.*

THEOREM 9. *If $f \in N(A_n)$ such that f has property Q, D is a complementary domain of* $\operatorname{Im} f$, *and H is a subset of $G(A_n)$ such that* $\operatorname{Bd} D = f(H^*)$, *then* (1) $\operatorname{Bd} D$ *is a simple closed curve, and* (2) *if $a_i \in A_n\cdot K\subset K\in H$ there is exactly one other element L of H containing an element a_j of A_n such that $f(a_i)=f(a_j)$, and, furthermore, K and L do not intersect.*

Proof. (1) By Theorem 6 no point separates $\operatorname{Im} f$ and by Theorem 46, p. 199 of [4], $\operatorname{Bd} D$ is a simple closed curve. (2) Let L and L' be the two elements of

$G(A_n)$ which contain a_j. Since f has property Q, neither L nor L' intersects K. By the proof of Theorem 4, Bd U must contain one of $f(L)$ and $f(L')$, but cannot contain both. Similarly Bd U cannot contain $f(K')$ for the element K' of $G(A_n)$ which contains a_i and is distinct from K.

4. **More definitions.** Given that W is decomposition of A_n into two element sets and $a_i, a_j \in A_n$, then a_i will be said to be equivalent to a_j modulo W ($a_i \equiv a_j(W)$) provided there exists $W_1 \in W$ such that $a_i, a_j \in W_1$. The subset H of $Q(A_n)$ will be said to contain the element W_1 of W modulo W provided there is an element I of H with an endpoint which is an element of W_1. Shorten the last definition to H contains(W) W_1, or write $W_1 \in H(W)$. Later definitions will simply be given in the abbreviated notation. The subset W_1 of W will be said to be a subset(W) of the subset H of $Q(A_n)$ ($W_1 \subset H(W)$) provided each $W' \in W_1$ satisfies $W' \in H(W)$. Two subsets H and K of $Q(A_n)$ will be said to intersect(W) provided there exists $W_1 \in W$ such that H and K both contain(W) W_1 and their intersection(W) (write $H \cdot K(W)$) is the set of all such W_1's. Two subsets H and K of $Q(A_n)$ will be said to be mutually separated(W) provided H and K do not intersect(W). The subset H of $Q(A_n)$ is said to be connected(W) provided H cannot be expressed as the sum of two mutually separated(W) sets K and L. Component(W) is defined in the obvious way.

The subset A of $Q(A_n)$ is said to be an arc(W) provided there exist two elements W_1 and W_2 of W such that A is connected(W) and contains(W) W_1 and W_2, but has no proper subset with the same property. In this case A is said to be an arc(W) from W_1 to W_2. The subset J of $Q(A_n)$ is said to be a simple closed curve(W) (write s.c.c.(W)) provided (1) J is degenerate and the endpoints of $I \in J$ are the elements of a set of W, or (2) J is connected(W) and if the element I of J contains(W) the element W_1 of W then there is exactly one other element I' of J which contains(W) the element W_1. If $Q(A_n)$ were infinite, the definition of J above could be altered to require also that J be finite.

If H is a subset of $Q(A_n)$ and W_1, W_2 and W_3 are three mutually exclusive subsets of W which are subsets(W) of H, then W_1 is said to separate(W) W_2 from W_3 in H provided it is true that if K is a connected(W) subset of H which contains(W) an element of W_2 and an element of W_3, then K also contains(W) an element of W_1.

The subset C of $Q(A_n)$ is called a cycle(W) if and only if C is connected(W) and (1) C is a degenerate s.c.c.(W), or (2) if a_i is an endpoint of the element I of C, there is another element I' of C with an endpoint a_j such that $\{a_i, a_j\} \in W$ and, either (a) each of a_i and a_j is an endpoint of only one element of C and I and I' have no common endpoint, or (b) there is no element $\{a_r, a_s\}$ of W such that r is between i and j and s is not, and every element of $Q(A_n)$ which has a_i or a_j for an endpoint belongs to C.

In the sequel, the sentence "a_i is an endpoint of the element I of $Q(A_n)$" will be shortened in many instances to "$a_i \in I \in Q(A_n)$."

5. **Topology modulo** W. This section is devoted to proving some more preliminary theorems to the main result, the theorems being in some cases just reformulations of well-known theorems about arcs and simple closed curves.

THEOREM 10. *If A is a connected(W) subset of $Q(A_n)$ which contains(W) the two elements W_1 and W_2 of W, then A contains an arc(W) from W_1 to W_2.*

Proof. If A is not an arc(W) from W_1 to W_2, A contains a proper subset A_1 which is connected(W) and contains(W) W_1 and W_2. A continuation of this process yields, after a finite number of steps, an arc(W) A_m from W_1 to W_2.

THEOREM 11. *If each of B, L, and M is a subset of $Q(A_n)$, B is connected(W), $B \subset L+M$, and L and M are mutually separated(W), then $B \subset L$ or $B \subset M$.*

Proof. If $B \cdot L$ and $B \cdot M$ both exist, they must intersect(W) because B is connected(W). This means that L and M intersect(W), a contradiction.

THEOREM 12. *If each of A and B is a connected(W) subset of $Q(A_n)$ such that $B \subset A$ and $A-B$ is the sum of two mutually separated(W) sets H and K, then $B+H$ and $B+K$ are connected(W).*

Proof. Suppose that $B+H$ is the sum of two mutually separated(W) sets L and M. By Theorem 11 B is a subset of one of L and M, so suppose $B \subset L$. Consider $K+L$ and M. K and M do not intersect(W) and neither do L and M. This means that $A=(K+L)+M$ is not connected(W).

THEOREM 13. *Suppose A is a subset of $Q(A_n)$ and W_1 and W_2 are two elements of W. Then, a necessary and sufficient condition that A be an arc(W) from W_1 to W_2 is that A can be expressed as $\{I_1, I_2, \ldots, I_m\}$ where* (1) *$W_1 \in I_p(W)$ for only $p=1$, $W_2 \in I_p(W)$ for only $p=m$,* (2) *I_p intersects(W) I_q if and only if $|p-q| \leqq 1$ for $1 \leqq p, q \leqq m$.*

Proof (Sufficiency). A clearly contains(W) W_1 and W_2. If A is the sum of two mutually separated(W) sets H and K, let k be an integer such that I_k does not belong to the same one of H and K as $I_1, I_2, \ldots, I_{k-1}$. But since I_k intersects(W) I_{k-1}, this means H intersects(W) K, a contradiction. If some proper connected(W) subset A' of A contains(W) W_1 and W_2 let I_k denote an element of $A-A'$. But $\{I_1, \ldots, I_{k-1}\} \cdot A'$ contains(W) W_1, and $\{I_{k+1}, \ldots, I_m\} \cdot A'$ contains(W) W_2, and these two sets are mutually separated(W).

(Necessity). If A is degenerate the theorem obviously holds. Suppose A is nondegenerate and let $I_1, I_2, I_3, \ldots, I_m$ denote a sequence of elements of A of maximum length which satisfies all the conditions (1) and (2), except that W_2 may be replaced by some $W_3 \in I(W)$.

Neither W_1 nor W_2 is contained(W) by two elements of A. For suppose there exist elements I and I' of A which both contain(W) W_1. But if $A-I$ is connected it contains an arc(W) from W_1 to W_2. If $A-I$ is the sum of two mutually separated(W) sets H and K, then either $I+H$ or $I+K$ contains an arc(W) from W_1 to W_2.

If $W_2 \in I_p(W)$ and $p \leqq m$ and there is an element I of A not in $\{I_1, I_2, \ldots, I_m\}$ then there is an arc(W) from W_1 to W_2 which is a proper subset of A. Therefore suppose that $W_2 \in I(W)$ but $I \notin \{I_1, \ldots, I_m\}$, and let B denote the component(W) of $A - \{I_1, I_2, \ldots, I_m\}$ which contains I. But B intersects(W) the set $\{I_1, \ldots, I_m\}$. If B intersects(W) I_p for $p < m$ then $I_1 + \cdots + I_p + B$ contains an arc(W) from W_1 to W_2, and if B intersects(W) I_m then m is not maximal. Either case gives a contradiction.

THEOREM 14. *If the subset J of $Q(A_n)$ is a s.c.c.(W) and W_1 and W_2 are two elements of W which J contains(W), then J can be written uniquely as the sum of two arcs(W), A_1 and A_2, both from W_1 to W_2, and such that $A_1 \cdot A_2(W) = W_1 + W_2$.*

Proof. By Theorem 10 there is a subset A_1 of J which is an arc(W) from W_1 to W_2. Apply Theorem 13 to express A_1 as $\{I_1, I_2, \ldots, I_m\}$ where conditions (1) and (2) are satisfied. Condition (1) of Theorem 13 reveals that there exist elements H_1 and K of $J - A_1$ so that $W_1 \in H_1(W)$ and $W_2 \in K(W)$. As in the proof of Theorem 13 let $H_1, H_2, \ldots, H_p$ denote a set of maximum length, each element of which belongs to $J - A_1$, and which satisfies conditions (1) and (2) of Theorem 13, except that W_2 may be replaced. There is an element H of J which intersects(W) H_p and is not H_{p-1}. H cannot be one of $I_1, I_2, \ldots, I_{m-1}$ and if $H \in J - A_1 - \{H_1, \ldots, H_p\}$ then p is not maximal. Therefore H must be the same as I_p. This means $H_m = K$ and that $\{H_1, \ldots, H_p\} = A_2$ is an arc(W) from W_1 to W_2. By construction $A_1 \circ A_2(W) = W_1 + W_2$ and if there is an element I of $J - (A_1 + A_2)$ it is clear that $A_1 + A_2$ and $J - (A_1 + A_2)$ cannot intersect(W). Therefore $J = A_1 + A_2$. The uniqueness follows easily by assuming that J contains a third arc(W) A_3 from W_1 to W_2 and expressing A_3 in the form used in Theorem 13.

THEOREM 15. *If J is a s.c.c.(W) and W_1, W_2, and W_3 are three elements of W such that $\{W_1, W_2, W_3\} \subset J(W)$, then W_1 does not separate W_2 from W_3 in J.*

Proof. The theorem follows from Theorem 14, in that there are two arcs(W) A_1 and A_2 from W_2 to W_3 which are subsets of J such that $(A_1 + A_2) \cdot J(W) = W_2 + W_3$.

THEOREM 16. *If J is a s.c.c.(W), and W_1, W_2, W_3, and W_4 are four elements of W which J contains(W) and $W_1 + W_2$ separates(W) W_3 from W_4 in J then $W_3 + W_4$ separates(W) W_1 from W_2 in J.*

Proof. If $W_3 + W_4$ does not separate(W) W_1 from W_2 in J, then there is an arc(W) A_1 from W_1 to W_2 in J which does not contain(W) either W_3 or W_4. Considerations of the proof of Theorem 13 yield that there is an arc(W) A_2 from W_1 to W_2 such that $J = A_1 + A_2$ and $A_1 \cdot A_2(W) = W_1 + W_2$. Theorem 13 now reveals that there is a connected(W) subset A_3 of A_2 which contains(W) W_3 and W_4, but does not contain(W) either W_1 or W_2. This contradicts the hypothesis.

THEOREM 17. *If C is a cycle(W) and W has property Q, then C is a s.c.c.(W) and no two elements of C have a common endpoint.*

Proof. If C is degenerate then C is a s.c.c.(W) by definition. If C is nondegenerate the fact that W has property Q yields that (2a) of the definition of cycle must hold, which clearly implies that C is a s.c.c.(W) and that no two elements of C have a common endpoint.

THEOREM 18. *Suppose $\{a_i, a_{2n}\} \in W$, $1<i<2n$ and no element $\{a_r, a_s\}$ of W satisfies $r<i<s<2n$. Suppose also that C is a cycle(W). Then C contains (a_{i-1}, a_i) and (a_{2n-1}, a_{2n}) if and only if C contains (a_i, a_{i+1}) and (a_{2n}, a_1). Furthermore, if J is a s.c.c.(W) then every $I \in J$ is of the form (a_p, a_{p+1}) for $i \leqq p < p+1 \leqq 2n$, or no $I \in J$ is of this form.*

Proof. If $i=2n-1$, the theorem follows easily from the definitions of cycle(W) and s.c.c.(W). Suppose $i<2n-1$ and that C contains (a_{i-1}, a_i) and (a_{2n-1}, a_{2n}), but neither (a_i, a_{i+1}) nor (a_{2n}, a_1). Suppose also that $n-i$ is a minimum for two elements that satisfy the same hypothesis but for which the theorem is denied. (I.e., if $r<s-1<s$ and $s-r<2n-i$ and $\{a_r, a_s\}$ is an element of W such that no element $\{a_t, a_u\}$ of W satisfies $t<r<u<s$ or $r<t<s<u$, and if C' is a cycle(W), then C' contains (a_{r-1}, a_r) and (a_{s-1}, a_s) if and only if C' contains (a_r, a_{r+1}) and (a_s, a_{s+1}).)

Let $I_1, I_2, \ldots, I_m$ denote a sequence, each element of which is an element of C, such that (1) $I_1=(a_{2n-1}, a_{2n})$, (2) each I_p is of the form (a_r, a_{r+1}), $i \leqq r < r+1 \leqq 2n$, (3) I_r intersects(W) I_s if and only if $|r-s| \leqq 1$, unless I_r and I_s both contain(W) $\{a_i, a_{2n}\}$, (4) m is maximal, (5) if $I_p=(a_r, a_{r+1})$ and a_s is such that $\{a_r, a_s\} \in W$ ($\{a_{r+1}, a_s\} \in W$) and $r<s$ ($s<r+1$), then there is an element $\{a_p, a_q\}$ of W such that $r<p<s$ ($s<p<r+1$) and $q<r$ or $s<q$ ($q<s$ or $r+1<q$), (6) no two elements of $I_1, \ldots, I_m$ have a common endpoint.

Suppose $I_m \neq (a_i, a_{i+1})$, but that $I_m=(a_p, a_{p+1})$. There is an element W_1 of W such that I_m contains(W) W_1 but no other I_q does for $q=1, \ldots, m-1$. Let a_r denote the other element of W_1 distinct from the one which is an endpoint of I_m. The definition of cycle(W) implies that either (a_{r-1}, a_r) or (a_r, a_{r+1}) is an element of C.

Case 1. $\{a_r, a_p\} \in W$. Let a_s be such that $\{a_{r-1}, a_s\} \in W$.

(1A) $(a_r, a_{r+1}) \in C$. The additional hypothesis implies that (a_{r-1}, a_r) and (a_{p-1}, a_p) are also in C and condition (5) above implies that $r<p$.

(1A)i $(a_{r-2}, a_{r-1})=I_q$ for some q. If $s<r-1$ condition (5) fails for I_q. If $r<s<p$, then the definition of cycle(W) is contradicted. So suppose $p<s$. Condition (5) implies that (a_{s-1}, a_s) cannot be an I_t, so let h denote the least integer k such that I_k has endpoints of the form (a_w, a_{w+1}) for $r-1<w<w+1<s$. Since $h>1$ there is an element $W_1=\{a_u, a_v\}$ such that $W_1 \in I_{h-1}(W)$, $W_1 \in I_h(W)$, and a_u is an endpoint of I_k. But $v<r-1$ or $v>s$, a contradiction.

(1A)ii $(a_{r-2}, a_{r-1}) \neq I_q$ for any q. The only reason (a_{r-1}, a_r) can not be I'_{m+1} is for condition (5) to fail, and it would have to fail because of a_{r-1} and a_s. So $s > r-1$. But (1A)i above shows this is impossible. On the other hand, the maximality of m is contradicted.

(1B) $(a_{r-1}, a_r) \in C$ and $(a_r, a_{r+1}) \notin C$. In this case one considers the two cases $r < p$ and $p < r$ and in each of these the two subcases (a) (a_{r-2}, a_{r-1}) is an I_q for some q and (b) (a_{r-2}, a_{r-1}) is not an I_q for any q. The arguments are entirely analogous to those in (1A).

Case 2. $\{a_{p+1}, a_r\} \in W$. This case is handled by arguments analogous to those in Case 1.

Since the s.c.c.(W) J is also a cycle(W), if J contains an element of the form (a_p, a_{p+1}) for $i \leqq p < p+1 \leqq 2n$ and an element not of the form, then the fact that J is connected(W) implies that the conditions of the first part are satisfied. But J cannot contain every element of $Q(A_n)$ which has a_i or a_{2n} for an endpoint, since no two intervals of J have a common endpoint. This is a contradiction.

6. **The main theorem.** This section is devoted to a somewhat lengthy proof of the main result. There are a number of lemmas, in addition to pictures and intuitive arguments to help the interested reader see what the author has in mind.

THEOREM 19. *Suppose n is an integer* $\geqq 1$ *and W is a decomposition of* A_n *into two element sets. Then, a necessary and sufficient condition that there exist an f in* $N(A_n)$ *such that the double point structure of f is determined by W is that* (1) *if* $\{a_i, a_j\} \in W$, *then* $j-i$ *is odd, and* (2) *there is a collection G of cycles(W) such that* (i) *if* $W_1 \in W$ *there is an element C of G which does not contain every element of* $Q(A_n)$ *which contains(W)* W_1, (ii) *each* $I \in Q(A_n)$ *belongs to exactly two cycles(W) of G*, (iii) *if* $a_p \in I \in Q(A_n)$, $a_q \in I' \in Q(A_n)$ *and* $\{a_p, a_q\} \in W$ *then some element of G contains I and I'*, (iv) *there is a decomposition of G into two sets H and K such that if C and C' are distinct elements of H or of K, then C and C' contain no common element of* $Q(A_n)$, *and* (v) *if J is a s.c.c.(W) which is a subset of the element C of G*, W_1, W_2, W_3, *and* W_4 *are four elements of W such that* (1) $W_i \in J(W)$ $i = 1, \dots, 4$, (2) $\{W_1, W_2\}$ *separates(W)* W_3 *from* W_4 *on J, and* (3) *there exist arcs(W)* A_1 *and* A_2 *from* W_1 *to* W_2 *and* W_3 *to* W_4, *respectively, then* A_1 *and* A_2 *intersect(W). Furthermore, in the case of the sufficiency it may be required that if* $C \in G$ *there is a complementary domain U of* Im f *such that* $B \in C$ *if and only if* $f(G(B)) \subset$ Bd U, *and in the case of the necessity, the collection G such that the subset C of* $Q(A_n)$ *is an element of G if and only if there is a complementary domain U of* Im f *such that* $f(G(C)^*) =$ Bd U *is a collection which satisfies condition* (2).

Proof (Sufficiency plus added condition).

Case 1. $n=1$. Given $A_1 = \{a_1, a_2\}$, then $W = \{\{a_1, a_2\}\}$ and the only possible cycles(W) are $\{(a_1, a_2)\}$, $\{(a_2, a_1)\}$, and $\{(a_1, a_2), (a_2, a_1)\}$. But for condition (2ii) of the hypothesis to hold G must contain all cycles(W). There is clearly a function f

mapping $[0, 1)$ onto a figure 8 curve in the plane such that f has the desired properties.

Case 2. Suppose the theorem holds for $1, \ldots, n-1$, where $n>1$.

(2A) W does not have property Q. Suppose there is an element $\{a_i, a_j\}$ of W such that $1<i<j=2n$ and no element $\{a_r, a_s\}$ of W satisfies $r<i<s<j$. If this is not the case, a cyclic permutation will give such a case. The proof given here will also suppose that $i<2n-1$, since the proof for $i=2n-1$ is easier.

The intuitive idea of the proof is to break A_n up into two sets $B_1=\{a_1, \ldots, a_{i-1}\}$ and $B_2=\{a_{i+1}, \ldots, a_{2n-1}\}$ and define W_i $(i=1, 2)$ such that $\{a_r, a_s\} \in W_i$ if and only if $\{a_r, a_s\} \in W$ and $\{a_r, a_s\} \subset B_i$. Then obtain mappings f_1 and f_2 in $N(B_1)$ and $N(B_2)$, respectively, whose double point structures are determined by W_1 and W_2, respectively. Finally, combine the functions f_1 and f_2 in such a way as to determine a function f with the desired properties (see Figure 3).

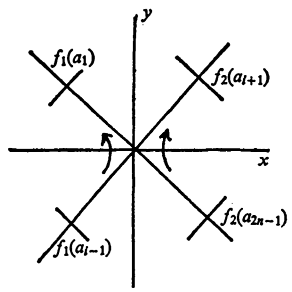


FIGURE 3

Conditions (2i) and (2iii) of the hypothesis and Theorem 18 may be used to obtain a cycle(W) C_1 of G such that C_1 contains (a_{i-1}, a_i) and (a_{2n}, a_1) but neither (a_i, a_{i+1}) nor (a_{2n-1}, a_{2n}). Condition (2iii) of the hypothesis, the definition of cycle(W), and the conditions of this case imply there is an element C_3 of G containing all four of the above, and finally, condition (2ii) of the hypothesis and the definition of cycle(W) show there is an element C_2 of G which contains the latter two elements of $Q(A_n)$, but neither of the first two. Also C_1, C_2, and C_3 are the only elements of G which contain(W) $\{a_i, a_{2n}\}$.

Condition (2iv) of the hypothesis yields that C_1 and C_2 both belong to the same one of H and K, and C_3 belongs to the other. Suppose $C_1, C_2 \in H$ and $C_3 \in K$. Also observe that Theorem 18 implies that C_1 contains no element of the form (a_p, a_{p+1}) for $i \leqq p<p+1 \leqq 2n$, and that C_2 contains no element of the form (a_p, a_{p+1}) for $1 \leqq p<p+1 \leqq i$, nor the element (a_{2n}, a_1).

Suppose B_1, B_2, W_1, and W_2 are defined as above. Let G_1 be defined such that $C \in G_1$ if and only if (1) C is an element of G which contains no element of the form (a_p, a_{p+1}) for $i-1 \leqq p<p+1 \leqq 2n$ nor (a_{2n}, a_1), or (2) C is the element C_1' formed by $C_1+\{(a_{i-1}, a_1)\}-\{(a_{i-1}, a_i), (a_{2n}, a_1)\}$, or (3) C is the element

$$C_3' = C_3-C_3 \circ \{(a_{i-1}, a_i), \ldots, (a_{2n-1}, a_{2n}), (a_{2n}, a_1)\}+\{(a_{i-1}, a_1)\}.$$

G_2, C_2'', and C_3'' are defined analogously. Let $D=\{C : C\in G$ and C contains an element of the form (a_p, a_{p+1}) for $p=1,\ldots, i-1\}$. It will be convenient to think of each element C of D as generating an element C' of G_1. The function $C\to C'$ will actually turn out to be one-to-one. Define $H_1\,(K_1)$ such that $C'\in G_1$ is an element of $H_1\,(K_1)$ provided $C\in H\,(K)$. It follows that $C\in H_1$ if and only if $C\in G_1\cdot H$ or $C=C_1'$, and $C\in K_1$ if and only if $C\in G_1\cdot K$ or $C=C_3'$. H_2 and K_2 are defined analogously.

The next part is to show that the system B_p, W_p, G_p, H_p, K_p $(p=1, 2)$ satisfies the induction hypothesis. The proof will be carried through only for $p=1$.

First define L: $Q(B_1)-\{(a_{i-1}, a_1)\}\to Q(A_n)$ by $L(I)=I$. Note that if

$$C\subset Q(B_1)-\{(a_{i-1}, a_i)\}$$

then C is connected(W_1), an arc(W_1), a s.c.c.(W_1) if and only if $L(C)$ is, respectively, connected(W), an arc(W), a s.c.c.(W).

(0) Each element C' of G_1 is a cycle(W_1).

Case A. $C'\neq C_1'$ nor C_3'. C' is easily seen to be connected(W_1) since it is $L^{-1}(C)$. Also if C' is degenerate C is degenerate, so C' is a s.c.c.(W_1). Assume C' nondegenerate.

Suppose $a_i\in I\in C'$. Use of the transformation L yields another element I' of C' with an endpoint a_j such that $\{a_i, a_j\}\in W_1$. Suppose there is no element $\{a_r, a_s\}$ of W_1 such that $r<i<s<j$ or $i<r<j<s$, but there exist two elements of C' which have a_i or a_j as an endpoint and which have a common endpoint. But by Theorem 18 every element of $Q(A_n)$ which has a_i or a_j for an endpoint must belong to C, so the corresponding situation holds for C'.

Case B. $C'=C_1'$. If C' is not connected(W_1) it is the sum of two sets M' and N' which do not intersect(W_1). But if $(a_{i-1}, a_1)\in M'$ define M and N such that $M=M'+\{(a_{2n}, a_1), (a_{i-1}, a_i)\}-\{(a_{i-1}, a_1)\}$ and $N=N'$. This means $C_1=M+N$ is the sum of two sets which do not intersect(W), a contradiction.

Suppose a_p is an endpoint of the element I' of C'. If $p\neq i-1$ or 1 then $L(I')$ is an element of C, and (assuming C' is nondegenerate) there is another element I_1 of C containing an element a_q so that $\{a_p, a_q\}\in W$. If $I_1\neq(a_{i-1}, a_i)$ or (a_{2n}, a_1) then $L^{-1}(I_1)$ is the desired element. Otherwise, (a_{i-1}, a_1) is the one. If, for example, $p=i-1$, then there exist elements (a_{i-1}, a_i) and I of C such that I contains an endpoint a_q such that $\{a_{i-1}, a_q\}\in W$. Also, since C' is nondegenerate, $q\neq 1$. Therefore, one of $L^{-1}(I)$ and (a_{i-1}, a_i) is I' and the other is the desired element.

Now assume that there are two elements of C' which have a_p or a_q as an endpoint and which have a common endpoint, where $\{a_p, a_q\}\in W_1$. This implies that there are three elements of C which have a_p or a_q as an endpoint and by Theorem 18 all the elements of $Q(A_n)$ which have a_p or a_q as endpoint must belong to C, which, in turn, implies the same thing for $Q(B_1)$ and C'. This completes the proof that C_1' is a cycle(W_1).

Case C. $C'=C_3'$. Analogous to Case B.

(1) If $\{a_p, a_q\} \in W_1$, then $p-q$ is odd. Since $\{a_p, a_q\} \in W$, $p-q$ is odd.

(2i) If $W' \in W_1$ there is an element C' of G_1 which does not contain every element of $Q(B_1)$ which contains(W_1) W'. If $i-1>2$, there is an element C of G such that $C \neq C_1$ nor C_3, and C does not contain every element of $Q(A_n)$ which contains(W) W'. The element C' fulfills the conclusion of the lemma.

If $i-1=2$, then one of C_1' and C_3' has the desired property.

(2ii) Each element of $Q(B_1)$ belongs to exactly two elements of G_1. First it is shown that $C_1' \neq C_3'$. There exist two elements of G, one containing (a_{2n}, a_1) and (a_{r-1}, a_r) and the other containing (a_{2n}, a_1) and (a_r, a_{r+1}), where $\{a_1, a_r\} \in W$. One of these is C_1 and the other is C_3, and neither contains both of (a_{r-1}, a_r) and (a_r, a_{r+1}). But (a_{r-1}, a_r) would then be an element of $Q(B_1)$ which belongs to one of C_1' and C_3' but not both. The preceding facts, together with the definition of G_1 yield that if C_p and C_q are two elements of D, then C_p' and C_q' are distinct elements of G_1.

If $I' \in Q(B_1)$ and $I' \neq (a_{i-1}, a_1)$, then there exist two elements C_p and C_q of G which contain I'. Therefore C_p' and C_q' are distinct elements of G_1 which contain I', and since each $C' \in G_1$ is generated by a $C \in G$, these are the only elements of G_1 which contain I'. If $I'=(a_{i-1}, a_i)$ then I' belongs (by definition) to C_1' and C_3', and these are the only ones I' belongs to.

(2iii) If $a_p \in I' \in Q(B_1)$, $a_q \in I'' \in Q(B_1)$ and $\{a_p, a_q\} \in W_1$, then some element of G_1 contains I' and I''. If neither I' nor I'' is (a_{i-1}, a_1) there is an element C of G which contains I' and I''. The element C' has the desired properties.

If, for example, $I'=(a_r, a_{r+1})$ and $I''=(a_{i-1}, a_1)$, where $a_q=a_1$, then let $I'''=(a_{2n}, a_1)$ and let C denote the element of G which contains I' and I'''. The element C' (which is C_1' or C_3') has the desired property.

(2iv) $\{H_1, K_1\}$ is a decomposition of G_1 such that if C_p' and C_q' are distinct elements of H_1 or of K_1, then C_p' and C_q' do not intersect. Suppose $C_p', C_q' \in H_1$ and $I' \in C_p' \cdot C_q'$. By definition $I' \neq (a_{i-1}, a_1)$, so $I' \in Q(A_n)$. But C_p and C_q must both belong to H and contain the common element I'. This involves a contradiction.

(2v) If J' is a s.c.c.(W_1) which is a subset of the element C' of G_1, W_1', W_2', W_3', and W_4' are four elements of W_1 such that $W_i' \in J'(W_1)$, $i=1, \ldots, 4$, (2) W_1' and W_3' separate(W_1) W_3' from W_4' on J', and (3) there exist arcs(W_1) A_1' and A_2' from W_1' to W_2' and W_3' to W_4', respectively, then A_1' and A_2' intersect(W_1). It may be supposed without loss of generality that $A_1' \cdot J'(W_1) = W_1' + W_2'$ and $A_2' \cdot J(W_1) = W_3' + W_4'$. The trick is to consider $A_1 = L(A_1')$, $A_2 = L(A_2')$ and $J = L(J')$ unless one of them contains (a_{i-1}, a_1), and if, for example, A_1' does, let

$$A_1 = L(A_1' - \{(a_{i-1}, a_1)\}) + \{(a_{2n}, a_1), (a_{i-1}, a_i)\}.$$

An application of (2v) concludes the argument here and also the general argument to show that the system B_1, W_1, G_1, H_1, K_1 satisfies parts (1) and (2) of the theorem.

In conclusion, there exists a map $f_1(f_2) \in N(B_1)(N(B_2))$ such that (1) the double point structure of f_1 (f_2) is determined by W_1 (W_2), (2) if $C \in G_1$ (G_2) there is a complementary domain U_1 (U_2) of $\operatorname{Im} f_1$ ($\operatorname{Im} f_2$) such that

$$f_1(G(C)^*) = \text{Bd } U_1(f_2(G(C)^*) = \text{Bd } U_2),$$

(3) $f_1([a_{i-1}, 1)+[0, a_1])$ is the sum of the intervals $[(-1, -1), (0, 0)]$ and $[(0, 0), (-1, 1)]$, where $f_1(a_{i-1})=(-1, -1)$, $f_1(a_i) = (0, 0)$, $f_1(a_1)=(-1, 1)$, and also $\operatorname{Im} f_1$ lies in the open half plane $x<0$, with the exception of $(0, 0)$, and (4) $f_2([a_{2n-1}, 1)+[0, a_{i+1}])$ is the sum of the intervals $[(1, -1), (0, 0)]$ and $[(0, 0), (1, 1])$, where $f_2(a_{2n-1})=(1, -1)$, $f_2(a_{2n})=(0, 0)$, $f_2(a_{i+1})=(1, 1)$, and also $\operatorname{Im} f_2$ lies in the half plane $x>0$ with the exception of $(0, 0)$. The map f is defined by

$$\begin{aligned} f(x) &= f_1(x), \quad 0 \leqq x \leqq a_i \\ &= tf_1(a_i)+(1-t)f_2(a_{i+1}), \qquad x = a_it+(1-t)a_{i+1}, \quad 0 \leqq t \leqq 1, \\ &= f_2(x), \quad a_{i+1} \leqq x \leqq a_{2n}, \\ &= tf_2(a_{2n})+(1-t)f_1(0), \qquad x = a_{2n}t+(1-t), \quad 0 < t \leqq 1 \end{aligned}$$

and satisfies the desired conditions.

This concludes the argument for Case (2A).

(2B) W has property Q. Suppose i and j are integers such that $1<i<j<2n$, $\{a_i, a_j\} \in W$, and $i<p<q<j$ implies that $\{a_p, a_q\} \notin W$.

The intuitive idea of the proof is to suppose for the moment that the figure can be drawn as desired (see Figure 4), to leave out a "crossing" as indicated, to check that the remaining system does satisfy the induction hypothesis, to draw it and then to show that the "crossing" can be put back in.

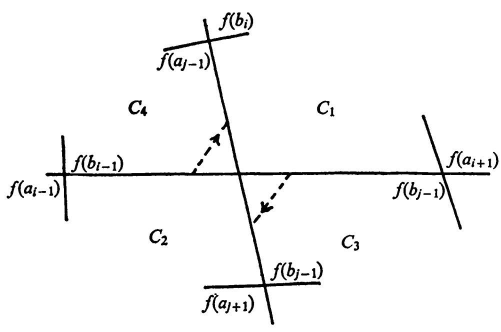


FIGURE 4

Let C_1, C_2, C_3, and C_4 denote the cycles of G which contain as subsets the sets $\{(a_i, a_{i+1}), (a_{j-1}, a_j)\}$, $\{(a_{i-1}, a_i), (a_j, a_{j+1})\}$, $\{(a_i, a_{i+1}), (a_j, a_{j+1})\}$, and $\{(a_{i-1}, a_i), (a_{j-1}, a_j)\}$, respectively. The definition of cycle(W) and the fact that W has property Q imply that C_1, C_2, C_3, and C_4 are four distinct cycles(W), and Theorem 17 yields that each $C \in G$ is a s.c.c.(W). By condition (2iv) of the hypothesis, C_1 and C_2 belong to one of H and K and C_3 and C_4 belong to the other. So suppose $C_1, C_2 \in H$.

Define $A_{n-1}=A_n-\{a_i, a_j\}=\{b_1, b_2, \ldots, b_{2n-2}\}$ where $b_p<b_q$ provided $p<q$. Define $F\colon A_{n-1}\to A_n$ by

$$\begin{aligned} F(b_p) &= a_p, && 1 \leqq p \leqq i-1,\\ &= a_{j+i-1-p}, && i \leqq p \leqq j-2,\\ &= a_{p+2}, && j-1 \leqq p \leqq 2n-2. \end{aligned}$$

Define W' such that $\{b_r, b_s\} \in W'$ if and only if $\{F(b_r), F(b_s)\} \in W$, let

$$A = \{(a_{i+1}, a_{i+2}), \ldots, (a_{j-2}, a_{j-1})\} \subset Q(A_n),$$

and let $B=Q(A_n)-A$.

Now define a collection G' such that $C' \in G'$ if and only if (1) there is an element C of G such that $C\neq C_i$ $(i=1,\ldots,4)$ and $(b_r, b_s) \in C'$ provided (i) $(F(b_s), F(b_r)) \in A\cdot C$ or (ii) $(F(b_r), F(b_s)) \in B\cdot C$, or (2) C' is one of the following three sets:

$$C'_{12} = \{(b_r, b_s) : (F(b_s), F(b_r)) \in (C_1+C_2)\cdot A \text{ or }$$

$$(F(b_r), F(b_s)) \in (C_1+C_2)\cdot B\}+\{(b_{i-1}, b_i), (b_{j-2}, b_{j-1})\}.$$

$$C'_3 = \{(b_r, b_s) : (F(b_s), F(b_r)) \in C_3\cdot A \text{ or } (F(b_r), F(b_s)) \in C_3\cdot B\}+\{(b_{j-2}, b_{j-1})\}.$$

$$C'_4 = \{(b_r, b_s) : (F(b_s), F(b_r)) \in C_4\cdot A \text{ or } (F(b_r), F(b_s)) \in C_4\cdot B\}+\{(b_{i-1}, b_i)\}.$$

Note that $C\to C'$ really defines a mapping $G\to G'$ which, in this case turns out to be one-to-one, with the exception that $C'_1=C'_2=C'_{12}$. Let H' and K' be such that $C' \in H'$ (K') if and only if $C \in H$ (K).

Let $L\colon (Q(A_{n-1})-\{(b_{i-1}, b_i), (b_{j-2}, b_{j-1})\})\to Q(A_n)$ be defined such that $L(b_r, b_s)=(F(b_s), F(b_r))$ for $i\leqq r<s=r+1\leqq j-2$ and $L((b_r, b_s))=(F(b_r), F(b_s))$ for $s\leqq i-1$ or $j-1\leqq r$. Note that if neither a_i nor a_j is an endpoint of I, $I' \in Q(A_n)$ then (1) I and I' have a common endpoint if and only if $L^{-1}(I)$ and $L^{-1}(I')$ do also, and (2) I intersects(W) I' if and only if $L^{-1}(I)$ intersects(W') $L^{-1}(I')$. Conditions (1) and (2) are sufficient to give L properties like a homeomorphism. In fact, it is easy to see

Lemma 0. *If M' is a subset of $Q(A_{n-1})-\{(b_{i-1}, b_i), (b_{j-2}, b_{j-1})\}$ then M' is connected(W'), an arc(W'), a s.c.c.(W') if and only if $L(M')$ is, respectively, connected(W) an arc(W), a s.c.c.(W).*

Lemma 1. *If $\{b_r, b_s\} \in W'$ then $r-s$ is odd. (Assuming the notion of even and odd is extended to all the integers.)*

Proof. Recall that for integers m, n that $m-n$ is odd if and only if $m+n$ and $-m-n$ are odd. Assuming that $r<s$, there are six cases to consider, three of which are done here. If $r, s \in \{1,\ldots,i-1\}$ then $F(b_r)=a_r$ and $F(b_s)=a_s$, and since $\{a_r, a_s\} \in W$, it follows that $r-s$ is odd. Suppose $r \in \{1,\ldots,i-1\}$ and $s \in \{i,\ldots,j-2\}$. $F(b_r)=a_r$ and $F(b_s)=a_{j+i-1-s}$. The fact that $(a_{j+i-1-s}, a_r) \in W$ implies that $j+i-1-s-r$ is odd. But $j-i$ is odd, which means that $j+i$ is odd also, and therefore

that $j+i-1$ is even. It then follows that $-s-r$ and therefore $r-s$ is odd. Now suppose that $r \in \{i, \ldots, j-2\}$, $s \in \{j-1, \ldots, 2n-2\}$. $F(b_r)=j+i-1-r$ and $F(b_s)=a_{s+2}$. Since $\{F(b_r), F(b_s)\} \in W$ it follows that $j+i-1-r-(s+2)$ is odd and this differs by 2 from the integer considered in the previous step.

LEMMA 2. *Each set $C' \in G'$ is a cycle(W').*

Proof. If C' is degenerate there is an element (b_r, b_{r+1}) of $Q(A_{n-1})$ such that $C'=\{(b_r, b_{r+1})\}$ where $r=i-1$ or $j-2$. If not, C' is $L^{-1}(C)$ for some C in G and the fact that W has property Q implies that C, and therefore C', is nondegenerate. Suppose $r=j-2$. Since C'_{12} is nondegenerate $C'=C'_3$. But for C'_3 to be degenerate every element of C_3 must have a_i or a_j for an endpoint. Since C_3 contains (a_i, a_{i+1}) and (a_j, a_{j+1}), the definition of cycle(W) yields that $\{a_{i+1}, a_{j+1}\} \in W$, which implies that $\{b_{j-2}, b_{j-1}\} \in W'$.

Suppose now that C' is nondegenerate.

In the case that $C'=C'_i$ for $i \neq 1, 2, 3$, or 4 use of Lemma 0 easily shows that since C_i is a cycle(W) which is a s.c.c.(W), then $C'=L^{-1}(C_i)$ is a cycle(W'). The case still remains where C' is nondegenerate and is one of C'_{12}, C'_3, and C'_4.

Suppose $C'=C'_{12}$, which is nondegenerate by definition. By Theorem 17 each of C_1 and C_2 is a s.c.c.(W), and by Theorem 14 C_1 is the sum of two arcs(W) from W_1 to W_2, where $a_{j-1} \in W_2$ and $a_{i+1} \in W_1$ (provided $W_1 \neq W_2$). One of these arcs(W) is $\{(a_{j-1}, a_j), (a_i, a_{i+1})\}$, so let the other one be called A_1. Similarly, C_2 is the sum of two arcs(W), A_2 and $\{(a_{i-1}, a_i), (a_j, a_{j+1})\}$, both from W_3 to W_4, where $a_{i-1} \in W_3$ and $a_{j+1} \in W_4$ (provided $W_3 \neq W_4$). The fact that no $\{a_r, a_s\} \in W$ satisfies $i<r<s<j$ implies that $W_1 \neq W_2$. The argument here will also assume that $W_3 \neq W_4$, but an understanding of this case will clearly show how to handle the case $W_3 = W_4$. The reader may observe, as he follows the proof for the case presented, that many of the difficulties that arise later would not appear at all in the latter case, because in that event C'_{12} is a s.c.c.(W'). Condition (1) of the hypothesis implies that $W_1 \neq W_3$ and $W_2 \neq W_4$.

Let $B_i=L^{-1}(A_i)$, $i=1, 2$. Since (1) each of B_1 and B_2 is an arc(W'), (2) each of $\{(b_{i-1}, b_i)\}$ and $\{(b_{j-2}, b_{j-1})\}$ is either an arc(W') or s.c.c.(W') depending on whether or not $W'_2=W'_3$ or $W'_1=W'_4$ and (3) $W_2 \in B_1 \cdot \{(b_{i-1}, b_i)\}(W')$, $W_1 \in B_1 \cdot \{(b_{j-2}, b_{j-1})\}(W')$, $W_4 \in B_2 \cdot \{(b_{j-2}, b_{j-1})\}(W')$, and $W_3 \in B_2 \cdot \{(b_{i-1}, b_i)\}(W')$, it follows that C'_{12} is connected(W'), and also that if b_p is an endpoint of $I'_1 \in C'_{12}$ there is another element I'_2 of C'_{12} which has an endpoint b_q such that $\{b_p, b_q\} \in W'$. Also, since (2iv) of the hypothesis implies that A_1 and A_2 have no common element, then $\{(b_{i-1}, b_i)\}+B_1$ and $\{(b_{j-2}, b_{j-1})\}+B_2$ have no common element. In addition, since C_1 and C_2 are both cycles(W) which are s.c.c.'s(W), it follows that if I'_1 and I'_2 are two elements of C'_{12} which have a common endpoint then (1) I_1 and I_2 belong to different ones of B_1 and B_2, (2) one of I_1 and I_2 is (b_{i-1}, b_i), the other is an element of B_1+B_2, and $W'_2=W'_3$ or (3) one of I_1 and I_2 is (b_{j-2}, b_{j-1}) and, the other is an element of B_1+B_2, and $W'_1=W'_4$.

Now suppose that $b_p \in I_1' \in C_{12}'$, $b_q \in I_2' \in C_{12}'$, $p<q$, and $\{b_p, b_q\} \in W$. The last statement of the preceding paragraph implies that if there are not three elements of C_{12}' which contain(W') $\{b_p, b_q\}$ then condition (2a) of the definition of cycle(W') holds for b_p and I_1'. So now suppose in addition that there are three such elements.

If $(b_p, b_q) \in Q(A_{n-1})$ then the fact that W has property Q implies that $p=i-1$ and $q=i$ or $p=j-2$ and $q=j-1$. Since there are only three elements of $Q(A_{n-1})$ which have b_p or b_q for an endpoint and since $q=p+1$, it clearly follows that case (2b) of the definition of cycle(W') does hold for b_p (or b_q) and any element of C_{12}' which has it for an endpoint.

The remaining case is now where $(b_p, b_q) \notin Q(A_{n-1})$. Let $F(b_p)=a_h$ and $F(b_q)=a_k$. Condition (2iv) of the hypothesis is the main factor in showing that every element of $Q(A_{n-1})$ which has b_p or b_q for an endpoint is an element of C_{12}'. See Figure 5. But now suppose by way of contradiction that there is an element $\{b_r, b_s\}$ of W' such that $r<p<s<q$ or $p<r<q<s$.

By Theorem 14 C_1 (C_2) is the sum of two arcs(W) M_1 and M_2 (N_1 and N_2) both from $W_5=\{a_i, a_j\}$ to $W_6=\{a_h, a_k\}$ such that $M_1 \cdot M_2(W)=W_5+W_6$,

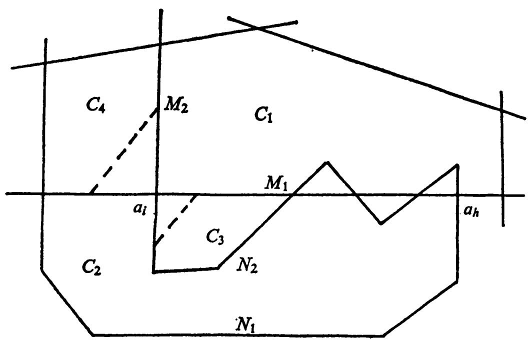


FIGURE 5

($N_1 \cdot N_2(W)=W_5+W_6$). Suppose further that $(a_{i-1}, a_i) \in N_1$, $(a_i, a_{i+1}) \in M_1$, $(a_{j-1}, a_j) \in M_2$, and $(a_j, a_{j+1}) \in N_2$.

Case 1. Neither h nor k is between i and j. The set

$$M = \{(a_{i+1}, a_{i+2}), \ldots, (a_{j-2}, a_{j-1})\}$$

is a connected(W) subset of $Q(A_n)$ which does not contain(W) W_5 nor W_6. There is an arc(W) N which is a subset of M and which is minimal with respect to the property of being connected(W) and intersecting(W) both M_1 and M_2 or both N_1 and N_2. Suppose $N=\{I_1, I_2, \ldots, I_m\}$ as in Theorem 13, where I_1 intersects(W) M_1 and I_m intersects(W) M_2. If N intersects(W) N_1 then it cannot intersect(W) N_2, because if it does, then either I_1 intersects(W) N_2 or I_m does and no other I_p does so. But in either case there is an element a_t of A_n and five different elements (two from M_1 or M_2, two from N_2, and I_m or I_1) which contain an element

$a \equiv a_t(W)$. This is a contradiction. In the other event, however, condition (2v) of the hypothesis is contradicted by using the arcs(W) N and N_2 and the s.c.c.(W) C_1.

Case 2. $i<h<j<k$. The arguments in the subcases here are devoted to obtaining a set such as the M of Case 1. The remainder of the argument is exactly the same.

(2A) If $(a_{h-1}, a_h) \in M_2$ let $M = \{(a_{i+1}, a_{i+2}), \ldots, (a_{h-2}, a_{h-1})\}$. Similarly if $(a_{h+1}, a_h) \in M_1$, $(a_{k-1}, a_k) \in N_1$ or $(a_k, a_{k+1}) \in N_2$.

(2B) $(a_{h-1}, a_h) \in N_1$, $(a_h, a_{h+1}) \in M_2$, $(a_{k-1}, a_k) \in N_2$ and $(a_k, a_{k+1}) \in M_1$. Using the properties of $\{b_r, b_s\}$ there must exist $\{a_t, a_u\} \in W$ such that (1) $h+1 \leqq t \leqq j-1$ and $j+1 \leqq u \leqq k-1$ or (2) $i+1 \leqq t \leqq h-1$ and $k+1 \leqq u$ or $u \leqq i-1$. Also since $k-h$ is odd and j is between h and k but i is not, there is an element $\{a_v, a_w\}$ of W such that (a) $h+1 \leqq v \leqq j-1$ and $w \geqq k+1$ or $w \leqq i-1$, or (b) $j+1 \leqq v \leqq k-1$ and $w \geqq k+1$ or $w \leqq i-1$. (Remember that no element $\{a_x, a_y\}$ of W satisfies $i<x<y<j$.) Let

$$\begin{aligned}
U_1 &= \{(a_{i+1}, a_{i+2}), \ldots, (a_{h-2}, a_{h-1})\},\\
U_2 &= \{(a_{h+1}, a_{h+2}), \ldots, (a_{j-2}, a_{j-1})\},\\
U_3 &= \{(a_{j+1}, a_{j+2}), \ldots, (a_{k-2}, a_{k-1})\},\\
U_4 &= \{(a_1, a_2), \ldots, (a_{i-2}, a_{i-1}), (a_{k+1}, a_{k+2}), \ldots, (a_{2n}, a_1)\}.
\end{aligned}$$

In cases (1a), (1b), (2a), (2b) use $M = U_2+U_4$, $U_2+U_3+U_4$, U_2+U_4, $U_1+U_3+U_4$, respectively. The type of argument used in this case is sufficient to settle the remaining cases, which are listed.

(2C) $(a_{h-1}, a_h) \in M_1$, $(a_h, a_{h+1}) \in N_2$, $(a_{k-1}, a_k) \in M_2$, and $(a_k, a_{k+1}) \in N_1$.

(2D) $(a_{h-1}, a_h) \in N_2$, $(a_h, a_{h+1}) \in M_2$, $(a_{k-1}, a_k) \in M_1$, and $(a_k, a_{k+1}) \in N_1$.

(2E) $(a_{h-1}, a_h) \in M_1$, $(a_h, a_{h+1}) \in N_1$, $(a_{k-1}, a_k) \in N_2$, and $(a_k, a_{k+1}) \in M_2$.

This completes the proof that $C' = C'_{12}$ is a cycle(W').

The case where $C' = C'_3$ or C'_4 is easy because in this case C' is a s.c.c.(W').

Lemma 3. *If $W'_1 \in W'$ there is an element C' of G' which does not contain every element of $Q(A_{n-1})$ which contains(W') W'_1.*

Proof. Suppose $W'_1 = \{b_p, b_q\}$ where $a_r = F(b_p)$ and $a_s = F(b_q)$. Since W has property Q and there are four distinct cycles(W) of G which contain(W) $W_1 = \{a_r, a_s\}$ there is an element C of G such that (1) C contains(W) W_1, (2) no two elements of C have a common endpoint and (3) C is neither C_1 nor C_2. It is easily verified that C induces an element C' of G' with the required property.

Lemma 4. *Each element I' of $Q(A_{n-1})$ belongs to exactly two cycles(W') of G.*

Proof. The validity of this lemma is evident in the case that $I' = (b_{i-1}, b_i)$ or (b_{j-2}, b_{j-1}) because of the definitions of C'_{12}, C'_3 and C'_4. So suppose I' is neither of these two. $L(I')$ is an element of exactly two elements C_p and C_q of G, so I' belongs to at most two elements of G'. But $C'_p \neq C'_q$ unless one of p and q is one and the other is two. This cannot happen because C_1 and C_2 have no common element. Therefore I' belongs to exactly two elements of G'.

LEMMA 5. *If* $b_p \in I'_1 \in Q(A_{n-1})$, $b_q \in I'_2 \in Q(A_{n-1})$ *and* $\{b_p, b_q\} \in W'$, *then some element* C' *of* G' *contains* I'_1 *and* I'_2.

Proof. If neither of I'_1 and I'_2 is one of (b_{i-1}, b_i) and (b_{j-2}, b_{j-1}), then let C denote an element of G containing $L(I'_1)$ and $L(I'_2)$. The element C' of G' has the desired property.

If, for example, $b_p = b_{i-1}$, $I'_1 = (b_{i-1}, b_i)$, $b_q = b_i$, and $I'_2 = (b_{i-1}, b_i)$, then $C' = \{(b_{i-1}, b_i)\}$. If $b_p = b_{i-1}$, $I'_1 = (b_{i-1}, b_i)$ and $I'_2 \neq I'_1$, then let C denote the element of G which contains (a_{i-1}, a_i) and $L(I'_2)$. The element C' has the desired property.

LEMMA 6. $G' = H' + K'$ *is a decomposition of* G' *into two sets such that if* C'_p *and* C'_q *are two elements of* H' (K'), *then* C'_p *and* C'_q *contain no common element.*

Proof. This is immediate from the definitions of G', H', and K' and the fact that similar properties are held by H and K with relation to the collection G.

LEMMA 7. *If* J' *is a s.c.c.*(W') *which is a subset of an element* C' *of* G', M'_1, M'_2, M'_3, *and* M'_4 *are four elements of* W' *such that* $\{M'_1, M'_2\}$ *separates*(W') M'_3 *from* M'_4 *on* J', *and* A'_1 *and* A'_2 *are arcs*(W') *from* M'_1 *to* M'_2 *and* M'_3 *to* M'_4, *respectively, then* A'_1 *and* A'_2 *intersect*(W').

Proof. Suppose A'_1 and A'_2 do not intersect(W'). Also, with the aid of Theorem 13 it can be assumed that there do not exist subarcs B'_1 and B'_2 of A'_1 and A'_2, respectively, at least one of which is a proper subset of the A'_i that contains it, and such that B'_1 and B'_2 also satisfy the hypothesis of the theorem with respect to J', and some new M'_i's. Also, let B_p $(p=1, 2)$ denote the maximal subarc(W) which is a subset of C_p and does not contain(W) the element $W_5 = \{a_i, a_j\}$ of W. Similarly, define B_3 (B_4), but only in case $W_1 \neq W_4$ $(W_2 \neq W_3)$. For each M'_i let M_i denote the corresponding element of W, and for each W_i, $1 \leqq i \leqq 5$, (as defined in Lemma 2) let W'_i denote the corresponding element of W'.

Case 1. $C' \neq C'_{12}$, C'_3, or C'_4.

(1A) Neither A'_1 nor A'_2 contains one of (b_{i-1}, b_i) and (b_{j-2}, b_{j-1}). Define $J = L(J')$, $A_1 = L(A'_1)$ and $A_2 = L(A'_2)$. This situation yields a contradiction because of (2v) of the hypothesis. In each of the following cases there is defined a J, A_1, and A_2 without further mention of there being a contradiction involved.

(1B) Only one of A'_1 and A'_2 contains one (and maybe both) of (b_{i-1}, b_i) and (b_{j-2}, b_{j-1}). Suppose A'_1 contains one or both, but A'_2 contains neither. Then, let $J = L(J')$, $A_2 = L(A'_2)$ and note that

$$L(A'_1 - \{(b_{i-1}, b_i), (b_{j-2}, b_{j-1})\}) + \{(a_{i-1}, a_i), (a_{j-1}, a_j)\}$$

(provided $(b_{i-1}, b_i) \in A'_1$) $+ \{(a_i, a_{i+1}), (a_j, a_{j+1})\}$ (provided $(b_{j-2}, b_{j-1}) \in A'_1$) contains an arc$(W)$ A_1 from M_1 to M_2 which does not intersect(W) A_2.

(1C) Each of A'_1 and A'_2 contains one of (b_{i-1}, b_i) and (b_{j-2}, b_{j-1}). Suppose $(b_{i-1}, b_i) \in A'_1$ and $(b_{j-2}, b_{j-1}) \in A'_2$.

If A_2' does not intersect(W') $L^{-1}(B_4)$ note that $L(A_1'-\{(b_{i-1}, b_i)\})+B_4$ contains an arc(W) A_1 from M_1 to M_2. Let

$$A_2 = L(A_2'-\{(b_{j-2}, b_{j-1})\})+\{(a_i, a_{i+1}), (a_j, a_{j+1})\} \text{ and } J = L(J').$$

This type of argument also handles the case where A_1' does not intersect(W') $L^{-1}(B_3)$.

Suppose A_2' intersects(W') $L^{-1}(B_4)$ and A_1' intersects(W') $L^{-1}(B_3)$. There is a subarc(W') A_3' of A_2' which intersects(W') B_4 and contains(W') either W_1' or W_4' and which contains no proper subarc(W') with the same property. A_3' does not contain (b_{j-2}, b_{j-1}) nor does it contain(W') both M_3' and M_4', so suppose, for example, that it does not contain(W') M_3' and does contain(W') W_4'. J' is the sum of two arcs(W') from M_1' to M_2' so let A_4' denote the one which contains(W') M_3'. Let $A_1=L(A_4')+L(A_1'-\{(b_{i-1}, b_i)\})$, $A_2=L(A_3')+\{(a_j, a_{j+1})\}$, and let $J=C_4$. See Figure 6.

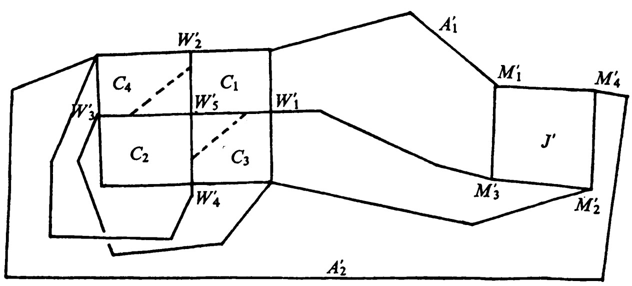


FIGURE 6

Case 2. $C'=C_{12}'$.

(2A) If the intersection(W') of $L^{-1}(B_1)$ and $A_1'+A_2'$ is $\{M_1', M_2', M_3', M_4'\}$ then let $A_1=L(A_1')$, $A_2=L(A_2')$, and $J=C_1$. Analogously for $L^{-1}(B_2)$.

(2B) Suppose the intersection(W') of $L^{-1}(B_1)$ and $A_1'+A_2'$ is $\{M_1', M_2', M_3'\}$ and the intersection(W') of $L^{-1}(B_2)$ and $A_1'+A_2'$ is $\{M_4'\}$. Let A_3 denote an arc(W) from M_4 to W_5 which is a subset of C_2. Then, $A_1=L(A_1')$, $A_2=A_3+L(A_2')$, and $J=C_1$. The other 1–3 possibilities are handled analogously.

(2C) Suppose that the intersection(W') of $L^{-1}(B_1)$ and $A_1'+A_2'$ is $\{M_1', M_3'\}$ and the intersection(W') of $L^{-1}(B_2)$ and $A_1'+A_2'$ is $\{M_2', M_4'\}$. Suppose that $L^{-1}(B_1)$ contains an arc(W') from W_1' to M_3' which contains(W') M_1'. It follows by an easy argument that the subarc(W') of B_2' from W_4' to M_2' contains(W') M_4'. If not then $\{M_1', M_2'\}$ would not separate M_3' from M_4' on J'.

(2C)a Suppose $M_2'\neq W_3'$ and $L(A_1')$ does not intersect(W') C_4. Let $A_1=L(A_1')+\{(a_i, a_{i+1})\}+$the subarc$(W)$ of B_1 from W_1 to M_1 (if $W_1\neq M_1$), $A_2=L(A_2')+B_4$ (if $W_2\neq W_3$)+the subarc(W) of B_1 from M_3 to W_2 (if $M_3\neq W_2$), and let $J=C_2$. A similar argument handles the case where $M_4'\neq W_4'$ and $L(A_2')$ does not intersect(W') C_3.

(2C)b A_1' intersection(W') $L^{-1}(B_4)$ contains some element distinct from W_3'.

Let A'_3 denote a subarc(W') of A'_1 which intersects(W') both $L^{-1}(B_1)$ and $L^{-1}(B_2)$, but which contains no proper subarc(W') with the same property. Let $A_1=L(A'_3)+\{(a_i, a_{i+1})\}+$the subarc of B_1 from W_1 to M_1 (if $W_1\neq M_1$), $A_2=L(A'_2)+$the subarc(W) of B_1 from M_3 to W_2 (if $M_3\neq W_2$)+the subarc(W) of B_2 from W_3 to M_4, and let $J=C_4$. Similarly for the case where A'_2 intersection(W') $L^{-1}(B_3)$ contains some element distinct from W'_4.

An inspection of the two previous cases implies that the only one left to consider is

(2C)c $A'_1+A'_2$ intersection(W') $C'_{12}+C'_3+C'_4=M'_1+M'_2+M'_3+M'_4$ and $W'_1=M'_1$, $W'_3=M'_2$, $W'_2=M'_3$, $W'_4=M'_4$. In this case $(a_{i-2}, a_{i-1})\in L(A_1)$, $(a_{i+1}, a_{i+2})\in L(A_1)$, $(a_{j-2}, a_{j-1})\in L(A_2)$, and $(a_{j+1}, a_{j+2})\in L(A_2)$. Let a_p, a_q, a_r, a_s denote the four points which pair with a_{i-1}, a_{i+1}, a_{j-1}, and a_{j+1} to form elements of W. Considerations of the definition of cycle(W) and the properties of C_1, C_2, C_3, C_4 imply that if t is one of p, q, r, s then (a_{t-1}, a_t) belongs to one of C_1, C_2, C_3, and C_4 and (a_t, a_{t+1}) belongs to another. There is some one of p, q, r, and s and some one of $i-1$, $i+1$, $j-1$, and $j+1$ such that none of the remaining six is between the given two. Take the case, for example, that $i+1<p$ but if t is any one of the remaining six then $t<i+1$ or $p<t$. Then $\{(a_{i+2}, a_{i+3}),\ldots,(a_{t-1}, a_t)\}$ is a connected(W) set A which intersects(W) $L(A_1)$ and one of C_1, C_2, C_3, and C_4, but does not contain(W) any one of $W_1,\ldots, W_5$. A contains an arc(W) A_3 which intersects(W) one of $L(A_1)$ and $L(A_2)$ and one of C_1, C_2, C_3+C_4 but which contains no proper subarcs(W) having the same property.

(2C)ci A_3 intersects(W) both C_1 and $L(A'_1)$. By considering a subarc(W) A_4 of $L(A'_1)+A_3$ which intersects(W) both B_1 and B_2, but contains(W) neither W_1 nor W_2 and applying the methods of (2C)a this case is disposed of and all other possibilities here are handled in the same way (see Figure 7).

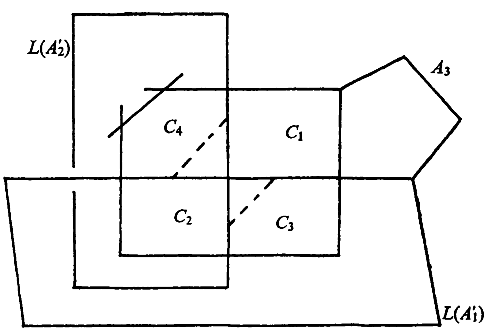


FIGURE 7

Case 3. $C'=C'_3$ or C'_4. Suppose $C'=C'_4$.

If neither A'_1 nor A'_2 contains (b_{j-2}, b_{j-1}) let $A_1=L(A'_1)$, $A_2=L(A'_2)$ and $J=C_4$.

If A'_1, for example, contains (b_{j-2}, b_{j-1}), let $A_2=L(A'_2)$, $J=C_4$, and note that $L(A'_1-\{(b_{j-2}, b_{j-1})\})+\{(a_j, a_{j+1}), (a_i, a_{i+1})\}$ contains an arc(W) A_1 with the desired properties so as to obtain a contradiction. This completes the argument for *Case* 3 and also the general argument to show that the induction hypothesis is satisfied for A_{n-1}, W', G', H', K'.

There is an element g of $N(A_{n-1})$ such that (1) W' determines the double point structure of g, and (2) if $C' \in G'$ there is a complementary domain U of $\operatorname{Im} g$ whose boundary is $g(G(C')^*)$. Also, $\operatorname{Im} g$ is the sum of a finite collection M of straight line intervals such that no point of $g(A_{n-1})$ is an endpoint of an interval of M. The sets $E_1 = g([b_{i-1}, b_i])$ and $E_2 = g([b_{j-2}, b_{j-1}])$ are either arcs or simple closed curves and do not intersect. $F_1 = g(G(A_1)^*)$ is an arc from $g(b_i)$ to $g(b_{j-2})$ such that $F_1 \cdot E_1 = g(b_i)$ $(= g(b_{i-1})$ also, if $W_2 = W_3)$ and $F_1 \cdot E_2 = g(b_{j-2})$. Similarly, $F_2 = g(G(A_2)^*)$ is an arc from $g(b_{j-1})$ to $g(b_{i-1})$ such that $F_2 \cdot E_1 = g(b_{i-1})$ and $F_2 \cdot E_2 = g(b_{j-1})$. Let U denote the complementary domain of $\operatorname{Im} g$ whose boundary is $g(G(C'_{12})^*)$, and let F denote a polygonal arc from $\operatorname{seg} E_1$ to $\operatorname{seg} E_2$ which lies except for its endpoints $Q_1 = g(a_i)$ and $Q_2 = g(a_j)$ in U. In addition let r_1, r_2, s_1, and s_2 denote numbers such that $a_{i-1} < r_1 < a_i < s_1 < a_{i+1}$ and $a_{j-1} < r_2 < a_j < s_2 < a_{j+1}$ and define $R_p = g(r_p)$, $S_p = g(s_p)$, $p = 1, 2$.

In case F_1 and F_2 intersect, let $\{P_1, \ldots, P_m\}$ denote the set $F_1 \cdot F_2$. There exists $e > 0$ such that if $C_1, \ldots, C_m$ denote the circles with center at $P_1, \ldots, P_m$ and radius e, respectively, then (1) each $C_p + \operatorname{Int} C_p$ contains only one double point of $\operatorname{Im} g$ and no endpoint of an interval in M nor a point of $F + \operatorname{arc} R_1Q_1S_1 + \operatorname{arc} R_2, Q_2S_2$, and (2) if $p \neq q$, then $C_p + \operatorname{Int} C_p$ and $C_q + \operatorname{Int} C_q$ do not intersect. From the proof of Theorem 4 and the fact that no two $\operatorname{arc}(W)$ of C_1 or of C_2 have a common endpoint it follows that each C_p, $p = 1, \ldots, m$, contains two mutually exclusive arcs D_p^1 and D_p^2 such that each D_p^1 has its endpoints on $H_1 = F_1 + \operatorname{arc} S_1 g(b_i) + \operatorname{arc} g(b_{j-2})R_2$, and each D_p^2 has its endpoints on $H_2 = F_2 + \operatorname{arc} S_2 g(b_{j-1}) + \operatorname{arc} g(b_{i-1})R_1$. Now let $F_i' = H_i + \sum_{p=1}^m D_p^i - \sum_{p=1}^m (H_i \cdot \operatorname{Int} C_p)$, $i = 1, 2$, and let $E_i' = \operatorname{arc} R_iQ_iS_i$, $i = 1, 2$. F_1' and F_2' are nonintersecting arcs.

The simple closed curve $J = E_1' + E_2' + F_1' + F_2'$ has the property that $E_1' \cdot F_1' = S_1$, $E_1' \cdot F_2' = R_1$, $E_2' \cdot F_1' = R_2$, $E_2' \cdot F_2' = S_2$, so S_1 and S_2 separate the points R_1 and R_2 on J. With the aid of Theorem 10, page 166 of [4] there can be constructed polygonal arcs R_1XR_2 and S_1XS_2 which (1) lie except for their endpoints in $D - \operatorname{Im} g$, where D is the complementary domain of J which contains $\operatorname{seg} F$, (2) have only the point X in common, and (3) both contain an open straight line interval containing X.

Define onto homeomorphisms h_1, h_2, and h_3 such that $h_1\colon [r_1, s_1] \to R_1XR_2$, $h_2\colon [s_1, r_2] \to [s_1, r_2]$, $h_3\colon [r_2, s_2] \to S_1XS_2$, and such that (1) $h_1(r_1) = R_1$, $h_1(a_i) = X$, and $h_1(s_1) = R_2$, (2) $h_2(s_1) = r_2$, $h_2(r_2) = s_1$, and $h_2(a_p) = a_{i+j-p}$, $p = i+1, \ldots, j-1$, and (3) $h_3(r_2) = S_1$, $h_3(a_j) = X$, and $h_3(s_2) = S_2$.

The required function f is now defined by

$$\begin{aligned} f(t) &= g(t), & t &\in [0, r_1] + [s_2, 1), \\ &= h_1(t), & t &\in [r_1, s_1], \\ &= g(h_2(t)), & t &\in [s_1, r_2], \\ &= h_3(t), & t &\in [r_2, s_2]. \end{aligned}$$

It is easy to verify that f is an element of $N(A_n)$ whose double point structure is determined by W. It also follows with little trouble that if $C \in G$ and C is not one of $C_1, \ldots, C_4$, then there is a complementary domain V of $\operatorname{Im} f$ such that Bd $V = f(G(C)^*)$.

Suppose, however, that $C = C_1$. By Theorem 5, page 143 of [4] seg R_1XR_2 separates D into two components D_1 and D_2 such that Bd D_1 is the simple closed curve $R_1XR_2 + E_1' + F_1'$ and Bd D_2 is the simple closed curve $R_1XR_2 + E_2' + F_2'$. An application of the same Theorem 5 to Bd D_1, the subarc S_1X of S_1XS_2, and D_1 yields domain T_1 whose boundary is $S_1X + R_2X + F_1'$. But $T_1 \subset U$, and as $e \to 0$ (e was defined above) Bd $T_1 \to f(G(C_1)^*)$, which implies there is a complementary domain V of $\operatorname{Im} f$ such that Bd $V = f(G(C_1)^*)$. Similar considerations will handle the cases $C = C_2$, C_3, or C_4. This concludes the proof for sufficiency plus added condition.

(Necessity plus added condition). By Theorem 7, for each such U there does exist such a C, which is the set of all $I \in Q(A_n)$ such that $f(G(I) - A_n \cdot G(I))$ contains a point of Bd U. Also, given such a C the U that determines C is unique.

LEMMA 8. *If $C \in G$, then C is a cycle(W).*

Proof. Let U denote the complementary domain of $\operatorname{Im} f$ which determines C.

By Theorem 23, page 176 of [4] Bd U is a compact continuum, so it follows by an easy argument that C is connected(W).

If C is a degenerate set $\{(a_r, a_s)\}$ then Theorem 4 implies that $\operatorname{Im} f$ must have at least three complementary domains. Therefore, $f(G(C))$ separates the plane, which yields that $f(a_r) = f(a_s)$, for, if not, then $f(G(C))$ is an arc, and (see Theorem 21, page 175 of [4]) does not separate the plane.

Suppose now that C is nondegenerate and $a_i \in I \in C$. Theorems 2 and 5 together imply that there is an element I' of $Q(A_n)$ with an endpoint a_j such that $\{a_i, a_j\} \in W$ and $I \neq I'$, and, furthermore, that if $n > 1$ then I' may be selected so as to have no endpoint common with I. (Suppose $i < j$.)

For $n = 1$, since C is nondegenerate both (a_1, a_2) and (a_2, a_1) belong to C, so condition (2b) of the definition of cycle(W) is satisfied for a_i and I. So suppose also that $n > 1$.

For the cases where $j = i + 1$ or $i = 1$ and $j = 2n$ if condition (2a) of the definition of cycle(W) does not hold for a_i and I (remember that for $n > 1$ I' was selected so as to have no endpoint common with I) then the fact that there is a third element I'' of C which contains(W) $\{a_i, a_j\}$ simply means that condition (2b) does hold.

So now suppose that $1 < i < i+1 < j < 2n$ and that there is an element I'' of C distinct from I and I' and which contains(W) $\{a_i, a_j\}$. Let U_1, U_2, U_3, and U_4 denote the complementary domains of $\operatorname{Im} f$ (see Theorem 5) whose boundaries contain, respectively, the sets $f([a_i, a_{i+1}] + [a_{j-1}, a_j])$, $f([a_{i-1}, a_i] + [a_j, a_{j+1}])$, $f([a_i, a_{i+1}] + [a_j, a_{j+1}])$, $f([a_{i-1}, a_i] + [a_{j-1}, a_j])$. By Theorem 2 either $U_1 = U_2 = U$ or $U_3 = U_4 = U$, so in either event C must contain every element of $Q(A_n)$ which has a_i or a_j for an endpoint. However, U_1 cannot be U_2 since there would exist a simple closed curve J containing a straight line interval $P\,f(a_i)\,Q$ and which lies except for $f(a_i)$ in U. It would follow by the Jordan Curve Theorem and some analytic geometry that $\operatorname{Im} f - f(a_i)$ is the sum of two mutually separated point sets H and K such that (1) $f(a_{j-1}) \in H$, (2) $f(a_{i+1}) \in K$ and (3) H and K are subsets of different complementary domains of J. This is impossible since $f([a_{i+1}, a_{j-1}])$ is a connected subset of $H+K$. Therefore, $U_4 = U_3 = U$.

Suppose, by way of contradiction, there is an element $\{a_r, a_s\}$ of W such that $r<i<s<j$ or $i<r<j<s$. But as above there is a simple closed curve J which separates $f((a_i, a_j))$ from $f([0, a_1)+(a_j, 1))$. These two sets would intersect since $f(a_r)=f(a_s)$. This yields a contradiction, and thus completes the proof of Lemma 8.

Proceeding now with the necessity. Part 1 follows from Theorem 8. (2i) follows from Theorems 2, 4, 5. (2ii) follows from Theorem 2. (2iii) follows from Theorem 5. (2v) follows from the fact that if J is a s.c.c.(W), then $f(G(J)^*)$ is a simple closed curve, and Theorem 11, page 147 of [4]. There remains only (2iv) which can be proved with the aid of

LEMMA 9. *If $f \in N(A_n)$ and G' denotes the collection of all complementary domains of* $\operatorname{Im} f$, *then there exists a decomposition of G' into two sets H' and K' such that if U_1 and U_2 are two elements of H' or of K' then* Bd U_1 *and* Bd U_2 *contain no common $f(G(I))$ for $I \in Q(A_n)$.*

If $n=1$ then $J_1=f([a_1, a_2])$ and $J_2=f([a_2, 1)+[0, a_1])$ are both simple closed curves such that $J_1-f(a_1)$ is a subset of one of the complementary domains of J_2, and $J_2-f(a_1)$ is a subset of one of the complementary domains of J_1. So suppose that $J_1-f(a_1) \subset \operatorname{Int} J_2$. There are 3 complementary domains of image f, $\operatorname{Int} J_1$, $\operatorname{Int} J_2 - \operatorname{Cl} \operatorname{Int} J_1$, $\operatorname{Ext} J_1$ whose boundaries are, respectively, $f([a_1, a_2])$, $\operatorname{Im} f$ and $f([a_2, 1)+[0, a_1])$. Let $H=\{\operatorname{Int} J_1, \operatorname{Ext} J_2\}$ and $K=\{\operatorname{Int} J_2 - \operatorname{Cl} \operatorname{Int} J_1\}$.

Now suppose true for $1, \dots, n-1$. The idea for this case is to leave out a crossing as in the proof of the sufficiency. As in the proof of Theorem 4 consider a circle of small radius e with center at $f(a_i)=f(a_j)$, $i \neq j$. Consider a new function g defined as before such that $\operatorname{Im} g$ is $\operatorname{Im} f +$ interval $AB +$ interval $CD -$ seg $AC -$ seg BD. See Figure 8.

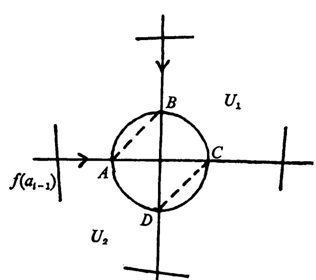


FIGURE 8

Since g is an element of $N(A_{n-1})$ there is a decomposition of the set G'' of all complementary domains of Im g into two sets H'' and K'' with the required properties. Suppose $f(a_i)$ is an element of the complementary domain U of Im g and that U is an element of H''. Then define H' and K' such that $U_1, U_2 \in H'$; $U_3, U_4 \in K'$ and if U is not one of $U_1, \ldots, U_4$ then U is an element of H' (K') provided it is an element of H'' (K'').

This concludes the proof of Theorem 19.

In a paper which is forthcoming the author proves a theorem about prime mappings which implies that in the case of Theorem 19, if W is also stipulated to have property P, then there is at most one collection G satisfying Conditions (1) and (2).

References

1. R. H. Crowell and R. H. Fox, *Introduction to knot theory*, Ginn, Boston, Mass., 1963.

2. C. F. Gauss, *Werke* (8), Teubner, Leipzig, 1900, 272, 282–286.

3. M. L. Marx, *Normal curves arising from light open mappings of the annulus*, Trans. Amer. Math. Soc. **120** (1967), 46–56.

4. R. L. Moore, *Foundations of point set theory*, Colloq. Publ., Vol. 13, Amer. Math. Soc., Providence, R. I., 1932.

5. J. V. Sz. Nagy, *Über ein topologisches Problem von Gauss*, Math. Z. **26** (1927), 579–592.

6. D. E. Penney, *An algorithm for establishing isomorphism between tame prime knots in E^3*, Doctoral Dissertation, Tulane Univ., New Orleans, La., 1965.

7. C. J. Titus, *A theory of normal curves and some applications*, Pacific J. Math. **10** (1960), 1083–1096.

8. ———, *The combinatorial topology of analytic functions on the boundary of a disk*, Acta Math. **106** (1961), 45–64.

Tulane University,
New Orleans, Louisiana