 FUNCTIONS STARLIKE OF ORDER $\alpha$

BY

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1. Introduction. Let $f(z)$ be regular in the unit disk, $\mathbb{U}$, with an expansion of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$  

We will say that $f(z)$ is starlike of order $\alpha$ in $\mathbb{U}$ if:

$$\text{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} > \alpha \quad \text{for all } z \in \mathbb{U};$$

we will denote by $S(\alpha)$ the class of all such functions for fixed $\alpha$. We consider all real $\alpha$: $-\infty < \alpha \leq 1$. The class $S(0)$ will be recognized as the class of functions starlike with respect to the origin.

For the natural number $p$ we define the related classes.

$$S_p(\alpha) = \{ f_p(z) = f(z^p) : f(z) \in S(\alpha) \}.$$  

Extremal problems for the coefficients in the power series expansions of functions in $S_p(0)$ and powers of such functions were recently investigated by J. T. Poole [4]. In this paper we will show that these coefficient problems are equivalent to extremal problems for the coefficients in the power series expansion of functions in $S(\alpha)$, $(\alpha < 0)$. This alternative approach leads to a substantial generalization.

We will also prove several theorems of the type commonly called "distortion theorems" for functions in the classes $S(\alpha)$. One such result is the:

**Theorem.** Let $f(z) \in S(\alpha)$, $-\infty < \alpha \leq 1$. Then for each natural number $n$ there is a point $z = e^{i\theta}$ on the unit circle such that:

$$\sum_{k=1}^{n} |f(e^{i\theta} \cdot \eta_k)| \geq n,$$

where $\{\eta_k\}$ $k = 1, 2, \ldots, n$ denote the $n$th roots of unity.

Let $f(z)$ be regular in the exterior of the unit circle $\mathbb{V}$, except for a simple pole at infinity, with an expansion of the form:

$$f(z) = z + \sum_{n=0}^{\infty} a_n z^{-n}.$$  

We will say that $f(z)$ is starlike of order $\alpha$ in $\mathbb{V}$ if:

$$\text{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} > \alpha \quad \text{for all } z \in \mathbb{V};$$

Received by the editors December 16, 1966.
we will denote by $\Sigma(\alpha)$ the class of all such functions for fixed $\alpha$. The class $\Sigma(0)$ is the class of functions whose compact complement is starlike with respect to the origin.

For the natural number $p$ we define the related classes:

$$\Sigma_p(\alpha) = \{ f_p(z) = f(z^p)^{1/p} : f(z) \in \Sigma(\alpha) \}.$$

We will obtain partial results for the coefficients in the power series expansions of functions in $\Sigma_p(\alpha)$ and powers of such functions by methods analogous to those employed with $\Sigma_p(\alpha)$.

2. Two lemmas for the classes $S(\alpha)$. The first lemma is, essentially, a result of M. S. Robertson [6, p. 386]. Though in his paper he speaks only of $S(\alpha)$, $0 \leq \alpha \leq 1$, the same result holds for all $-\infty < \alpha \leq 1$; his proof goes through without modification.

First a word about notation. If $x$ is a positive real number and $n$ a natural number, we will write:

$$\binom{-x}{n} = \frac{(-x)(-x-1)\cdots(-x-n+1)}{n!}.$$

**Lemma 2.1.** If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in $S(\alpha)$, $-\infty < \alpha \leq 1$, then:

$$|a_{n+1}| \leq \left| \binom{2(1-\alpha)}{n} \right|.$$

The second lemma is an extension of a theorem of Merkes, Robertson and Scott [1].

**Lemma 2.2.** Suppose

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

$$h(z) = z + \sum_{k=2}^{\infty} b_k z^k$$

and that

$$f(z) = z[h(z)/z]^{1/(1-\beta)}.$$

Then $f(z) \in S(\alpha)$ iff $h(z) \in S(\alpha + \beta - \alpha \beta)$.

**Proof.** From (2) we calculate:

$$\Re \{ z f'(z)/f(z) \} = 1 - (1/1-\beta) + (1/1-\beta) \cdot \Re \{ z h'(z)/h(z) \}$$

so that

$$(1-\beta) \Re \{ z f'(z)/f(z) \} + \beta = \Re \{ z h'(z)/h(z) \}.$$ 

Thus $f(z) \in S(\alpha)$, i.e., $\Re \{ z f'(z)/f(z) \} > \alpha$ iff $h(z) \in S(\alpha + \beta - \alpha \beta)$. 

3. A coefficient theorem for functions in $S_p(\alpha)$.

**Theorem 3.1.** Let $f_p(z) \in S_p(\alpha)$ and let $t$ be any positive real number. Then the coefficients of

$$[f_p(z)]^t = z^t + a_{p+1}z^{p+t} + a_{2p+t}z^{2p+t} + \cdots$$

are subject to the sharp bounds

$$|a_{np+t}| \leq \left| \left( \frac{-2(t/p)(1-\alpha)}{n} \right) \right|^t, \quad n = 1, 2, \ldots.$$

**Proof.** By definition, $f_p(z) = f(z^p)^{1/p}$, where $f(z) \in S(\alpha)$.

We use the representation (2) for the function $f(z)$, with $\beta = 1 - t/p$:

$$f(z) = z[h(z)/z]^\beta$$

or

$$f(z)^{1/p} = z^{(1-t)/p} h(z).$$

Let $\xi^t = z$. This becomes

(3) $$[f(\xi^t)^{1/p}]^t = \xi^{(t-\alpha)/p}. h(\xi^t).$$

Equation (3) shows that the coefficients of $[f_p(z)]^t$ are identical with those of a function:

$$h(z) \in S(\alpha + 1 - t/p - \alpha[1-t/p]) = S(1+t/p[\alpha-1]) \quad \text{by Lemma 2.2.}$$

Applying Lemma 2.1 to $h(z)$ we find that it is subject to the sharp coefficient bounds:

$$|b_{n+1}| \leq \left| \left( \frac{-2(t/p)(1-\alpha)}{n} \right) \right|.$$ 

Thus these are sharp bounds for the coefficients $|a_{np+t}|$ of the function $[f_p(z)]^t$ as asserted.

It should be remarked that the hypothesis: $f_p \in S_p(\alpha)$ is equivalent to the condition: $f_p \in S(\alpha)$. For a straightforward calculation shows:

(4) $$\Re \{z f_p^*(z)/f_p(z)\} = \Re \{z [f(z^p)^{1/p}]^*/f(z^p)^{1/p}\} = \Re \{z^p f^*(z^p)/f(z^p)\} = \Re \{\xi f^*(\xi)/f(\xi)\},$$

where $|z| < 1$ iff $|\xi| < 1$. Thus we have the

**Corollary 3.1.** If $f_p(z)$ is starlike with respect to the origin, the coefficients of $[f_p(z)]^t$ are subject to the sharp bounds:

$$|a_{np+t}| \leq \left| \left( \frac{-2t/p}{n} \right) \right|^t, \quad n = 1, 2, \ldots.$$

**Proof.** Set $\alpha = 0$ in Theorem 3.1. This is Poole's theorem, if we restrict $t > 0$ to integral values.
Corollary 3.2. If \( f_p(z) \) is convex, the coefficients of \( [f_p(z)]' \) are subject to the sharp bounds:

\[
|a_{n+p}| \leq \left( \frac{-t/p}{n} \right), \quad n = 1, 2, \ldots
\]

**Proof.** The well-known condition that a function \( g(z) \) be convex is:

\[
\text{Re} \left\{ \frac{z}{g(z)} \frac{g''(z)}{g'(z)} + 1 \right\} > 0 \quad \text{for all } z \in \mathbb{U}.
\]

Evaluating this functional for \( f_p(z) \) we find:

\[
\text{Re} \left\{ \frac{z}{f_p(z)} \frac{f''(z)}{f'(z)} + 1 \right\} = -p \cdot \text{Re} \left\{ \frac{z}{f(z)} \frac{f''(z)}{f'(z)} + 1 \right\} + (1-p) \cdot \text{Re} \left\{ \frac{z f'(z)}{f(z)} \right\}
\]

where \( |z| < 1 \text{ iff } |\xi| < 1 \).

Our hypothesis that \( f_p(z) \) is convex together with (5) and (6) means that in our case:

\[
p \cdot \text{Re} \left\{ \xi f''(\xi) f'(\xi) + 1 \right\} > (p-1) \cdot \text{Re} \left\{ \xi f'(\xi) f(\xi) \right\}.
\]

Furthermore, \( f_p(z) \) convex implies, a fortiori, that \( f_p(z) \) is starlike; therefore by (4), \( f(z) \) is starlike so that:

\[
p \cdot \text{Re} \left\{ \xi f''(\xi) f'(\xi) + 1 \right\} > 0.
\]

We have thus shown that \( f_p(z) \) convex implies that \( f(z) \) is convex.

E. Strohhacker has shown [8] that if \( f(z) \) is a convex function then:

\[
\text{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} > \frac{1}{2} \quad \text{for all } z \in \mathbb{U}.
\]

Consequently, \( f(z) \in S(1/2) \) so that by definition, \( f_p(z) \in S_p(1/2) \). Setting \( \alpha = 1/2 \) in Theorem 3.1 we obtain the stated result.

In the paper of Strohhacker referred to just above he proves that a convex function is also a star function of order not less than 1/2, but that the converse is false. This he illustrates with the example: \( f_1(z) = z + z^2/3 \); he shows that \( f_1(z) \in S(1/2) \) but nevertheless \( f_1(z) \) is not convex.

It is natural to ask whether there is an order of starlikeness \( \alpha > 1/2 \) which will guarantee convexity. We will show there is none; that is to say:

**Theorem 3.2.** For every \( \alpha < 1 \) there exists a function \( g(z) \in S(\alpha) \) which is not convex.

**Proof.** As we shall show presently, if \( f(z) \in S(\alpha), \alpha < 1 \), there is some point \( \xi \in \mathbb{U} \) for which

\[
\text{Re} \left\{ \xi f''(\xi) f'(\xi) + 1 \right\} - \text{Re} \left\{ \xi f'(\xi) f(\xi) \right\} = \delta < 0 \quad \text{and} \quad \text{Re} \left\{ \xi f'(\xi) f(\xi) \right\} < 2.
\]

Choose \( p \) so large that: \( 2 + p \cdot \delta < 0 \).
Now let \( f(z) \in S(\alpha) \) and consider the function \( f_p(z) \). By (6):

\[
\text{Re} \left\{ \frac{f_p(z)}{f(z)} \right\} = \text{Re} \left\{ \frac{f(z)}{f_p(z)} \right\} - p \cdot \text{Re} \left\{ \frac{f'(z)\overline{f'(z)}}{f(z)\overline{f(z)}} \right\} + \text{Re} \left\{ \frac{f''(z)}{f'(z)} \right\}
\]

\[
< \frac{p+2}{p} < 0 \quad \text{for} \quad z^p = \zeta.
\]

Thus \( f_p(z) \) is not convex. On the other hand we have by (4) that \( f_p(z) \in S(\alpha) \). This is the sought after function, \( g(z) \).

It remains to establish the inequality (7).

The difference on the left side can be expressed as:

\[
\text{Re} \left\{ z \frac{d}{dz} \log \left( \frac{f''(z)}{f(z)} \right) \right\} = \text{Re} \left\{ z \frac{d}{dz} \log (1 + b_1 z + b_2 z^2 + \cdots) \right\}
\]

\[
= \text{Re} \left\{ z \frac{b_1 + 2b_2 z + 3b_3 z^2 + \cdots}{1 + b_1 z + b_2 z^2 + \cdots} \right\}.
\]

The only function for which \( b_1 = b_2 = \cdots = 0 \) is \( f(z) = z \); this function is excluded by the hypothesis: \( \alpha < 1 \). The expression in braces is therefore a MacLaurin series equal to zero at \( z = 0 \) but not identically zero. By continuity it maps a neighborhood of the origin onto a neighborhood of the origin. Since \( \text{Re} \left\{ \frac{f''(z)}{f(z)} \right\} = 1 \) at \( \zeta = 0 \), this neighborhood can be chosen small enough that \( \text{Re} \left\{ \frac{f''(z)}{f(z)} \right\} < 2 \) at every point of it. This proves (7).

4. A coefficient theorem for functions in \( \Sigma_p(\alpha) \). In this section we obtain partial results for \( \Sigma_p(\alpha) \) of the same type as those which were obtained for \( S_p(\alpha) \) in Theorem 3.1. Our methods are modified owing to the lack of an equivalent to Lemma 2.1 for the classes \( \Sigma(\alpha) \). We have only a theorem of Pommerenke [3] which gives sharp coefficient bounds for \( f \in \Sigma(\alpha), \alpha \geq 0 \). His proof does not lend itself to an extension to cases where \( \alpha < 0 \). Pommerenke's theorem is:

**Lemma 4.1.** Let \( f(z) = z + \sum_{n=0}^{\infty} \alpha_n z^{-n} \) be in

\[ \Sigma(\alpha), \quad 0 \leq \alpha \leq 1. \]

Then \( |\alpha_n| \leq 2(1-\alpha)/(n+1) \), \( n = 0, 1, 2, \ldots \), with equality for the functions:

\[ f(z) = z(1 + z^{-n-1})^{\alpha(1-\alpha)/(n+1)}. \]

**Lemma 4.2.** Suppose

\[ f(z) = z[\tilde{h}(z)/z]^{1/(1-\beta)}. \]

Then \( f(z) \in \Sigma(\alpha) \) if and only if \( \tilde{h}(z) \in \Sigma(\alpha + \beta - \alpha \beta) \).

**Proof.** See the proof of Lemma 2.2.

**Theorem 4.1.** The coefficients of \( [f_p(z)]^t \), \( f_p \in \Sigma_p(\alpha) \) where \( t > 0 \) and \( (1-\alpha)t \leq p \) are subject to the sharp bounds:

\[ |\alpha_{n+1}| \leq \frac{t}{p} \cdot 2(1-\alpha)/(n+1), \quad n = 0, 1, 2, \ldots. \]
Proof. In a manner analogous to that of the proof of Theorem 3.1 we arrive at:

$$[\hat{f}_p(z)]^t = z^{t-p} \cdot \hat{h}(z^p) = z^t + \hat{b}_2 z^{t-2p} + \ldots$$

where $\hat{h}(z) \in \Sigma(1 + (t/p)(\alpha - 1))$. When $1 + (t/p)(\alpha - 1) \geq 0$, that is, when $(1 - \alpha)t \leq p$, the coefficients of $\hat{h}(z)$ are, by Lemma 4.1, subject to the sharp bounds:

$$|\hat{b}_n| \leq \frac{2(t/p)(1-\alpha)}{n+1}.$$ 

Thus these are sharp bounds for the coefficients: $|a_{(n+1)p-1}|$ of the function $[\hat{f}_p(z)]^t$.

5. Covering theorems for the classes $S(\alpha)$. A problem in conformal mapping is determining the largest disk about the origin $w=0$, covered by every mapping in a particular class. The classical result of Koebe-Bieberbach [2, p. 214] states that every mapping $w=f(z) \in S$ covers the disk: $|w| < 1/4$. The next theorem settles this problem for the classes under consideration here. Since radial limits exist for functions of these classes, we can speak about the value of a function at a point on the unit circle without any ambiguity.

**Theorem 5.1.** Suppose $f(z) \in S(\alpha)$, $-\infty < \alpha \leq 1$. Then the image of the circle $|z| = 1$ under $w=f(z)$ lies exterior to the disk:

$$|w| < (1/4)^{1-\alpha}.$$ 

**Proof.** We write

$$f(z) = z[h(z)/z]^{1-\alpha}.$$ 

By Lemma 2.2 $h(z) \in S(0)$. It is known that $S(0) \subseteq S$ [7] where $S$ denotes the class of functions regular and univalent in $\mathbb{U}$, normalized as in (1).

If $z=e^{i\theta}$ is a point on the circle $|z|=1$ we have $|f(e^{i\theta})| = |h(e^{i\theta})|^{1-\alpha} \geq (1/4)^{1-\alpha}$ by the Koebe-Bieberbach theorem.

Suppose the hypothesis $f(z)$ is convex is inserted in Theorem 5.1. Then according to the result of E. Strohhacker [8], $f(z) \in S(1/2)$. Consequently, the image of $\mathbb{D}$ under $w=f(z)$ covers the disk $|w| < (1/4)^{1-1/2} = 1/2$; a well-known result.

We have established a sharp lower bound for $|f(e^{i\theta})|$ where $f \in S(\alpha)$:

$$|f(e^{i\theta})| \geq (1/4)^{1-\alpha}.$$ 

Let $\eta_1, \eta_2, \ldots, \eta_n$ be the set of $n$th roots of unity. It follows trivially that:

$$\sum_{k=1}^n |f(e^{i\theta}, \eta_k)| \geq n(1/4)^{1-\alpha}.$$ 

The next theorem will improve on this lower bound. We will need two lemmas.

**Lemma 5.1.** If $f_1(z), f_2(z), \ldots, f_n(z)$ are each in $S(\alpha)$, then:

$$\left[ \prod_{k=1}^n f_k(z) \right]^{1/n} \in S(\alpha).$$
Proof. Let
\[ g(z) = \left( \prod_{k=1}^{n} f_k(z) \right)^{1/n}. \]

Straightforward calculation shows:
\[ \text{Re} \left( \frac{z g'(z)}{g(z)} \right) = \frac{1}{n} \sum_{k=1}^{n} \text{Re} \left( \frac{z f_k'(z)}{f_k(z)} \right) > \frac{1}{n} (n\alpha) = \alpha. \]

**Lemma 5.2 (Rengel [5]).** Let \( w=f(z) \) belong to class \( S \) and consider any system of \( n \) rays emerging from \( w=0 \) at equal angles. Then the maximal distance from \( w=0 \) of the nearest boundary points in the \( w \)-plane on these \( n \) rays is not less than \((1/4)^{1/n}\).

**Theorem 5.2.** Let \( f(z) \in S(\alpha) \) and \( z=e^{i\theta} \) a point on the circle \( |z|=1 \). Let \( \eta_1, \eta_2, \ldots, \eta_n \) be the set of \( n \)th roots of unity. Then
\[ \sum_{k=1}^{n} |f(e^{i\theta} \cdot \eta_k)| \geq n(1/4)^{(1-\alpha)/n}. \]

**Proof.** Consider the function
\[ g(z) = \left( \prod_{k=1}^{n} \frac{f(z \cdot \eta_k)}{\eta_k} \right)^{1/n}. \]

Each of the factors is in \( S(\alpha) \) and therefore by Lemma 5.1, \( g(z) \in S(\alpha) \). Consequenctly we can write, as we did in (8):
\[ (g(z) = z[h(z)/z]^{1-\alpha} \quad \text{where} \quad h(z) \in S(0) \subset S. \]

The mapping \( w=g(z) \) has the property that for every point \( r \cdot e^{i\theta} \) in the closed disk \( |z| \leq 1 \):
\[ g(r e^{i\theta} \cdot \eta_k) = \eta_k \cdot g(r e^{i\theta}), \quad k = 1, \ldots, n. \]

In other words, the set of \( n \)-fold symmetry \( \{r e^{i\theta} \cdot \eta_k\}, k=1, \ldots, n \) maps onto a set of \( n \)-fold symmetry. Using this fact, (9) and Lemma 5.2 we may conclude:
\[ |g(e^{i\theta})| = |h(e^{i\theta})^{1-\alpha}| \geq (1/4)^{(1-\alpha)/n} \]
or
\[ \left| \prod_{k=1}^{n} f(e^{i\theta} \cdot \eta_k) \right|^{1/n} \geq (1/4)^{(1-\alpha)/n}. \]

The inequality relating the arithmetic and geometric means then implies the inequality of the theorem.

**Theorem 5.3.** If \( \text{Re} \{z f'(z)/f(z)\} \) is bounded from below; that is to say, if \( f(z) \in S(\alpha) \) for some \( \alpha \), there is a point \( z=e^{i\theta} \) for which:
\[ \sum_{k=1}^{n} |f(e^{i\theta} \cdot \eta_k)| \geq n, \]
\[ n=1, 2, \ldots, z=\psi(n). \]
Proof. Given arbitrary $\epsilon > 0$ choose $s$ so large that $(1/4)^{(1-a)/ns} > 1 - \epsilon$. By the preceding theorem:

$$\sum_{k=1}^{ns} |f(e^{i\theta} \cdot \beta_k)| \geq ns(1/4)^{(1-a)/ns}$$

where $\beta_1, \beta_2, \ldots, \beta_{ns}$ are the $ns$ roots of unity. The set of points $\{e^{i\theta} \cdot \beta_k\}, k = 1, 2, \ldots, ns$ on the unit circle can be considered as $s$ disjoint sets of $n$ points each, every one of which is a set of $n$th roots of unity (rotated on the unit circle). The above inequality can therefore be written as:

$$\sum_{j=1}^{s} \sum_{k=1}^{n} |f(e^{i\theta} \cdot \phi_j \cdot \eta_k)| \geq ns(1/4)^{(1-a)/ns}$$

where $\{\eta_k\}, k = 1, \ldots, n$, are the $n$th roots of unity and $\{\phi_j\}, j = 1, \ldots, s$, are the first $s$ of the $ns$ roots of unity. This means that for some $\phi_j$:

$$\sum_{k=1}^{n} |f(e^{i\theta} \cdot \phi_j \cdot \eta_k)| \geq n(1/4)^{(1-a)/ns} > n(1-\epsilon).$$

Since $\epsilon$ is arbitrary the result follows.

References


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