FUNCTIONS STARLIKE OF ORDER α

BY
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1. **Introduction.** Let $f(z)$ be regular in the unit disk, $\mathscr{U}$, with an expansion of the form:

$$f(z) = z+\sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

We will say that $f(z)$ is starlike of order α in $\mathscr{U}$ if:

$$\operatorname{Re}\{zf'(z)/f(z)\} > \alpha \qquad \text{for all } z \in \mathscr{U};$$

we will denote by $S(\alpha)$ the class of all such functions for fixed α. We consider all real α: $-\infty < \alpha \leqq 1$. The class $S(0)$ will be recognized as the class of functions starlike with respect to the origin.

For the natural number p we define the related classes.

$$S_p(\alpha) = \{f_p(z) = f(z^p)^{1/p} : f(z) \in S(\alpha)\}.$$

Extremal problems for the coefficients in the power series expansions of functions in $S_p(0)$ and powers of such functions were recently investigated by J. T. Poole [4]. In this paper we will show that these coefficient problems are equivalent to extremal problems for the coefficients in the power series expansion of functions in $S(\alpha)$, $(\alpha < 0)$. This alternative approach leads to a substantial generalization.

We will also prove several theorems of the type commonly called "distortion theorems" for functions in the classes $S(\alpha)$. One such result is the:

THEOREM. *Let $f(z) \in S(\alpha)$, $-\infty < \alpha \leqq 1$. Then for each natural number n there is a point $z = e^{i\theta}$ on the unit circle such that*:

$$\sum_{k=1}^{n} |f(e^{i\theta}\cdot\eta_k)| \geqq n,$$

where $\{\eta_k\}$ $k=1, 2, \ldots, n$ denote the nth roots of unity.

Let $\tilde{f}(z)$ be regular in the exterior of the unit circle $\mathscr{V}$, except for a simple pole at infinity, with an expansion of the form:

$$\tilde{f}(z) = z+\sum_{n=0}^{\infty} \tilde{a}_n z^{-n}.$$

We will say that $\tilde{f}(z)$ is starlike of order α in $\mathscr{V}$ if:

$$\operatorname{Re}\{z\tilde{f}'(z)/\tilde{f}(z)\} > \alpha \qquad \text{for all } z \in \mathscr{V};$$

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we will denote by $\Sigma(\alpha)$ the class of all such functions for fixed α. The class $\Sigma(0)$ is the class of functions whose compact complement is starlike with respect to the origin.

For the natural number p we define the related classes:

$$\Sigma_p(\alpha) = \{\tilde{f}_p(z) = \tilde{f}(z^p)^{1/p} : \tilde{f}(z) \in \Sigma(\alpha)\}.$$

We will obtain partial results for the coefficients in the power series expansions of functions in $\Sigma_p(\alpha)$ and powers of such functions by methods analogous to those employed with $S_p(\alpha)$.

2. **Two lemmas for the classes** $S(\alpha)$. The first lemma is, essentially, a result of M. S. Robertson [6, p. 386]. Though in his paper he speaks only of $S(\alpha)$, $0 \leqq \alpha \leqq 1$, the same result holds for all $-\infty < \alpha \leqq 1$; his proof goes through without modification.

First a word about notation. If x is a positive real number and n a natural number, we will write:

$$\binom{-x}{n} = \frac{(-x)(-x-1)\cdots(-x-n+1)}{n!}.$$

LEMMA 2.1. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *is in* $S(\alpha)$, $-\infty < \alpha \leqq 1$, *then*:

$$|a_{n+1}| \leqq \left|\binom{-2(1-\alpha)}{n}\right|.$$

The second lemma is an extension of a theorem of Merkes, Robertson and Scott [1].

LEMMA 2.2. *Suppose*

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

$$h(z) = z + \sum_{k=2}^{\infty} b_k z^k$$

and that

$$f(z) = z[h(z)/z]^{1/(1-\beta)}. \tag{2}$$

Then $f(z) \in S(\alpha)$ *iff* $h(z) \in S(\alpha+\beta-\alpha\beta)$.

Proof. From (2) we calculate:

$$\operatorname{Re}\{zf'(z)/f(z)\} = 1-(1/1-\beta)+(1/1-\beta)\cdot\operatorname{Re}\{z\,h'(z)/h(z)\}$$

so that

$$(1-\beta)\operatorname{Re}\{zf'(z)/f(z)\}+\beta = \operatorname{Re}\{z\,h'(z)/h(z)\}.$$

Thus $f(z) \in S(\alpha)$, i.e., $\operatorname{Re}\{zf'(z)/f(z)\} > \alpha$ iff $h(z) \in S(\alpha+\beta-\alpha\beta)$.

3. A coefficient theorem for functions in $S_p(\alpha)$.

THEOREM 3.1. *Let $f_p(z) \in S_p(\alpha)$ and let t be any positive real number. Then the coefficients of*

$$[f_p(z)]^t = z^t + a_{p+t}z^{p+t} + a_{2p+t}z^{2p+t} + \cdots$$

are subject to the sharp bounds

$$|a_{np+t}| \leqq \left|\binom{-2(t/p)(1-\alpha)}{n}\right|, \qquad n = 1, 2, \ldots.$$

Proof. By definition, $f_p(z) = f(z^p)^{1/p}$, where $f(z) \in S(\alpha)$.

We use the representation (2) for the function $f(z)$, with $\beta = 1 - t/p$:

$$f(z) = z[h(z)/z]^{p/t}$$

or

$$f(z)^{t/p} = z^{(t-p)/p} \cdot h(z).$$

Let $\zeta^p = z$. This becomes

$$[f(\zeta^p)^{1/p}]^t = \zeta^{(t-p)} \cdot h(\zeta^p). \tag{3}$$

Equation (3) shows that the coefficients of $[f_p(z)]^t$ are identical with those of a function:

$$h(z) \in S(\alpha + 1 - t/p - \alpha[1 - t/p]) = S(1 + t/p[\alpha - 1]) \quad \text{by Lemma 2.2.}$$

Applying Lemma 2.1 to $h(z)$ we find that it is subject to the sharp coefficient bounds:

$$|b_{n+1}| \leqq \left|\binom{-2(t/p)(1-\alpha)}{n}\right|.$$

Thus these are sharp bounds for the coefficients $|a_{np+t}|$ of the function $[f_p(z)]^t$ as asserted.

It should be remarked that the hypothesis: $f_p \in S_p(\alpha)$ is equivalent to the condition: $f_p \in S(\alpha)$. For a straightforward calculation shows:

$$\begin{aligned} \operatorname{Re}\{z f_p'(z)/f_p(z)\} &= \operatorname{Re}\{z\,[f(z^p)^{1/p}]'/f(z^p)^{1/p}\} \\ &= \operatorname{Re}\{z^p f'(z^p)/f(z^p)\} = \operatorname{Re}\{\zeta f'(\zeta)/f(\zeta)\}, \end{aligned} \tag{4}$$

where $|z| < 1$ iff $|\zeta| < 1$. Thus we have the

COROLLARY 3.1. *If $f_p(z)$ is starlike with respect to the origin, the coefficients of $[f_p(z)]^t$ are subject to the sharp bounds*:

$$|a_{np+t}| \leqq \left|\binom{-2t/p}{n}\right|, \qquad n = 1, 2, \ldots.$$

Proof. Set $\alpha = 0$ in Theorem 3.1. This is Poole's theorem, if we restrict $t > 0$ to integral values.

COROLLARY 3.2. *If $f_p(z)$ is convex, the coefficients of $[f_p(z)]^t$ are subject to the sharp bounds*:

$$|a_{np+t}| \leqq \left|\binom{-t/p}{n}\right|, \qquad n = 1, 2, \ldots.$$

Proof. The well-known condition that a function $g(z)$ be convex is:

$$\text{(5)} \qquad \operatorname{Re}\{z\, g''(z)/g'(z)+1\} > 0 \quad \text{for all } z \in \mathscr{U}.$$

Evaluating this functional for $f_p(z)$ we find:

$$\begin{aligned}\operatorname{Re}\{z\,[f_p(z)]''/[f_p(z)]'+1\} &= \operatorname{Re}\{z\,[f(z^p)^{1/p}]''/[f(z^p)^{1/p}]'+1\}\\ \text{(6)} \qquad &= p\cdot \operatorname{Re}\{z^p f''(z^p)/f'(z^p)+1\}+(1-p)\operatorname{Re}\{z^p f'(z^p)/f(z^p)\}\\ &= p\cdot \operatorname{Re}\{\zeta f''(\zeta)/f'(\zeta)+1\}+(1-p)\cdot \operatorname{Re}\{\zeta f'(\zeta)/f(\zeta)\}\end{aligned}$$

where $|z|<1$ iff $|\zeta|<1$.

Our hypothesis that $f_p(z)$ is convex together with (5) and (6) means that in our case:

$$p\cdot \operatorname{Re}\{\zeta f''(\zeta)/f'(\zeta)+1\} > (p-1)\cdot \operatorname{Re}\{\zeta f'(\zeta)/f(\zeta)\}.$$

Furthermore, $f_p(z)$ convex implies, a fortiori, that $f_p(z)$ is starlike; therefore by (4), $f(z)$ is starlike so that:

$$p\cdot \operatorname{Re}\{\zeta f''(\zeta)/f'(\zeta)+1\} > 0.$$

We have thus shown that $f_p(z)$ convex implies that $f(z)$ is convex.

E. Strohhacker has shown [8] that if $f(z)$ is a convex function then:

$$\operatorname{Re}\{z f'(z)/f(z)\} > 1/2 \qquad \text{for all } z \in \mathscr{U}.$$

Consequently, $f(z) \in S(1/2)$ so that by definition, $f_p(z) \in S_p(1/2)$. Setting $\alpha=1/2$ in Theorem 3.1 we obtain the stated result.

In the paper of Strohhacker referred to just above he proves that a convex function is also a star function of order not less than $1/2$, but that the converse is false. This he illustrates with the example: $f_1(z)=z+z^2/3$; he shows that $f_1(z) \in S(1/2)$ but nevertheless $f_1(z)$ is not convex.

It is natural to ask whether there is an order of starlikeness $\alpha>1/2$ which will guarantee convexity. We will show there is none; that is to say:

THEOREM 3.2. *For every $\alpha<1$ there exists a function $g(z) \in S(\alpha)$ which is not convex.*

Proof. As we shall show presently, if $f(z) \in S(\alpha)$, $\alpha<1$, there is some point $\zeta \in \mathscr{U}$ for which

$$\text{(7)} \quad \operatorname{Re}\{\zeta f''(\zeta)/f'(\zeta)+1\}-\operatorname{Re}\{\zeta f'(\zeta)/f(\zeta)\} = \delta < 0 \quad \text{and} \quad \operatorname{Re}\{\zeta f'(\zeta)/f(\zeta)\} < 2.$$

Choose p so large that: $2+p\cdot\delta<0$.

Now let $f(z) \in S(\alpha)$ and consider the function $f_p(z)$. By (6):

$$\begin{aligned}&\operatorname{Re}\{z\,[f_p(z)]''/[f_p(z)]'+1\}\\&\quad = p\cdot\operatorname{Re}\{\zeta f''(\zeta)/f'(\zeta)+1\}-p\cdot\operatorname{Re}\{\zeta f'(\zeta)/f(\zeta)\}+\operatorname{Re}\{\zeta f'(\zeta)/f(\zeta)\}\\&\quad < p\cdot\delta+2 < 0 \qquad \text{for } z^p = \zeta.\end{aligned}$$

Thus $f_p(z)$ is not convex. On the other hand we have by (4) that $f_p(z) \in S(\alpha)$. This is the sought after function, $g(z)$.

It remains to establish the inequality (7).

The difference on the left side can be expressed as:

$$\begin{aligned}\operatorname{Re}\left\{z\frac{d}{dz}\log\left(z\frac{f'(z)}{f(z)}\right)\right\} &= \operatorname{Re}\left\{z\frac{d}{dz}\log\left(1+b_1z+b_2z^2+\cdots\right)\right\}\\ &= \operatorname{Re}\left\{z\,\frac{b_1+2b_2z+3b_3z^2+\cdots}{1+b_1z+b_2z^2+\cdots}\right\}.\end{aligned}$$

The only function for which $b_1=b_2=\cdots=0$ is $f(z)\equiv z$; this function is excluded by the hypothesis: $\alpha<1$. The expression in braces is therefore a MacLaurin series equal to zero at $z=0$ but not identically zero. By continuity it maps a neighborhood of the origin onto a neighborhood of the origin. Since $\operatorname{Re}\{\zeta f'(\zeta)/f(\zeta)\}=1$ at $\zeta=0$, this neighborhood can be chosen small enough that $\operatorname{Re}\{\zeta f'(\zeta)/f(\zeta)\}<2$ at every point of it. This proves (7).

4. **A coefficient theorem for functions in $\Sigma_p(\alpha)$.** In this section we obtain partial results for $\Sigma_p(\alpha)$ of the same type as those which were obtained for $S_p(\alpha)$ in Theorem 3.1. Our methods are modified owing to the lack of an equivalent to Lemma 2.1 for the classes $\Sigma(\alpha)$. We have only a theorem of Pommerenke [3] which gives sharp coefficient bounds for $\tilde{f}\in\Sigma(\alpha)$, $\alpha\geqq 0$. His proof does not lend itself to an extension to cases where $\alpha<0$. Pommerenke's theorem is:

LEMMA 4.1. *Let $\tilde{f}(z)=z+\sum_{n=0}^{\infty}\tilde{a}_nz^{-n}$ be in*

$$\Sigma(\alpha), \qquad 0 \leqq \alpha \leqq 1.$$

Then $|\tilde{a}_n|\leqq 2(1-\alpha)/(n+1)$, $n=0, 1, 2, \ldots$, with equality for the functions:

$$\tilde{f}(z) = z(1+z^{-n-1})^{2(1-\alpha)/(n+1)}.$$

LEMMA 4.2. *Suppose*

$$\tilde{f}(z) = z[\tilde{h}(z)/z]^{1/(1-\beta)}.$$

Then $\tilde{f}(z)\in\Sigma(\alpha)$ if and only if $\tilde{h}(z)\in\Sigma(\alpha+\beta-\alpha\beta)$.

Proof. See the proof of Lemma 2.2.

THEOREM 4.1. *The coefficients of $[\tilde{f}_p(z)]^t$, $f_p\in\Sigma_p(\alpha)$ where $t>0$ and $(1-\alpha)t\leqq p$ are subject to the sharp bounds*:

$$|\tilde{a}_{(n+1)p-t}| \leqq t/p\cdot 2(1-\alpha)/(n+1), \qquad n = 0, 1, 2, \ldots.$$

Proof. In a manner analogous to that of the proof of Theorem 3.1 we arrive at:

$$[\tilde{f}_p(z)]^t = z^{(t-p)}\cdot\tilde{h}(z^p) = z^t+\tilde{b}_0 z^{t-p}+\tilde{b}_1 z^{(t-2p)}+\cdots$$

where $\tilde{h}(z)\in\Sigma(1+(t/p)(\alpha-1))$. When $1+(t/p)(\alpha-1)\geqq 0$, that is, when $(1-\alpha)t\leqq p$, the coefficients of $\tilde{h}(z)$ are, by Lemma 4.1, subject to the sharp bounds:

$$|\tilde{b}_n| \leqq \frac{2(t/p)(1-\alpha)}{n+1}.$$

Thus these are sharp bounds for the coefficients: $|\tilde{a}_{(n+1)p-t}|$ of the function $[\tilde{f}_p(z)]^t$.

5. **Covering theorems for the classes $S(\alpha)$.** A problem in conformal mapping is determining the largest disk about the origin $w=0$, covered by every mapping in a particular class. The classical result of Koebe-Bieberbach [2, p. 214] states that every mapping $w=f(z)\in S$ covers the disk: $|w|<1/4$. The next theorem settles this problem for the classes under consideration here. Since radial limits exist for functions of these classes, we can speak about the value of a function at a point on the unit circle without any ambiguity.

Theorem 5.1. *Suppose $f(z)\in S(\alpha)$, $-\infty<\alpha\leqq 1$. Then the image of the circle $|z|=1$ under $w=f(z)$ lies exterior to the disk*:

$$|w| < (1/4)^{1-\alpha}.$$

Proof. We write

$$f(z) = z[h(z)/z]^{1-\alpha}. \tag{8}$$

By Lemma 2.2 $h(z)\in S(0)$. It is known that $S(0)\subset S$ [7] where S denotes the class of functions regular and univalent in $\mathscr{U}$, normalized as in (1).

If $z=e^{i\theta}$ is a point on the circle $|z|=1$ we have $|f(e^{i\theta})|=|h(e^{i\theta})|^{1-\alpha}\geqq(1/4)^{1-\alpha}$ by the Koebe-Bieberbach theorem.

Suppose the hypothesis $f(z)$ is convex is inserted in Theorem 5.1. Then according to the result of E. Strohhacker [8] $f(z)\in S(1/2)$. Consequently, the image of $\mathscr{U}$ under $w=f(z)$ covers the disk $|w|<(1/4)^{1-1/2}=1/2$; a well-known result.

We have established a sharp lower bound for $|f(e^{i\theta})|$ where

$$f\in S(\alpha) : |f(e^{i\theta})| \geqq (1/4)^{1-\alpha}.$$

Let $\eta_1,\eta_2,\ldots,\eta_n$ be the set of nth roots of unity. It follows trivially that:

$$\sum_{k=1}^{n} |f(e^{i\theta}\cdot\eta_k)| \geqq n(1/4)^{1-\alpha}.$$

The next theorem will improve on this lower bound. We will need two lemmas.

Lemma 5.1. *If $f_1(z), f_2(z),\ldots,f_n(z)$ are each in $S(\alpha)$, then*:

$$\left[\prod_{k=1}^{n} f_k(z)\right]^{1/n} \in S(\alpha).$$

Proof. Let

$$g(z) = \left[\prod_{k=1}^{n} f_k(z)\right]^{1/n}.$$

Straightforward calculation shows:

$$\operatorname{Re}\left\{z\frac{g'(z)}{g(z)}\right\} = \frac{1}{n}\sum_{k=1}^{n} \operatorname{Re}\left\{z\frac{f_k'(z)}{f_k(z)}\right\} > \frac{1}{n}(n\alpha) = \alpha.$$

LEMMA 5.2 (RENGEL [5]). *Let $w=f(z)$ belong to class S and consider any system of n rays emerging from $w=0$ at equal angles. Then the maximal distance from $w=0$ of the nearest boundary points in the w-plane on these n rays is not less than $(1/4)^{1/n}$.*

THEOREM 5.2. *Let $f(z) \in S(\alpha)$ and $z=e^{i\theta}$ a point on the circle $|z|=1$. Let $\eta_1, \eta_2, \ldots, \eta_n$ be the set of* nth *roots of unity. Then*

$$\sum_{k=1}^{n} |f(e^{i\theta}\cdot\eta_k)| \geqq n(1/4)^{(1-\alpha)/n}.$$

Proof. Consider the function

$$g(z) = \left[\prod_{k=1}^{n} \frac{f(z\cdot\eta_k)}{\eta_k}\right]^{1/n}.$$

Each of the factors is in $S(\alpha)$ and therefore by Lemma 5.1, $g(z) \in S(\alpha)$. Consequently we can write, as we did in (8):

$$g(z) = z[h(z)/z]^{1-\alpha} \qquad \text{where } h(z) \in S(0) \subset S. \tag{9}$$

The mapping $w=g(z)$ has the property that for every point $r\cdot e^{i\theta}$ in the closed disk $|z| \leqq 1$:

$$g(re^{i\theta}\cdot\eta_k) = \eta_k\cdot g(re^{i\theta}), \qquad k = 1, \ldots, n.$$

In other words, the set of n-fold symmetry $\{re^{i\theta}\cdot\eta_k\}$, $k=1, \ldots, n$ maps onto a set of n-fold symmetry. Using this fact, (9) and Lemma 5.2 we may conclude:

$$|g(e^{i\theta})| = |h(e^{i\theta})^{1-\alpha}| \geqq (1/4)^{(1-\alpha)/n}$$

or

$$\left|\prod_{k=1}^{n} f(e^{i\theta}\cdot\eta_k)\right|^{1/n} \geqq (1/4)^{(1-\alpha)/n}.$$

The inequality relating the arithmetic and geometric means then implies the inequality of the theorem.

THEOREM 5.3. *If $\operatorname{Re}\{zf'(z)/f(z)\}$ is bounded from below; that is to say, if $f(z) \in S(\alpha)$ for some α, there is a point $z=e^{i\theta}$ for which:*

$$\sum_{k=1}^{n} |f(e^{i\theta}\cdot\eta_k)| \geqq n,$$

$n=1, 2, \ldots,$ $z=\psi(n)$.

Proof. Given arbitrary $\varepsilon > 0$ choose s so large that $(1/4)^{(1-\alpha)/ns} > 1-\varepsilon$. By the preceding theorem:

$$\sum_{k=1}^{ns} |f(e^{i\theta}\cdot\beta_k)| \geqq ns(1/4)^{(1-\alpha)/ns}$$

where $\beta_1, \beta_2, \ldots, \beta_{ns}$ are the ns roots of unity. The set of points $\{e^{i\theta}\cdot\beta_k\}$, $k=1,2,\ldots, ns$ on the unit circle can be considered as s disjoint sets of n points each, every one of which is a set of nth roots of unity (rotated on the unit circle). The above inequality can therefore be written as:

$$\sum_{j=1}^{s}\sum_{k=1}^{n} |f(e^{i\theta}\cdot\phi_j\cdot\eta_k)| \geqq ns(1/4)^{(1-\alpha)/ns}$$

where $\{\eta_k\}$, $k=1,\ldots, n$, are the nth roots of unity and $\{\phi_j\}$, $j=1,\ldots, s$, are the first s of the ns roots of unity. This means that for some ϕ_j:

$$\sum_{k=1}^{n} |f(e^{i\theta}\cdot\phi_j\cdot\eta_k)| \geqq n(1/4)^{(1-\alpha)/ns} > n(1-\varepsilon).$$

Since ε is arbitrary the result follows.

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