ON FACTORIZATION OF MEROMORPHIC FUNCTIONS(1)

BY
FRED GROSS

1. **Introduction.** A meromorphic function $h(z)=f(g(z))$ is said to have $f(z)$ and $g(z)$ as left and right factors respectively, provided that $f(z)$ is nonlinear and meromorphic and $g(z)$ is nonlinear and entire (g may be meromorphic when $f(z)$ is rational). $h(z)$ is said to be prime (pseudo-prime) if every factorization of the above form implies that one of the functions $f(z)$ or $g(z)$ is linear (a polynomial or $f(z)$ is rational).

There are numerous questions that one can ask about factorization of meromorphic functions. We shall primarily be concerned with two of them.

1. How many factors does a given meromorphic function have?
2. Given certain properties of a meromorphic function, what are some related properties of its factors?

2. **Generalizations and extensions of previous results.** We begin with the simple

THEOREM 1. *Any transcendental meromorphic function of finite order which has at most a finite number of poles and zeros is pseudo-prime.*

We shall need the following lemma.

LEMMA 1 (EDREI AND FUCHS [2]). *If f is any meromorphic function and g is entire, then $f(g)$ is of finite order implies that either f is of finite order and g is a polynomial or that f is of zero order.*

Proof of theorem. Let $h=f(g)$ satisfy the hypotheses of the theorem and suppose that it is not pseudo-prime. Clearly $f(z)$ has at most one pole, say b, and at most one zero, say a. Thus it can be expressed as

$$f(z)=[(z-a)^n/(z-b)^m]e^{\alpha(z)},$$

where n and m are nonnegative integers and $\alpha(z)$ is entire. By Lemma 1, $\alpha(z)$ must be a constant and the proof is complete.

We note that $f(g)$ must have either infinitely many poles or zeros unless one of n, m is zero, so that $f(z)$ is of the form $c(z-a)^n$, where n is an integer and c is a constant.

This generalizes a result of Thron [11].

Received by the editors October 10, 1966 and, in revised form, January 12, 1967.

(1) This is an abstract of a talk which was given at the Mathematical Institute on Entire Functions and Related Topics at the University of California, at La Jolla, California, June 28, 1966.

Along these same lines we have

THEOREM 2. *Let h be an entire (meromorphic) function with at most a finite number of simple zeros. Either h has only polynomial right factors or every left factor has at most* 2 (4) *simple zeros.*

Proof. We use the standard notations of the Nevanlinna Theory. In particular $N(r, a; g)$ is the smoothed counting function of the a-points, a-points of multiplicity k being counted as k points; $N_1(r, a; g)$ is the smoothed counting function of multiple a-points, in which an a-point of multiplicity k is counted $k-1$ times. If

$$\theta(a; g) = \liminf_{r\to\infty} (m(r, a; g)+N_1(r, a, g))/T(r, g);$$

then [6, Theorem 14.7.1, p. 230]

$$(1) \qquad \sum_a \theta(a; g) \leqq 2 \quad (g \text{ meromorphic}), \qquad \sum_a \theta(a; g) \leqq 1 \quad (g \text{ entire}).$$

Under the hypotheses of the theorem, if a is a simple zero of $f(z)$, then $g(z)-a$ has only a finite number of simple zeros, so that $2N_1(r, a; g) \geqq N(r, a; g)+O(\log r)$. Therefore

$$\begin{aligned} m(r, a; g)+N_1(r, a; g) &\geqq \tfrac{1}{2}(m(r, a; g)+N(r, a; g))+O(\log r) \\ &\geqq \tfrac{1}{2}T(r, g)+O(\log r). \end{aligned}$$

Since g is transcendental, $\log r = o(T(r, g))$ and so $\theta(a; g) \geqq \frac{1}{2}$ and the theorem follows from (1).

For our next result we shall need

LEMMA 2 (HAYMAN [5]). *If f is any transcendental meromorphic function and $g(z)$ is a transcendental entire function, then $T(r, f(g))/T(r, g) \to \infty$ as $r \to \infty$.*

REMARK. Though Hayman states Lemma 2 for entire functions f, it remains valid for meromorphic f as well.

THEOREM 3. *Let $Q(z)$ be a nonzero polynomial and let τ be a nonzero constant. If $F(z)$ is entire, transcendental of exponential type and for some constant c satisfies $F(z+\tau)-F(z)=Q(z)e^{cz}$, then $F(z)$ is pseudo-prime.*

[REMARK. The theorem is also true if $Q(z)$ is identically zero (see Theorem 9), but the proof is different.]

Proof. Assume that $F(z)=f(g(z))$ where f is transcendental and meromorphic and g is transcendental and entire.

We have $f(g(z+\tau))-f(g(z))=Q(z)e^{cz}$. Therefore $g(z+\tau)-g(z)$ is an entire function with a finite number of zeros. By Lemma 2, $g(z)$ and so also $g(z+\tau)-g(z)$ is of order one, type zero, at most. Hence by Hadamard's factorization theorem

$g(z+\tau)-g(z)=$ polynomial. Hence $g^{(n)}(z)$ is periodic for some integer n, and since it is at most of order 1 type zero it must be constant. It follows that $g(z)$ must be a polynomial.

DEFINITION. z_0 is said to be a fix-point of a function $f(z)$ if $f(z_0)=z_0$.

LEMMA 3 (ROSENBLOOM [10]). *If $P(z)$ is a nonlinear polynomial and $f(z)$ is entire transcendental, then $P(f(z))$ has infinitely many fix-points. Equivalently $f(P(z))$ also has infinitely many fix-points.*

From Theorem 3 and Lemma 3 we get

COROLLARY. *e^z+z is prime.*

This last result was stated by Rosenbloom [10] without proof.

THEOREM 4. *Let F (F^*) denote the family of entire (meromorphic) functions with at most a finite number of fix-points. Then* (i) *every entire function has at most one factorization $f(g(z))$, f transcendental, $f\in F$, g entire*; (ii) *every meromorphic function has at most two distinct factorizations $f_i(g_i(z))$, f_i meromorphic, not rational, $f_i\in F^*$, g_i entire.*

Proof. We prove the second part only. The proof of the first part is similar. Suppose $h(z)$ has three factorizations $f_i(g_i(z))$ $(i=1, 2, 3)$ of the type described in statement (ii). Assume that g_i are all distinct.

By Lemma 2, $T(r, g_i)=o(T(r, h))$. Hence, by a well-known theorem of Nevanlinna [7],

$$(1+o(1))T(r, h) \leqq \sum_{i=1}^{3} \bar{N}\left(r, \frac{1}{h-g_i}\right)$$

outside a set of r of finite linear measure, or

$$(1+o(1))T(r, h) \leqq \sum_{i=1}^{3} \bar{N}\left(r, \frac{1}{f_i(g_i)-g_i}\right) \tag{2}$$

outside a set of r of finite linear measure.

Suppose that each of the functions f_i, $i=1, 2, 3$ has at most a finite number of fix-points. Say f_i has fix-points z_{ij}, $j=1, 2, \dots, K_i$. Then

$$\bar{N}\left(r, \frac{1}{f_i(g_i)-g_i}\right) \leqq \sum_{j=1}^{K_i} N\left(r, \frac{1}{g_i-z_{ij}}\right) \leqq cT(r, g_i) = o(T(r, h)), \tag{3}$$

where c is a constant.

Since (2) and (3) lead to a contradiction, our proof is complete.

Corollary 1 below is a generalization of the fact that $f_n(z)$, the nth iterate of $f(z)$, has infinitely many fix-points for $n>1$, a result also first proved by Rosenbloom. The corollary follows from Theorem 4 and the following lemma.

LEMMA 4 (BAKER [1]). *If $f(z)$ is a polynomial, then $P(f)=\{$entire $g; f(g(z))=g(f(z))\}$ contains transcendental functions if and only if $f(z)$ has one of the forms $f(z)=$* const *or $f(z)=\gamma z+\delta$, δ and γ a root of unity.*

COROLLARY 1. *Let $f(z)$ be a transcendental entire function and let $g(z)$ be nonlinear and entire. If $f\not\equiv g$ and $f(g)=g(f)$, then one of f, g has infinitely many fix-points.*

Proof. By Lemma 4, g cannot be a nonlinear polynomial. By the above theorem, f or g must have infinitely many fix-points since they both are left factors of the entire function $f(g)$.

COROLLARY 2. *If f and g are transcendental entire, then f or $f(g)$ must have infinitely many fix-points.*

Corollary 2 was first proved by Rosenbloom [10].

COROLLARY 3. *If f is transcendental meromorphic and g and h are transcendental entire then one of $f(z)$, $f(g(z))$, $f(g(h(z)))$ has infinitely many fix-points.*

Proof of Corollary 2. Let $\eta(z)$ be nonlinear entire, then $f(g(\eta(z)))$ has the two transcendental left factors f and $f(g)$ and, consequently, one must have infinitely many fix-points.

The proof of Corollary 3 is similar.

COROLLARY 4. *If f is a meromorphic (an entire) periodic function and $g(z)$ is entire, then $f(f(g))$ has infinitely many fix-points.*

Proof. Let $f\in F^*$. Suppose f is periodic with period τ. For any entire g, $f(g(z)+n\tau)=f(g(z))$ $(n=1, 2, 3, \ldots)$. This implies that f is not rational and the assertion follows from Theorem 4 (ii). Thus the corollary follows for f meromorphic. When f is entire periodic and g is entire, then $g(f)$ is periodic and must have infinitely many fix-points. It follows that $f(g)$ must have infinitely many fix-points.

COROLLARY 5. *If $f(g)$ is periodic and f is meromorphic with at most finitely many fix points, then g is periodic.*

Proof. Let $f\in F^*$. Suppose g is entire and $f(g(z))$ is periodic with period τ. We have $f(g(z+n\tau))=f(g(z))$, $n=1, 2, \ldots$. Again it follows from the last part of Theorem 4 that $g(z+n\tau)=g(z+m\tau)$ for some n and m with $n\not\equiv m$.

This generalizes a previous result of the author [4]. If $e^{g(z)}+g(z)$ is periodic for an entire function g, then g must be periodic.

It follows from the above discussion that if f is an entire function such that $f(f(z))$ is periodic, then f has infinitely many fix-points. An interesting related problem which the author has not been able to resolve is whether $f(f(z))$ is periodic if and only if $f(z)$ is. Another problem of this type is the following:

Let f be an entire function. How many entire solutions, g, does the functional equation $ff = gg$ have[2]? The methods of this paper do not seem to work for these problems. We do have, however,

COROLLARY 6. *Let f and g be entire functions and let $f_n(z)$ denote the nth iterate of $f(z)$. If for some integer $n > 1$, $f_n(z) = ag_n(z) + b$, then either $f = cg + d$ for some constants c and d, and f and g are polynomials or one of f, g has infinitely many fix-points.*

Proof. When f and g are transcendental, the assertion follows at once from Theorem 4 (i). If f and g are polynomials they must be of the same degree, as comparison of the highest powers of $f_n(z)$ and $ag_n(z) + b$ shows. Therefore f_{n-1} and g_{n-1} are also of the same degree and it is possible to choose the number λ so that $L = f_{n-1} - \lambda g_{n-1}$ is of degree lower than the degree of g_{n-1}. We show that L is a constant. The theorem then follows by induction on n.

Suppose that L is of degree m and that g_{n-1} is of degree $h > m$. If

$$f = A_0 + A_1 z + \cdots + A_k z^k, \qquad A_k \neq 0$$

and

$$g = B_0 + B_1 z + \cdots + B_k z^k, \qquad B_k \neq 0,$$

then $f(\lambda g_{n-1} + L) = ag(g_{n-1}) + b$. Comparison of the terms of degree $k \cdot h$ yields $A_k \lambda^k = aB_k$. After cancellation of the g_{n-1}^k-terms, the highest power of z on the right-hand side of the equation is $\leqq (k-1)h$; the highest power on the left is $(k-1)h + m$. Therefore $m = 0$, L is a constant.

We now give a generalization of a theorem of Rényi [9] and the author [3].

THEOREM 5. *Let $f(z)$ and $h(z)$ be arbitrary nonconstant meromorphic functions. The functional equation $f(g) = h$ has at most a denumerable number of solutions g.*

Proof. For given w_0 the number of solutions $g(z)$ of $f(g(z)) = h(z)$, $g(z_0) = w_0$ is finite by the inverse function theorem for entire functions. The possible values of w_0 must satisfy $f(w_0) = h(z_0)$. This gives a finite or denumerable set of w_0.

The same proof also shows:

COROLLARY 1. *For any rational function $P(w)$ and any meromorphic function $h(z)$ the functional equation $P(f(z)) = h(z)$ has at most a finite number of solutions $f(z)$.*

COROLLARY 2 (RÉNYI [9] AND THE AUTHOR [3]). *For any polynomial P and any entire function $f(z)$, $P(f(z))$ is periodic if and only if $f(z)$ is.*

COROLLARY 3. *Let f be an entire function. If $f(z_0 + n) = f(z_0)$ for an infinite number of integers n and some complex number z_0 and if for some meromorphic function, g, $g(f)$ is periodic with period* 1, *then f is periodic.*

---

(2) Subsequent to the completion of this paper, I. N. Baker and the author showed that there are at most denumerably many such g.

Theorem 5 leads to an interesting conjecture. For any factorization of a meromorphic function h, say

$$h = f(g), \tag{4}$$

$$h = fL(L^{-1}g) \tag{5}$$

is another such factorization, where L is a linear transformation. The only meromorphic functions with meromorphic inverses are the linear transformations. It is, therefore, reasonable to expect that if one considers such factorizations of h as (4) and (5) equivalent, then

CONJECTURE. Any meromorphic function has at most denumerably many non-equivalent factorizations.

THEOREM 6. *Let $g(z)=u(z)+iv(z)$ be entire. Let $f(z)$ be entire and periodic with real period. If $f(g(z))$ is periodic with real period and $v(z)$ is bounded on some horizontal half line L, then $g(z)=P(z)+Hz$, where $P(z)$ is periodic with real period and H is a real constant.*

Proof. Without loss of generality we may suppose that L is the positive x-axis. Let $f(w)$ have period $a>0$, $f(g(z))$, period $b>0$. Every point $g(mb)$ (m a positive integer) is congruent (mod a) to a point Z_m in $0\leqq x<a$, $|y|<K$. Also $f\{g(mb)\}=f(Z_m)$. If the point-set $\{Z_m\}$ is infinite, it has a limit point in $0\leqq x\leqq a$, $|y|\leqq K$ and therefore $f(z)\equiv$constant. If f is not constant, then the point set Z_m is finite, and there is an infinite set M of m such that there is a Z and an integer K_m with $g(mb)-k_m a=Z$, $(m\in M)$.

The equation

$$f(\gamma(z)) = f(g(z)), \qquad \gamma(0) = Z \tag{6}$$

has the solutions $\gamma(z)=g(z+mb)-K_m a$, $(m \in M)$. But (6) has at most a finite number of solutions (see the beginning of the proof of Theorem 5), so that

$$g(z+m_1 b)-K_1 a = g(z+m_2 b)-K_2 a.$$

This proves the theorem with $H=-(K_2-K_1)a/(m_2-m_1)b$.

As a further generalization of Corollary 2 of Theorem 5 we have

THEOREM 7. *If f is any entire function of order less than $\frac{1}{2}$ and g is entire, then $f(g)$ is periodic if and only if g is.*

Proof. Let $F(z)=f(g(z))$ and suppose that $F(z+\tau)=F(z)$. Let L be the line $z_0+\lambda\tau$, $-\infty<\lambda<\infty$. The periodic function $F(z)$ is bounded on L. If $g(z)$ is unbounded on L, then $g(L)$ is a path extending arbitrarily far from the origin on which $f(z)$ is bounded. This, however, is impossible, since by a well-known theorem of Wiman any entire function of order $\frac{1}{2}$ must be unbounded on every curve going to infinity. It follows that $g(z)$ is bounded on L. Choose a value z_0 on L such that $\alpha=f(g(z_0))$ is not an algebraic singularity of $f_{-1}(z)$, the inverse function of $f(z)$.

Now $\{g(z_0+n\tau)\}$, $n=1, 2, \ldots$ is bounded, say $|g(z_0+n\tau)| \leqq M$, while $f(g(z_0+n\tau)) = f(g(z_0)) = \alpha$. Thus all $g(z_0+n\tau)$ are among the finite set of solutions of $f(w)=\alpha$ which belong to $|w| \leqq M$. Hence for some $m \neq n$, $g(z_0+m\tau)=g(z_0+n\tau)$. Moreover, for all small ε, $f(g(z_0+\varepsilon+m\tau))=f(g(z_0+\varepsilon+n\tau))=\beta(\varepsilon)$, so that $g(z_0+\varepsilon+m\tau)$ and $g(z_0+\varepsilon+n\tau)$ are both equal to the unique root of $f(w)=\beta(\varepsilon)$ which lies near $g(z_0+m\tau)=g(z_0+n\tau)$. Thus we must have $g(z+m\tau)\equiv g(z+n\tau)$, and $g(z)$ has period $(m-n)\tau$.

Wiman's theorem mentioned in the proof can be generalized to lower order (see Whittaker [12]). Thus we have

THEOREM 7A. *If f is any entire function of lower order less than $\frac{1}{2}$ and g is entire, then $f(g)$ is periodic if and only if g is.*

NOTE. The function $\cos z$ illustrates that $\frac{1}{2}$ is the best upper bound in the above theorem.

Earlier we asked the question whether for an entire function f, ff is periodic if and only if f is. More generally one can ask:

If f and g are entire and $f(g)=g(f)=F$ is periodic, then can one expect that f and g are periodic? In other words, if f or g is nonperiodic can F be periodic?

Theorem 7A yields the following partial answer.

COROLLARY. *If f and g are entire functions, not both periodic, which commute and $F=f(g)$ has the property that for some $\varepsilon>0$, $M_F(r)<\exp(\exp(r^{1/2-\varepsilon}))$ for an infinite sequence of r approaching infinity, then F cannot be periodic.*

LEMMA 5 (PÓLYA [8]). *If $\phi(z)$, $g(z)$ and $h(z)$ are entire functions such that, $\phi(z)=g(h(z))$ and $h(0)=0$, then there is a positive constant c, independent of $g(z)$, $h(z)$ and r with $M_\phi(r)>M_g\{cM_h\{r/2\}\}$, where $M_f(r)=\max_{|z|=r}|f(z)|$.*

Proof of corollary. One can, after some simple transformations, apply Lemma 5 to F and conclude that either f or g must be of lower order less than $\frac{1}{2}$. Thus by the above theorem F cannot be periodic.

From the arguments of Rosenbloom [10] one can conclude that e^z is pseudo-prime. This also follows from Theorem 1. More generally we have

THEOREM 9. *Every periodic entire function of exponential type is pseudo-prime.*

Proof of Theorem. Let $F=f(g)$ satisfy the hypotheses of the theorem.

By Lemma 1 either g is a polynomial or f is of order zero. If f is of order zero, g must be periodic, by Theorem 7. But then $g(z)$ has to be of order $\geqq 1$ which by Lemma 2 contradicts the fact that $f(g(z))$ is of exponential type, unless f is a polynomial.

We conclude this paper with

THEOREM 10. *A meromorphic function and its derivative cannot have a common right factor other than one of the form $e^{cz+b}+d$ where c, b and d are constants.*

Proof. Suppose that $h=f(g)$ is meromorphic and $h'=l(g)$. Then $g'(z)f'(g(z)) = l(g(z))$.

Letting $H(w)=l(w)/f'(w)$ we have $g'=H(g)$.

Since $T(r, g')<(2+o(1))T(r, g)$ as $r\to\infty$, it follows from Lemma 2 that $H(w)=P(w)/Q(w)$, where $P(w)$, $Q(w)$ are relatively prime polynomials.

Thus for some constant c we have

$$(7)\qquad C(g-a_1)^{n_1}(g-a_2)^{n_2}\cdots(g-a_k)^{n_k} = g'(g-b_1)^{m_1}(g-b_2)^{m_2}\cdots(g-b_t)^{m_t},$$

where a_j and b_j are distinct complex numbers and the n's and m's positive integers. Each a_j and each b_j are Picard values of $g(z)$, as can be seen by considering the order to which each side of (7) vanishes at a root of $g(z)=a_j$ (or $g(z)=b_j$). Since there can only be one finite Picard value, the equation must be of the form $c(g-d)^k=g'$, where k is an integer.

Elementary integrations show that the only solutions $g(z)$ are of the desired form.

Added in Proof. After this paper was completed, the author discovered that Theorem 7 has already been proved by I. N. Baker. (See *On some results of A. Rényi and C. Rényi concerning periodic entire functions*, Acta Sci. Math. (Szeged) **27** (1966) 197–200.)

References

1. I. N. Baker, *Permutable entire functions*, Math. Z. **79** (1962), 243–249.

2. A. Edrei and W. H. Fuchs, *On the zeros of $f(g(z))$ where f and g are entire functions*, J. Analyse Math. **12** (1964), 243–255.

3. F. Gross, *Entire solutions of the functional equation* $\alpha(\beta(z))=\alpha(\gamma(z))+c$, J. Indian Math. Soc. (to appear).

4. ———, *On the periodicity of compositions of entire functions*, Canad. J. Math., **18** (1966), 724–730.

5. W. K. Hayman, *Meromorphic functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964; Chapter 2.

6. E. Hille, *Analytic function theory*, Vol. II, Ginn, Boston, Mass., 1962; Chapters 13, 14.

7. R. Nevanlinna, *Le théorème de Picard-Borel et la théorie des fonctions méromorphes*, Gauthier-Villars, Paris, 1929.

8. G. Pólya, *On an integral function of an integral function*, J. London Math. Soc. **1** (1926), 12.

9. A. Rényi and C. Rényi, *Some remarks on periodic entire functions*, J. Analyse Math. **14** (1965), 303–310.

10. P. C. Rosenbloom, *The fix-points of entire functions*, Medd. Lunds Univ. Mat. Sem., Tome Suppl. (1952), 186–192.

11. W. J. Thron, *Entire solutions of the functional equation, $f(f(z))=g(z)$*, Canad. J. Math. **8** (1956), 47–48.

12. J. M. Whittaker, *The lower order of integral functions*, J. London Math. Soc. **8** (1933), 20–27.

Bellcomm, Incorporated,
Washington, D. C.