POINT DERIVATIONS IN CERTAIN SUP-NORM ALGEBRAS

BY

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1. Let $A$ be a closed point-separating subalgebra of $C(X)$ containing the constants, where $X$ is a compact Hausdorff space. $M_A$ will denote the space of multiplicative linear functionals $\varphi$ on $A$, and to each such $\varphi$ we associate its kernel $A_\varphi$. The $A_\varphi$ are precisely the maximal ideals of $A$.

Under certain hypotheses, it is known that analytic discs can be embedded in $M_A$. Wermer [W1] showed that if $A$ is a Dirichlet algebra on $X$, then each Gleason part of $A$ is either a single point or an analytic disc. Hoffman [H] then generalized Wermer's result to logmodular algebras. Finally Lumer [L] observed that the conclusion is really "local": if $\varphi$ has a unique representing measure on $X$, then the part for $A$ containing $\varphi$ consists either of $\varphi$ alone or of an analytic disc.

Our objective in this paper is to take the weakest of these possible hypotheses, namely Lumer's, and show that in a broader sense the analytic disc at $\varphi$, if there is one, really does account for all the analytic structure at $\varphi$. Specifically, we show that all the bounded derivations and higher "derivatives" of $A$ at $\varphi$ are just differentiations with respect to the analytic structure of the analytic disc.

2. All measures will be regular Borel measures on $X$; they will be nonnegative real-valued unless they are called complex, in which case they will be complex-valued.

Let $\varphi \in M_A$. Then there is a measure $\mu$ representing $\varphi$, i.e., $\varphi(f) = \int f \, d\mu$ for all $f \in A$. If there is only one such measure, we will say that $\varphi$ satisfies condition (U) (for unique). Lumer [L] has shown that condition (U) guarantees the validity of essentially all the logmodular theory of Hoffman's paper [H, §§4-6] as applied to $\varphi$. We use this fact freely in the sequel. We shall require two additional facts based on Lumer's paper.

**THEOREM 1.** Suppose $\varphi \in M_A$ and $\mu$ is a measure representing $\varphi$. Then $\varphi$ satisfies (U) if and only if $\mu$ satisfies (U') $\int u \, d\mu = \sup \{\Re \varphi(f) : f \in A, \Re (f) \leq u\}$ for all $u \in C_b(X)$.

**Proof.** This result is, even strongly generalized, quite familiar (see, for example,
[A]). Clearly $(U') \Rightarrow (U)$, since if $\nu$ were a second measure representing $\varphi$ we would have for all $\mu \in C_R(X)$

$$\int u \, d\mu = \sup \{ \Re \varphi(f) : f \in A, \Re (f) \leq u \}$$

$$= \sup \left\{ \int \Re (f) \, d\nu : f \in A, \Re (f) \leq u \right\}$$

$$\leq \int u \, d\nu \leq \inf \left\{ \int \Re (f) \, d\nu : f \in A, \Re (f) \geq \mu \right\}$$

$$= \inf \{ \Re \varphi(f) : f \in A, \Re (f) \geq \mu \} = \int u \, d\mu,$$

hence $\int u \, d\nu = \int u \, d\mu$.

Conversely, suppose $(U')$ is false and select $\mu \in C_R(X)$ such that $\int u \, d\mu > \alpha = \sup \{ \Re \varphi(f) : f \in A, \Re (f) \leq u \}$. Define a (real) linear functional $L$ on the (real) subspace of $C_R(X)$ spanned by $\Re (A)$ and $u$, setting $L(\Re (f) + ru) = \Re \varphi(f) + ru$. It is easily seen that $L$ is a positive functional, hence has norm $L(1) = 1$. Then one finds a measure $\nu$ representing $L$, i.e., $\nu$ represents $\varphi$ and $\int u \, d\nu = \alpha$. This last equality shows that $\nu \neq \mu$, hence $(U)$ fails to hold. □

Recall that the relation $\varphi \sim \varphi' \iff \|\varphi - \varphi'\|_A < 2$ is an equivalence relation on points of $M_A$, and the equivalence classes are the (Gleason) parts for $A$ (see for example [H, §7]). We denote the part containing $\varphi$ by $P_\varphi$. The following fact is not actually necessary, but may make it easier for the reader to justify our use of some of Hoffman’s results; combined with the aforementioned validity of §4–6 of Hoffman’s paper, it immediately guarantees in addition the validity of §7 of Hoffman’s paper for any $\varphi$ satisfying condition (U).

**Theorem 2.** Suppose $\varphi$ satisfies condition (U) and $\varphi' \in P_\varphi$. Then $\varphi'$ satisfies condition (U), and if $\mu$ and $\mu'$ are the representing measures for $\varphi$ and $\varphi'$ respectively, then $\mu$ and $\mu'$ are mutually bounded absolutely continuous.

**Proof.** In the proof of Theorem 6 in [L], Lumer shows that if $\mu'$ is any representing measure for $\varphi'$ then $\mu'$ is boundedly absolutely continuous with respect to $\mu$. Suppose now $\mu_1$ and $\mu_2$ are two representing measures for $\varphi'$. Choose nonnegative real functions $h_1, h_2 \in L^\infty(\mu)$ with $d\mu_1 = h_1 \, d\mu$. Then $g = h_1 - h_2 \in L^1(\mu)$ and $f \in A \Rightarrow \int fg \, d\mu = 0$. Since $g$ is real-valued, $f \in A + \overline{A} \Rightarrow \int fg \, d\mu = 0$. By Theorem 6.7 of [H], $g = 0$ a.e. (\mu), so $\mu_1 = \mu_2$. □

3. A derivation of $A$ at $\varphi$ is a linear functional $D$ on $A$ satisfying the usual product rule for derivatives: $D(fg) = D(f) \cdot \varphi(g) + \varphi(f) \cdot D(g)$. The existence of a nonzero derivation at $\varphi$ is equivalent to $A_{\varphi} \neq A_{\varphi}^2$. [Note: If $I$ is an ideal in $A$, $I^n$ is the ideal generated by products $f_1 \cdots f_n$ with $f_i \in I$ if $n \geq 1$, and $I^0 = A$; also, $I$ denotes the closure of $I$.] Similarly, the properness of the inclusion $A_{\varphi}^2 \subset A_{\varphi}^3$ may be thought of as signifying a sort of second-derivative phenomenon, etc. In this same vein, the existence of a nonzero continuous derivation at $\varphi$ is equivalent to $A_{\varphi} \neq (A_{\varphi}^3)^-$, and so on.
We can now state the main result of this paper.

**Theorem 3.** Suppose $\varphi \in M_A$ satisfies condition (U). Then either $P_\varphi = \{\varphi\}$, in which case $A_\varphi = (A_\varphi^*)^-$ and hence $A_\varphi = (A_\varphi^*)^-$ for all $n \geq 1$, or there is a homeomorphism $h$ of the open unit disc $D$ in the complex plane onto $P_\varphi$ (in the $A^*$ metric topology) such that $(D, h)$ is an analytic disc at $\varphi$ (i.e., $h(0) = \varphi$ and $f \circ h$ is analytic for all $f \in A$, where $f$ is the Gelfand transform of $f$). In the latter case, $(A_\varphi^*)^-((A_\varphi^* + 1)^-)$ is 1-dimensional and $(A_\varphi^*)^-$ consists of those $f \in A$ for which $f \circ h$ vanishes at 0 to order at least $n$, for each $n \geq 0$.

The description of $P_\varphi$ as either a point or a disc is of course contained in the papers of Hoffman and of Lumer. Our contribution is the description of the ideals $(A_\varphi^*)^-$ for $n \geq 2$.

Theorem 3 is proved in the next section. In that section $\varphi$ will denote a multiplicative linear functional satisfying condition (U) and $\mu$ will denote its representing measure. Many of the arguments will look familiar, a point about which we shall make more comment later.

4. We begin with a mild variant of a standard result from the Dirichlet-logmodular theory [W2, Lemma 5].

**Lemma 4.** Suppose $I$ is an ideal of $A$, $f \in L^\infty(\mu)$, and $f$ lies in the $L^2(\mu)$ closure of $I$. Then we can find a sequence $\{f_n\}$ in $I$ such that

$$\|f_n\| = \sup \{|f_n(x)| : x \in X\} \leq \|f\|_{L^2(\mu)}$$

and $f_n \to f$ a.e. ($\mu$).

**Proof.** The proof is actually the same as Wermer's. One uses Theorem 1 instead of the Dirichlet property at a key point, and observes that if Wermer's $f_n$ are in $I$, so are his $h_n$. Glicksberg has also observed that this generalization holds [G, remark after Theorem 2.1].

Briefly, assume (as we may) that $\|f\|_{L^2(\mu)} = 1$ and let $\{k_n\}$ be a sequence in $I$ with $\|f - k_n\|_2 \to 0$. Define $u_n = -\log^+ |k_n| \in C_0(X)$. One shows that $\int u_n \, d\mu \to 0$. Use Theorem 1 to pick $g_n \in A$ with $\Re (g_n) \leq u_n$, $\Im (g_n) = 0$, and $\Re \varphi(g_n) > \int u_n \, d\mu / n$. Then $\|\exp (g_n)\| \leq 1$ and $\varphi(\exp (g_n)) \to 1$, from which one shows that $\exp (g_n) \to 1$ in $L^2(\mu)$. Setting $f_n = k_n \exp (g_n) \in I$, it follows that $f_n \to f$ in $L^1(\mu)$. Since $\|f_n\| \leq 1$, a subsequence of $\{f_n\}$ satisfies the conclusion of the lemma. □

**Lemma 5.** Suppose $L \in (A_\varphi^*)^+$ for a positive integer $n$, i.e., $L$ is a continuous linear functional on $A$ which annihilates $A_\varphi^*$. Let $\lambda$ be a complex measure representing $L$ and let $\lambda = \lambda_0 + \lambda_2$ be its Lebesgue decomposition with respect to $\mu$. Then $\lambda_0$ represents $L$, i.e., $\lambda_0$ annihilates $A$.

**Proof.** We use induction on $n$. The case $n = 1$ is the F. and M. Riesz theorem [H, Theorem 6.5]. Suppose $N > 1$ and we know the lemma for $1 \leq n < N$. Let $L \in (A_\varphi^*)^+$ have the complex representing measure $\lambda$. 
Let \( g \in A^n_{0}^{-1} \) and define \( L_1 \in A^n_{0} \) by \( L_1(f) = L(fg) \). If \( d\lambda_1 = g \ d\lambda \) then \( \lambda_1 \) represents \( L_1 \), so the case \( n = 1 \) tells us that \( (\lambda_1)_s \) annihilates \( A \), i.e., \( f \in A \Rightarrow \int fg \ d\lambda_s = \int f (d\lambda)_s = 0 \). Taking \( f = 1 \) we obtain \( \int g \ d\lambda_s = 0 \).

Thus \( \lambda_s \) annihilates \( A^n_{0}^{-1} \), so by our induction assumption \( \lambda_s = (\lambda_n)_s \) annihilates \( A \). □

Remark. A rather different and in some ways more satisfactory route to Lemma 5 is available. Ahern [A] has observed that, in considerably more generality than we need here, the F. and M. Riesz theorem can be made to follow from a lemma patterned after a theorem of Forelli [F, Theorem 1]. Glicksberg [G] has carried out this program in a form quite convenient for us: his proof of the F. and M. Riesz theorem [G, Theorem 1.1] from a Forelli-type lemma [G, Lemma 1.2] can be trivially modified to give our Lemma 5 for all \( n \) simultaneously.

We have instead used the proof above for two reasons. First, Hoffman’s paper [H] is our basic text, and our proof seems to be the quickest route from Hoffman’s paper to Lemma 5. Second, this (admittedly trivial) proof is evidently applicable to a perhaps much larger class of situations: we have a projection \( \lambda \to \lambda_s \) satisfying certain conditions (the conclusion of the F. and M. Riesz theorem) relative to an algebra \( A \) and an ideal \( I \), and we draw the same conclusion for the ideals \( I^n \); it is possible to reformulate the entire affair in purely algebraic terms, and conclude that if a certain kind of projection “satisfies an F. and M. Riesz theorem” with respect to a suitable algebra \( A \) and ideal \( I \), then it also “satisfies an F. and M. Riesz theorem” with respect to \( A \) and \( I^n \).

\( H^p(\mu) \) denotes the closure of \( A \) in \( L^p(\mu) \), \( 1 \leq p < \infty \), and \( H^{a}(\mu) = H^2(\mu) \cap L^\infty(\mu) \). Lemma 4 and bounded convergence show that \( H^{a}(\mu) \) consists of the a.e. \( (\mu) \) pointwise limits of bounded sequences in \( A \), hence is a Banach algebra when endowed with a.e. \( (\mu) \) pointwise operations and the \( L^\infty \) norm. Let \( H^2(\mu) = \{ f \in H^p(\mu) : \int f \ d\mu = 0 \} \), \( 1 \leq p \leq \infty \). In particular, if \( 1 \leq p < \infty \), then \( H^2(\mu) \) is the \( L^p \) closure of \( A^p \).

\( \varphi' \in P_v \) has a unique representing measure \( \mu' \), and \( \mu \) and \( \mu' \) are mutually boundedly absolutely continuous. Thus the spaces \( L^p(\mu) \) and \( L^p(\mu') \) are identical as function spaces, as are \( H^p(\mu) \) and \( H^p(\mu') \), \( 1 \leq p \leq \infty \), and the respective pairs of norms are equivalent.

For each \( f \in L^v(\mu) \) we can therefore define a function \( f^\ast \) on \( P_v \) by \( f^\ast(\varphi') = \int f \ d\mu' \). Clearly this agrees with the usual notion of \( f^\ast | P_v \) if \( f \in A \). Further, \( f \to f^\ast(\varphi') \) is a bounded linear functional on \( L^v(\mu) \) (hence on \( L^p(\mu) \), \( 1 \leq p \leq \infty \)) for each \( \varphi' \in P_v \).

Lemma 6 (see [H, Theorem 5.1]). \( f \to f^\ast(\varphi') \) is multiplicative on \( H^2(\mu) \) for each \( \varphi' \in P_v \), in the sense that if \( f, g \in H^2(\mu) \) then \( fg \in H^2(\mu) \) and \( (fg)^\ast(\varphi') = f^\ast(\varphi') \cdot g(\varphi') \). In particular, \( f \to f^\ast(\varphi') \) is multiplicative on \( H^a(\mu) \).

Lemma 7. If \( L \in (A^n_{0})^{-1} \) for a positive integer \( n \), \( L \) extends to \( L_1 \in L^a(\mu)^* \) in such a way that \( \| L_1 \| = \| L \| \), \( L_1 \) annihilates \( (H^a(\mu))^a \), and \( L_1 \) is weakly continuous, i.e., if \( \{ f_i \} \) is a bounded sequence in \( L^a(\mu) \) and \( f_i \to g \) a.e. \( (\mu) \) then \( L_1(f_i) \to L_1(g) \).
**Proof.** Select \( \lambda \) a complex measure representing \( L \) such that \( \|\lambda\| = \|L\| \). Lemma 5 shows that \( d\lambda = h \, d\mu \) for some \( h \in L^1(\mu) \). Lemma 4 then can be used to see that \( L_1(f) = \int f \, d\lambda \) will do. □

**Theorem 8.** Suppose \( P_0 \neq \{\varphi\} \). Then there is a homeomorphism \( h \) of the open unit disc \( D \) onto \( P_0 \) (in the \( A^* \) metric topology) such that \( (D, h) \) is an analytic disc at \( \varphi \).

If \( h_1 \) and \( h_2 \) are two such functions then \( h_1^{-1} \circ h_2 \) is an analytic homeomorphism of \( D \) onto itself. Any such function \( h \) satisfies:

(a) \( f \circ h \) is analytic for each \( f \in H^2(\mu) \).

(b) If \( L \in (A_0^*)^1 \) for a positive integer \( n \) and \( L_1 \) is the extension of \( L \) to \( L^n(\mu) \) guaranteed by Lemma 7, then \( L_1|H^n(\mu) \) has the form \( L_1(f) = \sum_{k=0}^{n-1} a_k (d^k/dz^k)(f \circ h)(0) \) for appropriate constants \( a_0, \ldots, a_{n-1} \).

**Proof.** Theorems 7.6 and 7.4 of [H] imply the existence of a \( Z \in H^n(\mu) \) with the following properties:

(i) \( |Z| = 1 \) a.e. \( (\mu) \).

(ii) \( Z \) maps \( P_0 \) (in the \( A^* \) metric topology) homeomorphically onto \( D \).

(iii) The function \( f - \sum_{k=0}^{n-1} (\int Z^k f \, d\mu) Z^k \) is in \( Z^{n+1}H^2(\mu) \) whenever \( f \in H^2(\mu) \) and \( n \) is a nonnegative integer.

(iv) \( \varphi' \in P_0, f \in H^2(\mu) \Rightarrow \int \varphi' \, d\mu = \sum_{k=0}^{n-1} (\int Z^k f \, d\mu) Z^k \).

Then \( h = (Z|P_0)^{-1} \) satisfies everything except (b), and we now verify (b). Define \( a_k = L_1(Z^k)/k! \) for \( 0 \leq k \leq n-1 \). Then \( a_k (d^k/dz^k)(f \circ h)(0) = (\int Z^k f \, d\mu) L_1(Z^k) \) for \( f \in H^2(\mu) \). If \( f \in H^n(\mu) \) then (i) and (iii) show that \( f - \sum_{k=0}^{n-1} (\int Z^k f \, d\mu) Z^k \) is in \( Z^nH^n(\mu) \subset (H^n(\mu))^n \), so \( L_1(f - \sum_{k=0}^{n-1} (\int Z^k f \, d\mu) Z^k) = 0 \) and therefore

\[
L_1(f) = \sum_{k=0}^{n-1} (\int Z^k f \, d\mu) L_1(Z^k) = \sum_{k=0}^{n-1} a_k (d^k/dz^k)(f \circ h)(0).
\]

This proves (b).

Now let \( h' \) be a second function mapping \( D \) homeomorphically and analytically onto \( P_0 \). Then \( h^{-1} \circ h' \) is a homeomorphism of \( D \) onto itself. Lemma 4 permits us to find a bounded sequence \( \{f_j\} \) in \( A_0 \) such that \( f_j \to Z \) a.e. \( (\mu) \). Then \( f_j|P_0 \to Z \) pointwise, so \( f_j \circ h \to Z \circ h \) pointwise. Since \( f_j \circ h \) is analytic by hypothesis, so is \( Z \circ h = h^{-1} \circ h' \). Thus \( h^{-1} \circ h' \) is a diffeomorphism. This also implies that (a) and (b) hold for \( h' \), completing the proof. □

**Proposition 9.** Let \( Z \) be as in Theorem 8, \( h = (Z|P_0)^{-1} \), \( n \) a positive integer. Define \( L \) on \( A_0 \) by \( L(f) = (d^{n-1}/dz^{n-1})(f \circ h)(0) \). Then \( L \in (A_0^*)^1 \), but \( L \notin (A_0^*)^{n-1} \).

**Proof.** Using the Cauchy integral representation for derivatives, \( |L(f)| \leq (n-1)! \|f\| \), so \( L \in A^n \). \( f \in A_0 \Rightarrow f = f - (\int Z^0 f \, d\mu) Z^0 = f - (\int ZH^n(\mu) \) by (iii), so \( f \in A_0^* \Rightarrow f = Z^g \) where \( g \in H^n(\mu) \). Therefore \( (f \circ h)(z) = z^n \cdot (g \circ h)(z) \) where \( g \circ h \) is analytic on \( D \), so \( L(f) = 0 \). Therefore \( L \in (A_0^*)^1 \).

On the other hand, Lemma 4 enables us to select a bounded sequence \( \{f_j\} \) in \( A_0 \) such that \( f_j \to Z \) a.e. \( (\mu) \). Then \( f_j|P_0 \to Z \) pointwise, so \( f_j \circ h \to Z \circ h \) pointwise. It follows that \( L(f_j^{n-1}) \to (n-1)! \) while \( f_j^{n-1} \in A_0^{n-1} \). Thus \( L \notin (A_0^{n-1})^1 \). □
Theorem 8 and Proposition 9 prove that portion of Theorem 3 dealing with the case \( P_\varphi \neq \{ \varphi \} \). The case \( P_\varphi = \{ \varphi \} \) will now be covered by showing that if \( A_\varphi \neq (A_\varphi^2)^- \) then there is a nontrivial analytic disc at \( \varphi \).

Let \( V \) denote the closure in \( L^2(\mu) \) of \( A_\varphi^2 \). Clearly \( V \subset H_\varphi^2(\mu) \).

**Lemma 10.** \( A_\varphi \neq (A_\varphi^2)^- \Rightarrow H_\varphi^2(\mu) \neq V \).

**Proof.** Select \( f_0 \) in \( A_\varphi \) but not in \( (A_\varphi^2)^- \). Then \( f_0 \in H_\varphi^2(\mu) \).

We can find \( L \in (A_\varphi^2)^- \) such that \( L(f_0) \neq 0 \). Suppose \( f_0 \in V \). By Lemma 4 we can select a bounded sequence \( \{ f_i \} \) in \( A_\varphi^2 \) such that \( f_i \to f_0 \) a.e. (\( \mu \)). On the one hand \( L(f_i) = 0 \). On the other hand, Lemma 7 implies that \( L \) is weakly continuous, hence \( L(f_i) \to L(f_0) \neq 0 \), a contradiction. Thus \( f_0 \notin V \). □

**Theorem 11.** If \( A_\varphi \neq (A_\varphi^2)^- \) there exists \( G \in H_\varphi^2(\mu) \) satisfying

(i) \( |G| = 1 \) a.e. (\( \mu \)).

(ii) \( G \) spans \( H_\varphi^2(\mu) \cap V^\perp \).

(iii) \( GH_\varphi^2(\mu) = H_\varphi^2(\mu) \).

**Proof.** This will be a familiar invariant subspace argument, and will really be the proof of the following more general theorem (see, e.g., [SW, Theorem 3.1]): If \( M \) is a singly invariant closed subspace of \( L^2(\mu) \) (i.e., the closed linear span of \( A_\varphi M \) is a proper subspace of \( M \)) then \( M = GH_\varphi^2(\mu) \) where \( |G| = 1 \) a.e. (\( \mu \)).

By Lemma 10 we can select \( G \in H_\varphi^2(\mu) \cap V^\perp \) such that \( \|G\| = 1 \). We show that (i), (ii) and (iii) hold.

\[ f \in A_\varphi \Rightarrow Gf \in V \Rightarrow Gf \perp G \Rightarrow \int |G|^2 \, d\mu = 0. \]

Thus \( |G|^2 \, d\mu \) represents \( \varphi \), so uniqueness of \( \mu \) gives (i).

If (ii) is false we can find orthonormal \( G_1, G_2 \in H_\varphi^2(\mu) \cap V^\perp \). Let \( (a_1, a_2) \) be a pair of complex constants such that \( |a_1|^2 + |a_2|^2 = 1 \). Then \( a_1G_1 + a_2G_2 \in H_\varphi^2(\mu) \cap V^\perp \) and \( \|a_1G_1 + a_2G_2\| = 1 \), so again \( |a_1G_1 + a_2G_2| = 1 \) a.e. (\( \mu \)). It is easily seen that this cannot hold simultaneously for all such pairs \( (a_1, a_2) \). Therefore (ii) must be true.

In view of (i) and Lemma 6, \( GH_\varphi^2(\mu) \) is a closed subspace of \( H_\varphi^2(\mu) \). Suppose \( g \in H_\varphi^2(\mu) \cap (GH_\varphi^2(\mu))^\perp \). Then \( f \in A \Rightarrow Gf \in GH_\varphi^2(\mu) \Rightarrow Gf \perp g \Rightarrow \int Gf \, d\mu = 0 \). On the other hand, \( f \in A \Rightarrow fg \in V \Rightarrow fg \perp G \Rightarrow \int Gf \, d\mu = 0 \). Thus \( Gf \, d\mu \) annihilates \( \overline{A} + A_\varphi \) = \( \overline{A} + A_\varphi \), so by Theorem 6.7 of [H] \( Gf \, d\mu = 0 \) a.e. (\( \mu \)). Because of (i), \( g = 0 \) a.e. (\( \mu \)). Thus (ii) holds. □

**Lemma 12.** Suppose \( A_\varphi \neq (A_\varphi^2)^- \) and \( G \) is as in Theorem 11. Then whenever \( f \in H_\varphi^2(\mu) \) and \( n \) is a nonnegative integer, we have \( g_n \in G^{n+1}H_\varphi^2(\mu) \) where

\[ g_n = f - \sum_{k=0}^{n} \left( \int \overline{G^kf} \, d\mu \right) G^k. \]

**Proof.** Induction on \( n \). □

With hypotheses as in Lemma 12, for each \( z \in D \) define a linear functional \( \bar{z} \) on
$L^2(\mu)$ by $\tilde{z}(f) = \sum_{n=0}^\infty (\int G^nf \, d\mu)z^n$. $\tilde{z}$ is bounded (with norm at most $(1 - |z|)^{-1}$) and for each $f \in L^2(\mu)$ the function $z \rightarrow \tilde{z}(f)$ is analytic on $D$.

**Theorem 13.** The map $z \rightarrow \tilde{z}|A$ is a nontrivial (in fact 1-1) analytic disc at $\phi$. In particular, $A_\phi \neq (A_\phi^\circ)^{-} \Rightarrow P_\phi \neq \{\phi\}$.

**Proof.** Using Lemma 12 it is easy to see that $\tilde{z}$ is multiplicative on $H^\infty(\mu)$, hence on $A$; since also $\tilde{z}(1)=1$, $\tilde{z}|A \in M_A$. Clearly the map is "analytic", and Theorem 11 leads to $\tilde{0}|A=\phi$. Thus $z \rightarrow \tilde{z}|A$ is an analytic disc at $\phi$. Finally, select a sequence $\{f_j\} \subset A$ such that $f_j \rightarrow g$ in $L^2(\mu)$. Then for each $z \in D$, $\tilde{z}(f_j) \rightarrow \tilde{z}(g)=z$. Thus the map is 1-1. □

Remark. Lemma 12 and Theorem 13 are essentially the argument used by Wermer in [W] and repeated by Hoffman in [H] to put a disc in $M_A$. Theorem 13 completes the proof of Theorem 3.

5. In this section we use our characterization of the ideal $(A_\phi^\circ)^{-}$ to see how certain behavior on $X$ of functions in $A$ can imply their belonging or at least being close to $(A_\phi^\circ)^{-}$. Until further notice, we do not assume $\phi$ satisfies condition (U).

If $\phi \in M_A$, a Jensen measure for $\phi$ on $X$ is a measure $\mu$ of a total mass 1 such that the "Jensen inequality" $|\log |\phi(f)|| \leq \int |\log |f|| \, d\mu$ holds for all $f \in A$. Such a $\mu$ is easily seen to be an Arens-Singer measure for $\phi$ (i.e., $J \log |\phi(f)| = \int |\log |f|| \, d\mu$ for all invertible $f \in A$) and therefore a representing measure for $\phi$ (since $\Re(A) \subset \log |A^{-1}|$). Bishop has shown [B] that $\phi$ always has a Jensen measure on $X$.

**Lemma 14.** Suppose $\mu$ is a Jensen measure for $\phi$ and $E$ is a Borel set in $X$ such that $\mu(E) > 0$. Suppose $\{f_j\}$ is a sequence in $A$ and $g \in \gamma(A)$ is such that $|f_j| \leq g$ a.e. $(\mu)$, and suppose $f_j \rightarrow 0$ pointwise on $E$. Then $\varphi(f_j) \rightarrow 0$.

**Proof.** If $\epsilon > 0$ is given, select $\delta > 0$ so small that $(\log (\epsilon) - \int g \, d\mu) \log (\delta) < \mu(E)$ and $\delta < 1$. If $E_j=\{x \in E : |f_j(x)| < \delta\}$ we can find $J$ so large that $j \geq J = \mu(E_j) > (\log (\epsilon) - \int g \, d\mu)/\log (\delta)$. Then

$$j \geq J \Rightarrow |\log |\varphi(f_j)|| \leq \int |\log |f_j|| \, d\mu$$

$$= \int_{E_j} |\log |f_j|| \, d\mu + \int_{X-E_j} |\log |f_j|| \, d\mu$$

$$\leq \mu(E_j) \log (\delta) + \int g \, d\mu < \log (\epsilon),$$

so $|\varphi(f_j)| < \epsilon$. □

**Lemma 15.** Let $U$ and $V$ be open subsets of the complex plane with respective coordinates $u$ and $v$. Let $\tau : U \rightarrow V$ be analytic. Then for $1 \leq k < \infty$ and $1 \leq l \leq k$, there exist polynomials $Q_{k,l}$ in $k$ variables $x_1, \ldots, x_k$ such that whenever $f$ is an analytic function on $V$, we have

$$(f \circ \tau)^{(k)}(u) = \sum_{i=1}^k Q_{k,i}(\tau'(u), \ldots, \tau^{(k)}(u))f^{(i)}(\tau(u)), \quad 1 \leq k < \infty.$$

**Proof.** Induction on $k$. □
If \( \{T_a\} \) is a family of subspaces of \( A \), a sequence \( \{f_i\} \) in \( A \) will be said to converge to \( \{T_a\} \), written \( f_i \to \{T_a\} \), if \( \lim_i (\sup_a \inf \{ \|f_i - g\| : g \in T_a \}) = 0 \). An easy application of the Hahn-Banach theorem shows that this is equivalent to the following: if \( S \) is the closed unit ball in \( A^* \), then \( f_i \to 0 \) uniformly on \( \bigcup_a (S \cap T_a^*) \) where \( f_i \) is interpreted as being in \( A^{**} \).

**Theorem 16.** Suppose \( \varphi \) satisfies condition (U) and \( \mu \) is a representing measure for \( \varphi \). Suppose \( E \) is a Borel set in \( X \) and \( \mu(E) > 0 \). Suppose \( \{f_i\} \) is a bounded sequence in \( A \) such that \( f_i \to 0 \) pointwise on \( E \). Then for each positive integer \( n \),

\[
f_i \to \{A^n_{\varphi'} : \varphi' \in F\}
\]

where \( F \) is any metrically compact subset of \( P_\varphi \).

**Proof.** Let \( \varphi' \in P_\varphi \) have representing measure \( \mu' \). By Theorem 2 \( \varphi' \) satisfies condition (U) and since \( \varphi' \) has a Jensen measure on \( X \), \( \mu' \) must be that Jensen measure. Lemma 14 then implies that \( \varphi'(f_i) \to 0 \). Thus \( f_i \to 0 \) pointwise on \( P_\varphi \).

If \( P_\varphi = \{\varphi\} \) then Theorem 3 says \( (A^\varphi)^{-1} = A_\varphi \) and we are done.

Assume \( P_\varphi \neq \{\varphi\} \) and let \( Z \) be as in Theorem 8, \( h = (\tilde{Z}|P_\varphi)^{-1} \), \( \{f_i \circ h\} \) is a bounded sequence of analytic functions on \( D \) such that \( f_i \circ h \to 0 \) pointwise, hence for every nonnegative integer \( k \), \( (f_i \circ h)^{(k)} \to 0 \) uniformly on any compact subset of \( D \).

We must show that \( f_i \to 0 \) uniformly on \( \bigcup \{ S \cap (A^\varphi)^{-1} : \varphi' \in F\} \). We will accomplish this by finding a sequence of positive constants \( \{c_k\} \) such that \( L \in S \cap (A^\varphi)^{-1} \) for some \( \varphi' \in F \) implies \( |L(f_i)| \leq \sum_{s=0}^{n-1} c_k (f_i \circ h)^{(s)}(|\tilde{Z}(F)|)^{s} \). Since \( \tilde{Z}(F) \) is a compact subset of \( D \), \( (f_i \circ h)^{(k)} \to 0 \) uniformly on \( \tilde{Z}(F) \), and the theorem will be proved. We define the \( c_k \) inductively by \( c_0 = 1 \) and \( c_k = (1/k!) + \sum_{s=0}^{k-1} c_s/(k-s)! \) for \( k > 0 \).

Suppose \( \varphi' \in F \). Let \( Z' \) be constructed for \( \varphi' \) as in Theorem 8 and set \( h' = (\tilde{Z}'|P_\varphi)^{-1} \). Define an analytic function \( \tau = \tilde{Z} \circ h' : D \to D \). ((a) in Theorem 8 shows that \( \tau \) is indeed analytic.)

Let \( L \in S \cap (A^\varphi)^{-1} \) be given and let \( L_1 \) be the extension of \( L \) guaranteed by Lemma 7. By Theorem 8 \( L_1|H^\varphi(\mu) \) has the form \( L_1(f) = \sum_{s=0}^{n-1} a_k (f \circ h)^{(s)}(0) \). Lemma 15 then implies that \( L_1|H^\varphi(\mu) \) has the form \( L_1(f) = \sum_{s=0}^{n-1} b_s (f \circ h)^{(s)}(\tilde{Z}(\varphi')) \).

We will be done if \( |b_k| \leq c_k \) for all \( k \). This we show by induction on \( k \).

Observe that \( |L_1(Z^k)| \leq \|L_1\| = \|L\| \leq 1 \) for all \( k \geq 0 \). Applied to the case \( k = 0 \) this gives \( |b_0| \leq 1 = c_0 \).

Suppose \( 1 \leq K \leq n - 1 \) and \( |b_k| \leq c_k \) for \( 0 \leq k < K \). Then

\[
1 \geq |L_1(Z^K)| = \left| \sum_{s=0}^{n-1} b_s (\tilde{Z}^K \circ h)^{(s)}(\tilde{Z}(\varphi')) \right| = \left| \sum_{s=0}^{K} K! b_s (\tilde{Z}(\varphi'))^{K-s}/(K-s)! \right| = K! \left| b_K + \sum_{s=0}^{K-1} b_s (\tilde{Z}(\varphi'))^{K-s}/(K-s)! \right|
\]
so that

\[ |b_k| \leq \frac{1}{K!} + \left| \sum_{s=0}^{K-1} b_s(\hat{g}(\varphi'))(K-s)! \right| \leq \frac{1}{K!} + \sum_{s=0}^{K-1} c_s!(K-s)! = c_K. \]

**Corollary 17.** Let \( \varphi, \mu \) and \( E \) be as in Theorem 16. Suppose \( f \in A \) and \( f|E=0. \) Then

\[ f \in \bigcap \{(A^e_\varphi)^- : \varphi' \in P_\varphi, 1 \leq n < \infty\}. \]

**References**


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