THE MINIMAL BOUNDARY OF $C(X)$

BY

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1. **Introduction.** Let X be a nonvoid completely regular Hausdorff space, let $C(X)$ be the space of all continuous, complex-valued functions on X, and let $E \subseteq C(X)$ be a subspace which separates points of X and contains the constant functions and is an algebra under pointwise multiplication. In [1] Bishop proved that if X is compact and metrizable, and if as a subspace of $C(X)$ with the supremum norm, E is a Banach algebra, then there exists a smallest subset M of X with the property that each function in E takes on its maximum absolute value on M. He called M the minimal boundary and exhibited a few of its properties. Bishop also showed that if X was compact but not metrizable, then the minimal boundary need not exist.

In §2 we remove the restriction of metrizability on X, and consider spaces of type $C(X)$. We say that $C(X)$ (or sometimes X) has a minimal boundary M_X if M_X is the smallest subset of X with the property that each function which is bounded and attains its maximum absolute value on X in fact attains this maximum absolute value on M_X. In our fundamental Theorem 2.7 we characterize those spaces X for which M_X exists. As a consequence we observe that if M_X does exist, it is dense in X and it consists precisely of all the G_δ points in X. If X is metrizable or dispersed, then M_X always exists, while if X is an infinite compact extremally disconnected space, then M_X cannot exist. We complete the section by giving an example in which the minimal boundary M_X exists, but is not even a Borel subset of X.

§3 deals with inheritance properties of minimal boundaries with respect to subspaces and products of given spaces. The main theorem of the section, Theorem 3.1, says the following. An arbitrary completely regular Hausdorff space Y can be embedded topologically in a completely regular Hausdorff space X such that M_X exists and such that if M_Y exists, then $M_Y = M_X \cap Y$. It follows from this theorem that if X is such that M_X exists, then it is by no means true that for each closed subspace Y of X, M_Y must exist. However, it is true for any subspace Y which is a G_δ (closed or not) in X. Finally we show that countable products of spaces with minimal boundaries also have minimal boundaries, while uncountable products of nontrivial spaces never have minimal boundaries.

§4 concludes the paper. In it we demonstrate that if X is not compact, then the

Stone-Čech compactification βX of X has a minimal boundary if and only if X is pseudo-compact and X itself has a minimal boundary. From this result we deduce that $C(M_X)$ has a natural identification with $C(X)$ only when M_X is pseudo-compact and $\beta M_X = X$.

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2. **Characterization of the minimal boundary.** We set about to characterize the minimal boundary of $C(X)$, and list a few of its properties and consider several examples.

2.1. DEFINITION. Let X be an arbitrary topological space. Let f be a real-valued continuous function on X. Let

$$B_f = \{x \in X : |f(x)| = \sup\{|f(y)| : y \in X\}\}.$$

Let $C_s(X)$ consist of all continuous, real-valued functions f on X such that $B_f \neq \varnothing$.

Then $f \in C_s(X)$ if and only if f attains its maximum absolute value on X. Note that $C_s(X) = C(X)$ if and only if X is pseudo-compact. Furthermore, $C_s(X)$ need not be closed under multiplication of functions. For let $X = [-1, 1)$ and let

$$f(x) = |x|, \quad x \in X, \quad g(x) = \tfrac{1}{2}x + \tfrac{1}{2}, \qquad x < 0,$$
$$= \tfrac{1}{2}, \qquad x \geq 0.$$

Then $f, g \in C_s(X)$ but $fg \notin C_s(X)$. On the other hand, if $f \geq 0$ and $h \geq 0$ and if $f, h \in C_s(X)$, and if $B_f \cap B_h \neq \varnothing$, then $fh \in C_s(X)$. This observation will be useful in what follows.

2.2. DEFINITION. A subspace M of X is a boundary for $C(X)$ if and only if for every $f \in C_s(X)$ we have $M \cap B_f \neq \varnothing$. If there is a smallest such subspace, we call it the minimal boundary of $C(X)$, and denote it by M_X.

When X is compact this definition coincides with that given in [1]. It is to be noted that if X is not pseudo-compact, there is no analogous concept of minimal boundary defined by all the continuous functions on X, since some of those are unbounded. However, it might be possible to consider a boundary defined by the larger class of continuous functions on X which are bounded but which do not necessarily attain their maximum absolute values. We shall not develop this idea further here. We also mention that we could just as well consider complex-valued functions. The notion of minimal boundary would be the same, and all the theorems parallel to those given here.

It is obvious that X itself is always a boundary, so that the notion of "smallest" always makes sense, even if such a set might not exist. However, it is perfectly conceivable that a "minimal" boundary might exist without it being the "smallest" boundary. Thus we need to justify the terminology.

First we reduce our study to topological spaces which are both completely regular and Hausdorff. For an arbitrary topological space X, the completely regular Hausdorff space Y identified with X is defined in [3, p. 41], and $C(X)$

and $C(Y)$ are isomorphic in every sense; so are $C_s(X)$ and $C_s(Y)$. In particular, if $x \in X$, let x_1 be the equivalence class consisting of all $y \in X$ such that for each $f \in C(X)$, we have $f(y) = f(x)$. Then the Y defined above is in fact the set of all such equivalence classes, endowed with the topology inherited from X.

2.3. THEOREM. *Let X be a topological space and let Y be the completely regular Hausdorff space derived above. Then M_X exists if and only if M_Y exists, and when they do exist they are identifiable.*

Proof. Let M_X exist, and let $M = \{x_1 :$ there exists an $x \in X$ such that $x \in x_1$ and $x \in M_X\}$. Then certainly M is a boundary for $C(Y)$. That it is the smallest boundary for $C(Y)$ follows from the fact that M_X is the smallest boundary on $C(X)$ and from the construction of Y. So $M = M_Y$; M_X and M_Y are identified in the obvious way. Now let M_Y exist, and for each $x_1 \in M_Y$ let x_0 be a fixed element of x_1. Let $M = \{x_0 \in X : x_1 \in M_Y\}$. By the isomorphism of $C_s(X)$ and $C_s(Y)$, M is a boundary for X, and the fact that M_Y is the smallest boundary for Y and the uniqueness of x_0 in each x_1 yields M as the minimal boundary for X. Thus $M = M_X$. Once again M_X and M_Y can be identified in the natural way.

N.B. Henceforth, we shall always assume that any topological space we discuss is completely regular and Hausdorff.

Next we determine which spaces are endowed with a minimal boundary. For a subspace D of X, let $X - D$ be the complement in X of D.

2.4. LEMMA. *If A is a nonvoid G_δ subset of X, then there exists an $f \in C_s(X)$ such that $B_f \subseteq A$.*

Proof. Let $A = \bigcap_{n=1}^{\infty} U_n$, U_n open in X, and let $x \in A$ be fixed. Then the complete regularity of X yields a function $f_n : X \to [0, 1/2^n]$ which is continuous, such that $f_n(x) = 1/2^n$ and $f_n(X - U_n) = 0$. Let $f = \sum_{n=1}^{\infty} f_n$. Then $f \in C_s(X)$ and $B_f \subseteq A$.

2.5. COROLLARY. *If $\{x\}$ is a G_δ, then $B_f = \{x\}$ for some $f \in C_s(X)$.*

2.6. LEMMA. *If M is a boundary for $C(X)$ and if x is not a G_δ point, then $M - \{x\}$ is a boundary for $C(X)$.*

Proof. Assume without loss of generality that $x \in M$, and let $f \in C_s(X)$, $f \geq 0$, such that $x \in B_f$. We must show that $(M - \{x\}) \cap B_f \neq \varnothing$. Inasmuch as B_f is a closed G_δ [3, p. 15] and x is assumed to be a non-G_δ point, there exists a point $z \in B_f$ such that $z \neq x$. The complete regularity of and Hausdorff topology on X yield a continuous function $h : X \to [0, 1]$ such that $h(z) = 1$ and $h(x) = 0$. Then $f + h \in C_s(X)$ and $B_{f+h} \subseteq B_f - \{x\}$. But M is a boundary for $C(X)$, so $M \cap B_{f+h} \neq \varnothing$. Therefore we have $(M - \{x\}) \cap B_f \neq \varnothing$. Since the $f \in C_s(X)$ was arbitrary, the lemma follows.

2.7. THEOREM. *For any space X, the minimal boundary M_X exists if and only if every nonempty closed G_δ subset of X contains a G_δ point.*

Proof. We first show that if M_X exists, then every nonempty closed G_δ in X contains a G_δ point. Assume that A is a nonvoid closed G_δ without any G_δ points, and let M be an arbitrary boundary for $C(X)$. By Lemma 2.4, there is an $f \in C_s(X)$ such that $B_f \subseteq A$. Then $M \cap B_f \neq \varnothing$, so that there is an $x \in M \cap B_f \subseteq A$. By hypothesis, x is not a G_δ, so Lemma 2.6 yields $M - \{x\}$ as a boundary. Therefore M cannot be the smallest boundary. Conversely, assume the condition holds, and let $M = \{x \in X : \{x\}$ is a G_δ point$\}$. Then M is certainly a boundary because the maximum absolute value of any function in $C_s(X)$ is a closed G_δ; M is the smallest boundary by virtue of Lemma 2.5.

We note immediately that if X is metrizable, then M_X exists, since each element of X is a G_δ. In fact, the same is true if the first countability axiom prevails on X. That this is a strictly larger class of spaces is assured by the existence of a compact Hausdorff space for which the first countability axiom holds but which is not metrizable [6, p. 164]. A space X for which M_X exists and is different from X is the ordinals less than or equal to the first uncountable Ω, with the order topology. Then $M_X = X - \{\Omega\}$. Finally, if X is a nonmetrizable topological group which is T_0, then M_X does not exist [4, Theorem 8.3], and if X is an uncountable product of nontrivial spaces with the product topology, then also M_X does not exist. In each case there are no G_δ points in the space.

2.8. COROLLARY. *If M_X exists, then it consists precisely of the G_δ points in X.*

2.9. COROLLARY. *If M_X exists, then it is dense in X.*

Proof. Simply note the definition of complete regularity.

2.10. COROLLARY. *If M is a boundary which contains properly no other boundary, then $M = M_X$.*

Proof. If M contains no other boundary, then it is precisely the collection of G_δ points, so $M = M_X$.

Corollary 2.10 tells us that the boundaries for $C(X)$ form a lattice, that the concept of minimal and of smallest boundary coincide, so the term "minimal boundary" is grammatically sound. It is easy to show that if X is compact and Hausdorff, then the Silov boundary [9], Choquet boundary [8], and strong boundary [9] of $C(X)$ exist, are identical, and are equal to X, so that when the minimal boundary exists, it is dense in each of these boundaries. It is also apparent that if X and Y are homeomorphic, then M_X and M_Y are coexistent if either deigns to exist, and when they exist, they are homeomorphic. Nevertheless, if X and Y are given, and if $f: X \to Y$ is a one-to-one and onto and continuous map, and if M_X exists, this in no way implies that M_Y exists. Just let Y be a space for which M_Y does not exist, and let X be the same space as a set, but with the discrete topology, and let f be the identity function. Then $M_X = X$, and there is no M_Y.

A natural question to ask is if the process of obtaining minimal boundaries can continue; i.e. if M_X is the minimal boundary for $C(X)$, does $M_{(M_X)}$ exist? If

$M_{(M_X)}$ does exist, may $M_{(M_X)} \neq M_X$? Since each point in M_X is a G_δ, $M_{(M_X)}$ exists and $M_{(M_X)} = M_X$, so the process happily stops after the initial step.

A topological space X is said to be dispersed if it contains no nonvoid perfect subspace [7]. For such spaces we have the following

2.11. THEOREM. *If X is dispersed, then M_X exists.*

Proof. Let A be a nonvoid G_δ subset of X. Since any subspace of a dispersed space is also dispersed, so is A, and this means that there is an isolated point $x \in A$. Thus there is an open set U in X such that $U \cap A = \{x\}$. On the other hand, $A = \bigcap_{n=1}^\infty U_n$, where U_n is open in X. This means that $\{x\} = \bigcap_{n=1}^\infty (U \cap U_n)$, so that x is a G_δ point in X. Now refer to Theorem 2.7, and the proof is complete.

We have shown that dispersed spaces and metrizable spaces each have minimal boundaries. These types of spaces are different. For the one-point compactification of a discrete space is dispersed, and if uncountable, then it is not metrizable. The same is true for the space of ordinals less than or equal to the first uncountable. Next we look at spaces for which the minimal boundary does not exist. A space X is called extremally disconnected if and only if the closure of each open set is open [3, p. 22].

2.12. THEOREM. *If X is an infinite compact extremally disconnected space, then M_X fails to exist.*

Proof. Assume that M_X exists. Then each point of M_X is a G_δ, which means rather more—namely, that each point of M_X is isolated. Since M_X is dense in X which is infinite, M_X contains an infinite set $(x_n)_{n=1}^\infty$. Define $f \colon M_X \to [0, 1)$ by $f(x_n) = 1 - 1/n$, all n, and $f(x) = 0$, for all other $x \in M_X$. Then f is continuous on M_X, and has a continuous extension to X [3, p. 96]. Then $B_f \cap M_X = \varnothing$, so that M_X cannot be a boundary.

The theorem is not true for all extremally disconnected spaces, since if X is discrete, then it is extremally disconnected but $M_X = X$. On the other hand, there are numerous infinite compact extremally disconnected spaces. Let Y be an infinite compact (Hausdorff) space, and let $C^{**}(Y)$ be the second conjugate space of $C(Y)$. Then $C^{**}(Y)$ is isometrically (and algebraically!) isomorphic to $C(X)$, where X is an appropriate compact, extremally disconnected space [5, p. 1022]. Furthermore, Y is embedded as a dense discrete space in X. Consequently, even if X has a dense subspace of isolated points, M_X need not exist.

It is clear that if M_X exists, it need not be open in X. For let Y be such that M_Y exist but $M_Y \neq Y$, and for each n, let $Y_n = Y$, and set $X = \prod_{n=1}^\infty Y_n$, where X is endowed with the product topology. Then M_X exists and $M_X = \prod_{n=1}^\infty M_{Y_n}$ (this will follow directly from Theorem 3.5). However, things are worse. If we consider the smallest σ-algebra generated by the closed and open sets of X to be the Borel sets of X, then our next example shows that M_X can exist without being a Borel set.

2.13. EXAMPLE. Let $Y = [0, 1]$ with the usual topology, and let Z be a non-Borel subset of Y, which by the axiom of choice exists. Let the elements of Z be denoted by $\{y_\lambda : \lambda \in \Lambda\}$, Λ an index set. For each $\lambda \in \Lambda$, let Y_λ be the collection of ordinals less than or equal to the first uncountable under the order topology, and let the first uncountable ordinal be y_λ itself. Think of the Y_λ as being distinct entities. Let $X = Y \cup \bigcup_{\lambda \in \Lambda} Y_\lambda$. For $x \in X$ such that $x \in Y_\lambda$ for some $\lambda \in \Lambda$, and such that $x \neq y_\lambda$, let a neighborhood basis consist of those sets in Y_λ which are neighborhoods of x in the order topology of Y_λ. For $x \in Y$, let a neighborhood V of x contain a neighborhood U of x in $[0, 1]$ (with its own usual topology) together with a neighborhood in the order topology of Y_λ for each y_λ in U. It is simple to verify that X is a topological space and is Hausdorff. Furthermore, Y and each Y_λ are embedded topologically in X. To show that X is completely regular, let A be a nonvoid closed subset of X, and $x \notin A$. If $x \notin Y$, then there is a $\lambda \in \Lambda$ such that $x \in Y_\lambda$. Since Y_λ is completely regular, there is a continuous function $f_0 : Y_\lambda \to [0, 1]$ such that $f_0(y) = 0$ for $y \in Y_\lambda$ with $y > x$, and $f_0(x) = 1$, and $f_0(A \cap Y_\lambda) = 0$. Now define $f : X \to [0, 1]$ by $f(y) = f_0(y)$, for all $y \in Y_\lambda$, and $f(y) = 0$, all other $y \in X$. Then f is continuous on X, and $f(x) = 1$, while $f(A) = 0$. Next, if $x \in Y$, then there is an open set U in Y with $x \in U$, and with the property that for each y_λ in U there is an open set V_λ in Y_λ, with $y_\lambda \in V_\lambda$, and such that $(U \cup \bigcup_{y_\lambda \in U} V_\lambda) \cap A = \varnothing$. Since Y is completely regular, there is a continuous function $f_0 : Y \to [0, 1]$ such that $f_0(x) = 1$ while $f_0(Y - U) = 0$. Furthermore, for each $\lambda \in \Lambda$, there is a continuous function $f_\lambda : Y_\lambda \to [0, f_0(y_\lambda)]$ such that $f_\lambda(y_\lambda) = f_0(y_\lambda)$ and $f_\lambda(Y - V_\lambda) = 0$. Now define $f : X \to [0, 1]$ by $f(y) = f_0(y)$, for all $y \in Y$, and $f(y) = f_\lambda(y)$, for all $y \in Y_\lambda$, for each $\lambda \in \Lambda$. Then f is continuous on X, and $f(x) = 1$, and $f(A) = 0$, which is just what we want. Thus X is completely regular.

Next we identify the G_δ points in X. Certainly if $x \in Y_\lambda$ and $x \neq y_\lambda$, then x is a G_δ point in Y_λ, and hence in X. If $x \in Y$ with $x \neq y_\lambda$, for all $\lambda \in \Lambda$, then let $(U_n)_{n=1}^\infty$ be an open neighborhood system for x in Y, and let $V_n = U_n \cup \bigcup_{\lambda \in \Lambda_n} Y_\lambda$, where Λ_n consists of all $\lambda \in \Lambda$ such that $y_\lambda \in U_n$. Then V_n is open in X, and $\bigcap_{n=1}^\infty V_n = \{x\}$, so that x is a G_δ in X. Finally, if $x = y_\lambda$ for some $\lambda \in \Lambda$, then x is not a G_δ in X, since it is not a G_δ in Y_λ, which is topologically embedded in X. Hence the G_δ points in X consist of the set $X - Z$. However, if C is a nonvoid closed G_δ set in X, and if $y_\lambda \in C$, then there exists $y \in Y_\lambda$, with $y \neq y_\lambda$, such that $y \in C$. But such a y is, by what we have just said, a G_δ in X. Thus C contains a G_δ point. Therefore M_X exists, and $M_X = X - Z$. Note that if $X - Z$ were in fact a Borel set in X, then Z would also be Borel in X, so that $Z = Z \cap Y$ would be Borel in Y, which by assumption it is not. Thus M_X is not a Borel subset of X.

We also have an example of a space X such that not only M_X exists and is dense in X, but also $X - M_X$ is dense in X. Ours is too complicated to print here. Obviously no point can be isolated. The example reminds one of the subject of study in the "corona problem" [10].

3. **Inheritance properties of the minimal boundary.** Suppose that for a given space X, M_X exists, and let Y be a closed subspace of X. Does M_Y exist? And if M_Y does exist, does $M_Y = M_X \cap Y$? In this section we answer these questions. We conclude by giving a criterion for products to have minimal boundaries.

3.1. THEOREM. *Let Y be an arbitrary space. Then there exists a (completely regular Hausdorff) space X with the following properties*:
1. $Y \subseteq X$ *(homeomorphically)*.
2. M_X *exists*.
3. *If M_Y exists, then $M_Y = M_X \cap Y$.*

Proof. Let $Z = (z_\lambda)_{\lambda \in \Lambda}$, where Λ is an uncountable index set and $0 \in \Lambda$, and let Z have the discrete topology. Let $X = Y \times Z$, and define the following as neighborhoods in X, for all $(y, z) \in X$:
1. If $z \neq z_0$, then $\{(y, z)\}$ is a neighborhood of itself.
2. If $z = z_0$, then W is a neighborhood of (y, z_0) if $W \supseteq U$, where $U = (V - \{y\}, Z)$ $\cup \{(y, z_0)\}$ for some neighborhood V of y in Y.

In order to show that we have described a topology on X, we may show that U is a neighborhood of each of its points, for U a neighborhood of (y_0, z_0). If $(y, z) \in U$ and $z \neq z_0$, then by 1, U is a neighborhood of (y, z). If $y \neq y_0$ but $z = z_0$, then since Y is assumed to be Hausdorff, there is a neighborhood V_1 of y contained in V such that $y_0 \notin V_1$. But then $(V_1, Z) \subseteq U$ and by the definition in 2, (V, Z) is a neighborhood of (y, z). Thus we have a topology on X. Since Y is Hausdorff and Z has the discrete topology, X is Hausdorff. It is straightforward to show that X is completely regular, and we omit the proof.

Next we remark that (Y, z_0) is homeomorphic to Y. For U is a neighborhood of (y, z_0) if and only if $[U \cap (Y, z_0)] \supseteq (V, z_0)$ for some neighborhood V of y in Y. Thus Y is embedded topologically in X. Toward showing that M_X exists, we note that if $z \neq z_0$, then (y, z_0) is a G_δ, for all $y \in Y$. Next, if y is a G_δ in Y and if $\bigcap_{n=1}^\infty V_n = \{y\}$, with V_n open in Y, then $\bigcap_{n=1}^\infty [(V_n - \{y\}, Z) \cup (y, z_0)] = \{(y, z_0)\}$, so that (y, z_0) is a G_δ in X. Now if y is not a G_δ in Y, then for any countable collection $(V_n)_{n=1}^\infty$ of neighborhoods for y in Y, there is $y_1 \in \bigcap_{n=1}^\infty V_n$, whence $(y_1, Z - \{z_0\})$ $\subseteq \bigcap_{n=1}^\infty (V_n - \{y\}, Z) \cup (y, z_0)$, which itself is an arbitrary G_δ in X containing (y, z_0). However, every element of $(y_1, Z - \{z_0\})$ is discrete, so that any (not necessarily closed) nonvoid G_δ in X which contains (y, z_0) also contains a G_δ point. Thus M_X exists. From our proof we see that if M_Y also exists, then $M_X \cap Y = M_Y$.

It is interesting to observe that X is never compact. To prove it, let $y_0 \in Y$ and let $U = (Y - \{y_0\}, Z) \cup (y_0, z_0)$, so that U is open in X. But then $X - U = (y_0, Z - \{z_0\})$, which is an uncountable discrete set, so that X cannot possibly be compact. The same reasoning shows that X is not even locally compact.

Inasmuch as Y is embedded as a closed subspace of X, Theorem 3.1 gives the answer "not always" to the first question at the beginning of this section. However,

Theorem 3.1 does not settle the second question. Trivially the answer to it is again "not always." For let X be such that M_X exists and $M_X \neq X$, and let $y \in X - M_X$ and $Y = \{y\}$. Then $M_Y = \{y\}$ and $M_X \cap Y = \varnothing$. Perhaps this is not a satisfying answer. A more fruitful settlement of the question is contained in

3.2. THEOREM. *Let Y be an arbitrary space. Then there exists a compact Hausdorff space with the following properties*:
1. $Y \subseteq X$ *(homeomorphically)*.
2. M_X *exists*.
3. $M_X \cap Y = \varnothing$.

Proof. For the time being, assume that Y is compact. Define $X = Y \times Z$ as a set, just as in Theorem 3.1. Let the following be neighborhoods in X, for the arbitrary element (y, z):
1. If $z \neq z_0$, then $\{(y, z)\}$ is a neighborhood of itself.
2. If $z = z_0$, then W is a neighborhood of (y, z_0) if $W \supseteq U$, where

$$W = [(V, Z - \text{finite subset of } X] \cup \{(y, z_0)\},$$

for some neighborhood V of y in Y.

Indeed this describes a topology on X, as is easy to check, and once again, since Y is Hausdorff and Z has the discrete topology, X is Hausdorff too. Now we show that X is compact—whence completely regular. Let $(W_\gamma)_{\gamma \in \Gamma}$ be an open cover of X. For each (y, z_0) in X, there is an open neighborhood U_y of y in Y such that $(y, z_0) \in [(U_y, Z) - \text{finite set}] \subseteq W_\gamma$, for some $\gamma \in \Gamma$ (depending of course on the chosen y). Since Y is assumed to be compact, there are $y_1, \ldots, y_n$, such that $\bigcup_{k=1}^n U_{y_k}$ covers Y. But then $X - \bigcup_{k=1}^n [(U_{y_k}, Z) - \text{finite set}]$ is a finite point set. Therefore X can be covered by a finite subcollection of $(W_\gamma)_{\gamma \in \Gamma}$. This means X is compact.

Now we show that M_X exists. If $z \neq z_0$, then (y, z) is isolated, so is a G_δ point. Next, let (y, z_0) be an element of any G_δ set $\bigcap_{n=1}^\infty W_n$, where each W_n is open in X. Then there are $V_n, n = 1, 2, \ldots$, open in Y, such that $(y, z_0) \in [(y, Z) - \text{countable set}] \cup \{(y, z_0)\} \subseteq \bigcap_{n=1}^\infty [(V_n, Z) - \text{countable set}] \subseteq \bigcap_{n=1}^\infty W_n$. But $[(y, Z) - \text{countable set}]$ contains an uncountable collection of isolated points. Thus $\bigcap_{n=1}^\infty W_n$ contains G_δ points. It is then apparent that M_X exists. With a simple calculation we find that Y is homeomorphic to (Y, z_0); by construction (Y, z_0) has no G_δ points in X. Thus $M_X \cap Y = \varnothing$—provided that Y is compact. Now let Y be an arbitrary completely regular Hausdorff space, and let βY be the Stone-Čech compactification of Y. Finally, let $X = \beta Y \times Z$, and put the topology defined above on X, with respect to βY and Z. Then X is compact and M_X exists. Furthermore, βY is homeomorphic to $(\beta Y, z_0)$, so that Y is homeomorphic to (Y, z_0), with the result that Y is embedded topologically in X. Once again $M_X \cap Y = \varnothing$, and the proof is complete.

Although Theorems 3.1 and 3.2 give a negative answer to the questions at the

outset of this section, we now show that under certain circumstances the answers to both questions are yes.

3.3. LEMMA. *If Y is a subspace of X, and if both M_X and M_Y exist, then $M_Y \supseteq M_X \cap Y$.*

Proof. This follows immediately from Corollary 2.8 and from the observation that a point G_δ in X which is in Y is a point G_δ in Y.

3.4. THEOREM. *Let X be an arbitrary space, and Y a nonvoid G_δ in X. If M_X exists, then M_Y exists and $M_Y = M_X \cap Y$.*

Proof. If A is a nonvoid closed G_δ in Y, then A is at least a G_δ in X, so that by Lemma 2.4, there is an $f \in C_s(X)$ such that $B_f \subseteq A$. By hypothesis and by Theorem 2.7, B_f contains a G_δ point y (in X). Then y is a G_δ in Y, and therefore M_Y exists. Since G_δ points in the G_δ subspace Y are G_δ points in X, $M_Y = M_X \cap Y$.

Theorem 3.4 tells us that some of the closed subspaces—at least those which are closed G_δ sets—inherit minimal boundaries. So also for all open subspaces. In the final theorem of this section, we characterize those product spaces which have minimal boundaries, in terms of their coordinate spaces.

3.5. THEOREM. *Let Λ be an index set, and for each $\lambda \in \Lambda$, let Y_λ be a nontrivial topological space. Let $X = \prod_{\lambda \in \Lambda} Y_\lambda$, endowed with the product topology. Then X has a minimal boundary if and only if Λ is at most countable and for each $\lambda \in \Lambda$, Y_λ has a minimal boundary. If X has a minimal boundary, then $M_X = \prod_{\lambda \in \Lambda} M_{Y_\lambda}$.*

Proof. If Λ is uncountable, then no points of the product space X are G_δ points, so that there can be no minimal boundary. Next, if Λ is at most countable, but some coordinate space, say Y_1, has no minimal boundary, then there is a nonvoid closed G_δ set A_1 in Y_1 without any G_δ points. Let $A_k = Y_k$ for all $k \neq 1$, and let $A = \prod_{k \in \Lambda} A_k$. Then A is a nonvoid closed G_δ in X. Since the projection of a G_δ point in X onto any of its coordinates must be a G_δ point in the coordinate space, we find that A has no G_δ points in X. Thus in this case M_X fails to exist. We have remaining the case in which Λ is at most countable and each Y_k has a minimal boundary. We will show that M_X exists and $M_X = \prod_{k \in \Lambda} M_{Y_k}$. A routine check shows that if for each k, y_k is a G_δ in Y_k, then $x = (y_k)_{k \in \Lambda}$ is a G_δ in X. Now let A be a nonvoid closed G_δ in X. We will find a G_δ point x in A. To this end, note that $A = \bigcap_{n=1}^{\infty} W_n$, where $W_n = \bigcup_{i=1}^{m_n} \prod_{k \in \Lambda} W_{n,k,i}$, and where each $W_{n,k,i}$ is at least open in Y_k. Fix $y = (y_k)_{k \in \Lambda}$ in A. Then for each n, there exists i_n with $i_n \leq m_n$, such that $\prod_{k \in \Lambda} W_{n,k,i_n}$ contains y. But for each n and k, there is a continuous function f_{nk} such that $f_{nk} \colon Y_k \to [0, 1/2^n]$, $f_{nk}(y_k) = 1/2^n$ and $f_{nk}(Y_k - W_{n,k,i_n}) = 0$. Define $f_k = \sum_{n=1}^{\infty} f_{nk}$, so that $f_k \in C_s(Y_k)$ and $B_{f_k} \subseteq \bigcap_{n=1}^{\infty} W_{n,k,i_n}$. By hypothesis, there is a G_δ point x_k in B_{f_k}, for each k. Let $x = (x_k)_{k \in \Lambda}$. Then by our remark above, x is a G_δ in X, and $x \in \prod_{k \in \Lambda} B_{f_k} \subseteq \bigcap_{n=1}^{\infty} \prod_{k \in \Lambda} W_{n,k,i_n} \subseteq \bigcap_{n=1}^{\infty} W_n = A$. Therefore M_X exists. Evidently $M_X = \prod_{k \in \Lambda} M_{Y_k}$.

4. Minimal boundaries for compactifications.

4.1. THEOREM. *Let X_0 be a locally compact, noncompact Hausdorff space, and let X be the one-point (x_∞) compactification of X_0. Then M_X exists if and only if M_{X_0} exists, and if they exist, then $M_X = M_{X_0} \cup \{x_\infty\}$ if and only if X is σ-compact.*

Proof. Since X is Hausdorff, X_0 is open in X, so we can invoke Theorem 3.4 to show that if M_X exists, then M_{X_0} exists. Now we prove the reverse implication. First we note that if x is a G_δ point in X_0, then it is a G_δ in X since X_0 is open in X. Assume that M_{X_0} exists, and let A be a nonvoid closed G_δ subset of X. Then $A_0 = A \cap X_0$ is a closed G_δ in X_0. If $A_0 = \varnothing$, then $A = \{x_\infty\}$, which is thus a G_δ point in X. Otherwise $A_0 \neq \varnothing$, and by the hypothesis, there is a G_δ point y in A_0. Then y is a G_δ point in X, with the result that M_X exists. By the definition of the topology on one-point compactification, x_∞ is a G_δ point if and only if X is σ-compact, so that the last statement of the theorem is a consequence of Corollary 2.8.

We mention that if X_0 is the space of ordinals less than the first uncountable, with the order topology, then X is the space of ordinals less than or equal to the first uncountable, and $M_X = X_0 = M_{X_0}$, while if $X_0 = (0, 1]$ with the usual topology, then $X = [0, 1]$, and $M_X = [0, 1] \neq (0, 1] = M_{X_0}$.

Next we turn to the Stone-Čech compactification βX of an arbitrary completely regular Hausdorff space X. This compactification always exists for such X [3, p. 86].

4.2. LEMMA. *Let X be pseudo-compact. Then x is a G_δ in X if and only if x is a G_δ in βX.*

Proof. If $x \in \beta X$ and is a G_δ, then $x \in X$, since $\beta X - X$ has no G_δ points [3, p. 132]. But this means that x is a G_δ in X. On the other hand, if $x \in X$ is a G_δ, then surely x is a G_δ in βX, since otherwise the Hausdorff topology on βX would yield a function $f \in C_s(\beta X)$ such that $B_f \cap X = \varnothing$. But since X is pseudocompact, this is impossible.

4.3. THEOREM. *Assume that X is not compact. We have the following:*

1. *If X is not pseudo-compact, then $M_{\beta X}$ does not exist.*

2. *If X is pseudo-compact, then $M_{\beta X}$ exists if and only if M_X exists, and in this case, $M_{\beta X} = M_X$.*

Proof. 1. If X is not pseudo-compact, then there is an unbounded function h on X such that h is continuous. Let $g = \max(1, h)$, $f = 1/g$, so that $f: X \to (0, 1]$ is continuous, and $[f(X)]^- = [0, 1]$. Let f_1 be the continuous extension of f to βX. Then $(1 - f_1) \in C_s(\beta X)$ and $B_{1-f_1} \cap X = \varnothing$. However, no point in $\beta X - X$ can ever be a G_δ. Thus the closed nonempty G_δ set B_{1-f_1} in βX has no G_δ point, and βX therefore has no minimal boundary.

2. Assume that M_X exists, and let A be a nonvoid closed G_δ in βX. Then $A \cap X$

is a nonvoid closed G_δ in X, so by hypothesis contains a point which is a G_δ in X. But such points are G_δ in βX, by Lemma 4.2. Hence βX has a minimal boundary. Conversely, assume that $M_{\beta X}$ exists, and let A be a nonvoid closed G_δ in X. By Lemma 2.4 there is a function $f \in C_s(X)$ such that $B_f \subseteq A$. If f_1 is its continuous extension to βX, then $B_{f_1} \cap X \subseteq A$, and by hypothesis there exists a G_δ point x in B_{f_1}. However, x cannot be in $\beta X - X$ since x is a G_δ, so that $x \in A$ and is a G_δ in X. Therefore M_f exists. Furthermore, since $\beta X - X$ has no G_δ points, Corollary 2.8 provides us with the final equality $M_{\beta X} = M_X$.

One of the consequences of Theorem 4.3 is that if N, Q, and R are the spaces of integers, rationals, and reals, respectively, under their ordinary topologies, then $M_{\beta N}$, $M_{\beta Q}$, and $M_{\beta R}$ do not exist, although $M_N = N$, $M_Q = Q$, and $M_R = R$.

For an arbitrary space Y, let $C_b(Y)$ consists of all real-valued bounded continuous functions on Y, with supremum norm and pointwise multiplication. Henceforth assume that X is compact and that M_X exists. We ask under what conditions each function in $C_b(M_X)$ has a unique continuous extension to X. In other words, when are $C_b(M_X)$ and $C(X)$ naturally identifiable?

4.4. Theorem. *The following are equivalent*:

1. $C_b(M_X)$ *is identifiable with* $C(X)$.

2. M_X *is pseudo-compact and* $\beta M_X = X$.

3. M_X *is pseudo-compact and* $f \in C(MX)$ *is uniformly continuous with respect to the uniform structure on* X *restricted to* M_X.

Proof. If $\beta M_X \neq X$, then by the Stone-Čech compactification theorem, $C_b(M_X)$ is not identifiable with $C(X)$. Furthermore, if $C_b(M_X)$ and $C(X)$ are identifiable, then [2, Theorem IV.6.26] shows that X and βM_X are homeomorphic, so that $M_{\beta M_X}$ exists. But then Theorem 4.3 tells us that M_X is pseudo-compact. Hence 1 implies 2. The converse is true by the Stone-Čech compactification theorem. Next we show that 2 implies 3. If $f \in C_b(M_X) = C(M_X)$, then let f_0 be the continuous extension of f to $\beta M_X = X$. Since X is compact, f_0 is uniformly continuous on X, so f is uniformly continuous on the restricted uniformity on M_X. Hence 2 implies 3. Conversely, assume that 3 holds, and let $f \in C(M_X)$. Then f is uniformly continuous with respect to the given uniform structure on M_X, so has a unique continuous extension f_0 to the compact space X by [6, Theorem 6.26]. But this is tantamount to the statement that $X = \beta M_X$. Thus 3 implies 2.

Let us note that there is at least one nontrivial example which satisfies the conditions of Theorem 4.4: if X is the space of ordinals less than or equal to the first uncountable, with its order topology, then $\beta M_X = X$ and M_X is pseudo-compact, and $M_X \neq X$, and $C_b(M_X)$ is identifiable with $C(X)$. However, X may be such that M_X is pseudo-compact without $C_b(M_X)$ and $C(X)$ being identifiable. To create such a space, we let Y and Z each be distinct copies of the ordinals less than or equal to the first uncountable, endowed with the order topology, and let the first uncountables be denoted by y_0 and z_0 in Y and Z respectively. Assume that $y_0 = z_0$.

Let $X = Y \cup Z$, with neighborhoods of any $y < y_0$ and any $z < z_0$ just the neighborhoods they normally have in Y and Z respectively. Let the neighborhoods of $y_0 = z_0$ be the union of neighborhoods of y_0 in Y and z_0 in Z. Then $M_X = X - \{y_0\}$ $= X - \{z_0\}$, and M_X inherits its pseudo-compactness from Y and Z. However, $\beta M_X \neq X$. For take $f: M_X \rightarrow [0, 1]$ defined by $f(Y - \{y_0\}) = 0$ and $f(Z - \{z_0\}) = 1$. Then f is continuous on M_X and has no continuous extension to X, so that $\beta M_X \neq X$. However, βM_X is the two-point compactification of $(Y - \{y_0\}) \cup (Z - \{z_0\})$. By [2, Theorem IV.6.26], βM_X and X are not homeomorphic; therefore $C_b(M_X)$ and $C(X)$ are not identifiable.

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