

THE WORD PROBLEM AND CONSEQUENCES FOR THE BRAID GROUPS AND MAPPING CLASS GROUPS OF THE 2-SPHERE⁽¹⁾

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1. Introduction. Artin [2] solved the word problem for the group $B_n(E^2)$ of n -string braids on the plane with presentation

$$\langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} : \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \ (i+1 < j) \rangle$$

by giving a normal form for such braids together with an effective procedure for putting a braid into this form. Fox and Neuwirth [9] showed that no braid in $B_n(E^2)$ has finite order. Fadell and one of the authors [8] obtained a presentation of the group $B_n(S^2)$ of n -string braids on the 2-sphere by adding the relation

$$\sigma_1 \sigma_2 \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_2 \sigma_1 = 1$$

to the presentation of $B_n(E^2)$, and showed that finite order braids exist for each $n > 1$. Magnus [10] derived a presentation of the group M_n of mapping classes of the n -punctured 2-sphere from that of $B_n(S^2)$ by adding the relation $(\sigma_1 \sigma_2 \cdots \sigma_{n-1})^n = 1$. A mapping class of the n -punctured 2-sphere can be visualized as an n -string braid between concentric 2-spheres, where the inner sphere is free to execute full revolutions.

In this paper we modify Artin's effective procedure for putting a braid on E^2 into normal form in order to solve the word problems for $B_n(S^2)$ and M_n . The resulting normal forms are then used to prove the following theorems.

THEOREM. *If $n > 1$, $B_n(S^2)$ has a unique braid of order 2 which generates its center. If $n > 2$, $B_n(S^2)$ has a braid of finite order m if and only if m divides $2n$, $2n-2$ or $2n-4$; and if $n > 3$ and $m > 2$, there are infinitely many braids of each such order m .*

THEOREM. *If $n > 2$, M_n is centerless. If $n > 2$, M_n contains a mapping class of period m if and only if m divides n , $n-1$ or $n-2$. Further, if $n > 3$ and $m > 1$, then there are infinitely many mapping classes of each such period m .*

Given that $B_n(E^2)$ has no elements of finite order, our results are algebraic consequences of the presentations of $B_n(S^2)$ and M_n .

2. Preliminaries. The configuration space of a manifold X is the space $F_n(X)$ of n -tuples of distinct points of X . The symmetric group Σ_n of degree n acts freely

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on $F_n(X)$ by permuting coordinates, and this action induces a regular covering of the orbit space $F_n(X)/\Sigma_n$ by the configuration space $F_n(X)$ (we will assume that X is a connected 2-manifold). The fundamental group of $F_n(X)/\Sigma_n$ is called the *n-string braid group of X* , and is denoted $B_n(X)$ [7], [9]. The covering map $F_n(X) \rightarrow F_n(X)/\Sigma_n$ induces an exact sequence

$$1 \longrightarrow K_n(X) \longrightarrow B_n(X) \xrightarrow{\alpha} \Sigma_n \longrightarrow 1$$

where $K_n(X)$ is the fundamental group of $F_n(X)$. The group $K_n(X)$ will be identified with its image as a normal subgroup of $B_n(X)$. If $b \in B_n(X)$, we will speak of $\alpha(b)$ as the *permutation associated with b* .

Let Σ'_{n-1} be the subgroup of Σ_n consisting of the permutations which leave the symbol 1 fixed, and let $D_n(X)$ be the fundamental group of $F_n(X)/\Sigma'_{n-1}$. There are associated covering maps

$$F_n(X) \rightarrow F_n(X)/\Sigma'_{n-1} \rightarrow F_n(X)/\Sigma_n$$

so that $D_n(X)$ may be viewed as a subgroup of $B_n(X)$ containing $K_n(X)$. The projection of $F_n(X)$ onto $F_{n-1}(X)$ which deletes the first coordinate induces a map of $F_n(X)/\Sigma'_{n-1}$ onto $F_{n-1}(X)/\Sigma_{n-1}$ which induces a homomorphism $j: D_n(X) \rightarrow B_{n-1}(X)$. The kernel of j will be denoted $A_n(X)$, and the action of j will be referred to as *killing the first string*. Algebraically, $D_n(X)$ is the inverse image of Σ'_{n-1} under the epimorphism α .

The definition of $B_n(X)$ generalizes the concept of an n -string braid on the plane E^2 which was introduced by Artin in 1925 [1], [2]. Artin showed that if an n -string braid on E^2 is viewed as a motion of n points p_1, \dots, p_n in E^2 (by means of a representative path in $F_n(E^2)$), then this motion extends to an isotopy $\{f_t: 0 \leq t \leq 1\}$ of E^2 having compact support, where $f_0 = 1$ and f_1 permutes the points p_1, \dots, p_n among themselves [2, Theorem 6]. The homeomorphism f_1 induces an automorphism ψ of the fundamental group of the n -punctured plane $E^2 - \{p_1, \dots, p_n\}$; that is, an automorphism of a free group $\mathcal{F}_n = \langle t_1, \dots, t_n \rangle$. The automorphism ψ has the following properties:

- (i) Each $\psi(t_i)$ is a conjugate $T_i^{-1}t_iT_i$ of some t_j , where $T_i \in \mathcal{F}_n$.
- (ii) $\psi(t_1t_2 \cdots t_n) = t_1t_2 \cdots t_n$.

In this way, $B_n(E^2)$ is isomorphic with the group of all automorphisms of \mathcal{F}_n having these two properties [2, Theorems 14, 15 and 16]. (The analogous representation for the n -string braid group of the 2-sphere is not faithful because it annihilates the center. Compare Theorem I.4.1 [5].)

For $1 \leq i < n$, let σ_i be the braid in $B_n(E^2)$ which corresponds to the automorphism of \mathcal{F}_n taking t_i to t_{i+1} and t_{i+1} to $t_i^{-1}t_it_{i+1}$, while leaving all other t_j fixed. For $1 \leq i < j \leq n$, define a_{ij} to be the braid

$$\sigma_i^{-1}\sigma_{i+1}^{-1} \cdots \sigma_{j-2}^{-1}\sigma_{j-1}^2\sigma_j^{-2} \cdots \sigma_{i+1}\sigma_i$$

where $a_{i,i+1}$ is understood to be σ_i^2 , and a_{ji} is defined to be a_{ij} . In these terms, the

epimorphism α has the explicit expression $\alpha(\sigma_i) = (i\ i+1)$, so that each a_{ij} is in the kernel $K_n(E^2)$ of α .

The group $B_n(E^2)$ is generated by $\sigma_1, \dots, \sigma_{n-1}$ subject to the following defining relations:

- (i) $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $1 < i+1 < j < n$;
- (ii) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1 \leq i < n-1$

[1], [3], [4], [8], [9]. The group $A_n(E^2)$ is freely generated by a_{12}, \dots, a_{1n} , and the exact sequence

$$1 \longrightarrow A_n(E^2) \longrightarrow D_n(E^2) \xrightarrow{j} B_{n-1}(E^2) \longrightarrow 1$$

splits, where $D_n(E^2)$ is generated by $a_{12}, \dots, a_{1n}, \sigma_2, \dots, \sigma_{n-1}$, and where $j(\sigma_i) = \sigma_i$ for $i=2, \dots, n-1$ [4, p. 654]. Notice that here $B_{n-1}(E^2)$ is generated by $\sigma_2, \dots, \sigma_{n-1}$. As a result, the *fundamental exact sequence* of $B_n(E^2)$,

$$1 \longrightarrow A_n(E^2) \longrightarrow K_n(E^2) \xrightarrow{j} K_{n-1}(E^2) \longrightarrow 1,$$

obtained by restricting j to $K_n(E^2)$, is split exact and therefore each braid k in $K_n(E^2)$ is uniquely expressible in the form $k = \mathcal{A}_n \mathcal{A}_{n-1} \cdots \mathcal{A}_2$, where \mathcal{A}_{n-t+1} is a word in the free generators $a_{i,t+1}, \dots, a_{i,n}$ [2]–[4]. We will refer to this expression as the *Artin normal form* for k . Beginning with k as a word in $\sigma_1, \dots, \sigma_{n-1}$ there are effective procedures [2]–[4] for explicitly computing the Artin normal form for k . We will refer to such a procedure as *combing the braid* k . Observe that the word problem for the presentation of $B_n(E^2)$ given above is solved in two ways. For to determine whether a braid b (given as a word in $\sigma_1, \dots, \sigma_{n-1}$) is the identity, one can compute explicitly the automorphism of \mathcal{F}_n determined by b [11, p. 174], or, having determined whether b is in $K_n(E^2)$ by computing $\alpha(b)$, one can find the Artin normal form for b .

It is known that all of the above results for $B_n(E^2)$ are obtainable by algebraic means from the presentation [3], [4].

Viewing the 2-sphere S^2 as the compactified plane, there is an embedding of $F_n(E^2)$ in $F_n(S^2)$ which induces a homomorphism β from $B_n(E^2)$ to $B_n(S^2)$, where it may be assumed that β preserves permutations; $\alpha\beta = \alpha$. For simplicity we will usually use the same notation for a braid in $B_n(E^2)$ as for its image in $B_n(S^2)$ under β . We will usually omit the symbol “ S^2 ” and write B_n for $B_n(S^2)$, etc.

It is shown in [8, p. 245] that a presentation for B_n is obtained by adjoining the single relation $\sigma_1\sigma_2 \cdots \sigma_{n-1}\sigma_{n-1} \cdots \sigma_2\sigma_1 = 1$ to the presentation for $B_n(E^2)$. Furthermore, if $n \geq 4$, A_n is the free group of rank $n-2$ having presentation

$$\langle a_{12}, \dots, a_{1n} : a_{12} \cdots a_{1n} = 1 \rangle,$$

and D_n is generated by $a_{12}, \dots, a_{1n}, \sigma_2, \dots, \sigma_{n-1}$ [8] (for algebraic proofs see [13, p. 36]). The sequence

$$1 \longrightarrow A_n \longrightarrow D_n \xrightarrow{j} B_{n-1} \longrightarrow 1$$

does not generally split, as we shall see in §4. The restriction of j to K_n induces the *fundamental exact sequence of B_n* :

$$1 \longrightarrow A_n \longrightarrow K_n \xrightarrow{j} K_{n-1} \longrightarrow 1.$$

Using topological methods it can be shown that this sequence splits if $n \geq 4$ [8, p. 256]. Letting $s: K_{n-1} \rightarrow K_n$ denote a splitting map, there is a semidirect product decomposition $K_n = A_n s(K_{n-1})$. However, the splitting map s cannot be induced by the inclusion map $K_{n-1}(E^2) \rightarrow K_n(E^2)$ [8, p. 256]. Therefore the question of *effectively* expressing a braid $k \in K_{n-1}$ as a product $k = \mathcal{A}_n \cdot sj(k)$, $\mathcal{A}_n \in A_n$, depends on explicit formulas for s on the generators of K_{n-1} and ultimately on a procedure, such as Artin's, for expressing a braid in K_{n-1} in terms of the a_{ij} 's. Rather than pursue this, we shall use Artin's combing procedure to derive a normal form for braids in K_n .

The center of $B_n(E^2)$ is the infinite cyclic group generated by $(\sigma_1 \sigma_2 \cdots \sigma_{n-1})^n$ [2, p. 121], [4, p. 658] and its image under the homomorphism $\beta: B_n(E^2) \rightarrow B_n$ is the cyclic group of order 2 generated by the *Dirac braid* $\Delta_n = (\sigma_1 \sigma_2 \cdots \sigma_{n-1})^n$ [6, p. 239], [8, p. 248]. Thus the center of B_n contains Δ_n . From the fact that the k -string braid group on E^2 generated by $\sigma_i, \sigma_{i+1}, \dots, \sigma_{i+k-2}$ has $(\sigma_i \cdots \sigma_{i+k-2})^k$ as the generator of its center and has defining relations which also hold in the subgroup of B_n generated by $\sigma_i, \sigma_{i+1}, \dots, \sigma_{i+k-2}$ ($i \leq n-k+1$), we obtain, by computations found in [8, p. 247], the following

LEMMA 2.1. *In B_n ,*

(i) $(\sigma_i \cdots \sigma_{i+k-2})^k = a_{i,i+1} \cdots a_{i,i+k-1} (\sigma_{i+1} \cdots \sigma_{i+k-2})^{k-1}$ and commutes with σ_j ($i \leq j \leq i+k-2$).

(ii) *The Dirac braid $\Delta_n = (\sigma_1 \cdots \sigma_{n-2})^{n-1} = (a_{12} a_{13} \cdots a_{1,n-1})(a_{23} \cdots a_{2,n-1}) \cdots (a_{n-2,n-1})$.*

LEMMA 2.2. *In B_n ,*

(iii) $(\sigma_i \cdots \sigma_{k-1})^r \sigma_j (\sigma_i \cdots \sigma_{k-1})^{-r} = \sigma_{j+r}$ ($i \leq j < k-r$),

(iv) $(\sigma_2 \cdots \sigma_{k-1})^r a_{1,m} (\sigma_2 \cdots \sigma_{k-1})^{-r} = a_{1,m+r}$ ($m \leq k-r$),

and with the indices taken mod n ,

(v) $a_{i,i+1} a_{i,i+2} \cdots a_{i,i+n-1} = 1$,

(vi) $(\sigma_1 \cdots \sigma_{n-1})^r a_{ij} (\sigma_1 \cdots \sigma_{n-1})^{-r} = a_{i+r,j+r}$,

(vii) *Any $n-2$ of $a_{1,i}, \dots, a_{i-1,i}, a_{i,i+1}, \dots, a_{in}$*

freely generate a subgroup of B_n , if $n \geq 4$.

Proof. From the easily derived

$$(\sigma_1 \cdots \sigma_{k-1}) \sigma_j (\sigma_1 \cdots \sigma_{k-1})^{-1} = \sigma_{j+1} \quad (j < k-1)$$

we obtain (iii) and

$$(*) \quad (\sigma_1 \cdots \sigma_{k-1}) a_{ij} (\sigma_1 \cdots \sigma_{k-1})^{-1} = a_{i+1,j+1} \quad (i, j < k).$$

Now (iv) follows from the fact that, by (*),

$$(\sigma_1 \cdots \sigma_{k-1})a_{1,j}(\sigma_1 \cdots \sigma_{k-1})^{-1} = a_{2,j+1} = \sigma_1 a_{1,j+1} \sigma_1^{-1} \quad (j < k).$$

We next observe that, taking indices mod n ,

$$a_{i,i+1}a_{i,i+2} \cdots a_{i,i+n-1} = \sigma_i \sigma_{i+1} \cdots \sigma_{n-1} \sigma_{n-1} \cdots \sigma_1 \sigma_1 \cdots \sigma_{i-1} = 1,$$

since, by [2, p. 119], if $i < j$, then

$$a_{i,j} = a_{j,i} = \sigma_i^{-1} \cdots \sigma_{j-2}^{-1} \sigma_{j-1}^2 \sigma_{j-2} \cdots \sigma_i = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}.$$

We now obtain (vi) from (*) and the fact that $(\sigma_1 \cdots \sigma_{n-1})a_{in}(\sigma_1 \cdots \sigma_{n-1})^{-1} = a_{1,i+1}$ since, by (v), $a_{in} = a_{i,n-1}^{-1} \cdots a_{i,i+1}^{-1} a_{i-1,i}^{-1} \cdots a_{1,i}^{-1}$. Finally, we recall that any $n-2$ of a_{12}, \dots, a_{1n} freely generate A_n , $n \geq 4$, and note that since, by (vi), the inner automorphism determined by $(\sigma_1 \cdots \sigma_{n-1})^{i-1}$ maps $a_{1,j}$ onto $a_{i,i+j-1}$, (vii) follows.

COROLLARY. In B_4 , $(\sigma_1 \sigma_2 \sigma_3) a_{12} (\sigma_1 \sigma_2 \sigma_3)^{-1} = a_{14} \Delta_4$ and $(\sigma_1 \sigma_2 \sigma_3) a_{14} (\sigma_1 \sigma_2 \sigma_3)^{-1} = a_{12}$.

Proof. We only need observe that

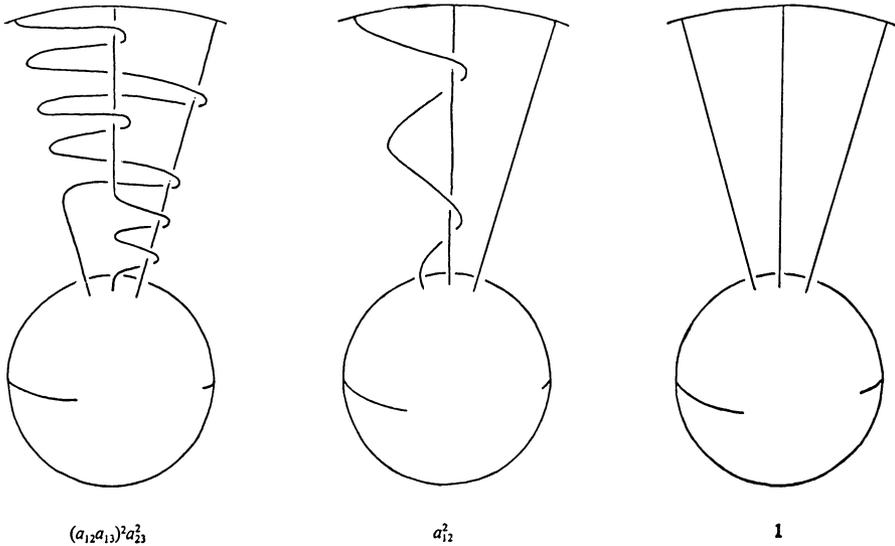
$$(\sigma_1 \sigma_2 \sigma_3) a_{12} (\sigma_1 \sigma_2 \sigma_3)^{-1} = a_{23} = a_{13}^{-1} a_{12}^{-1} \Delta_4 = a_{14} \Delta_4$$

by (ii) of Lemma 2.1, and (v) and (vi) of Lemma 2.2.

Let $Q_n = \{p_1, \dots, p_n\}$ be a set of n distinct points of S^2 and let $H(S^2, Q_n)$ be the group of orientation preserving homeomorphisms h of S^2 such that $h(Q_n) = Q_n$. Let $D(S^2, Q_n)$ be the normal subgroup of $H(S^2, Q_n)$ consisting of homeomorphisms which are isotopic to the identity, keeping Q_n (pointwise) fixed. The mapping class group M_n of the n -punctured sphere is defined to be the quotient group $H(S^2, Q_n)/D(S^2, Q_n)$. Obviously $M_0 = M_1 = 1$, while M_2 is cyclic of order 2. Magnus has shown [10, p. 624] that a presentation for M_n is obtained by adjoining to the presentation for $B_n(S^2)$ the single relation $(\sigma_1 \cdots \sigma_{n-1})^n = 1$. There is thus an epimorphism $\delta_n: B_n \rightarrow M_n$ with kernel generated by the Dirac braid Δ_n .

3. Normal form and center. In this section we solve the word problem for the n -string braid group of the 2-sphere by producing a normal form for braids in $B_n(S^2)$ and an effective procedure for transforming a word in the generators $\sigma_1, \dots, \sigma_{n-1}$ into this form. Using the normal form we show that for $n \geq 3$ the Dirac braid Δ_n is the only braid of order 2 in $B_n(S^2)$ and that Δ_n generates the center of $B_n(S^2)$. Similarly we solve the word problem for the mapping class group of the n -punctured 2-sphere and show that for $n \geq 3$, M_n is centerless. Everything proved here depends ultimately on the presentation for $B_n(S^2)$ given in [8] and on the algorithm given by Artin for computing the normal form of a braid in $B_n(E^2)$, where that braid is given in terms of $\sigma_1, \dots, \sigma_{n-1}$.

We observe that, since each relation in the generators of $B_n(E^2)$ holds for the corresponding generators of $B_n(S^2)$, Artin's "combing" procedure for putting a plane braid into normal form, when applied to a braid in $B_n(S^2)$, yields an



equivalent n -string braid on S^2 . However, equivalent braids in S^2 may be transformed into distinct Artin forms. For example, the braids shown in the figure are in Artin normal form and distinct when viewed as 3-string braids on E^2 , yet are equivalent as braids in $B_3(S^2)$.

THEOREM 3.1 (NORMAL FORM). *If b is an n -string braid on the 2-sphere, $n \geq 3$, then b is uniquely expressible in the form $b = m \mathcal{A}_n \mathcal{A}_{n-1} \cdots \mathcal{A}_4 \Delta_n^\varepsilon$, where m depends only on the permutation $\alpha(b)$, \mathcal{A}_i is in the group freely generated by $a_{n-i+1, n-i+2}, \dots, a_{n-i+1, n-1}$, and ε is 0 or 1. If b is expressed in terms of $\sigma_1, \dots, \sigma_{n-1}$, then there is an effective procedure for putting b into this form with m expressed in terms of $\sigma_1, \dots, \sigma_{n-1}$.*

Proof. We will first show that each braid in B_n can be effectively expressed in the prescribed form and then show that this form is, in fact, unique. The normal form we obtain will be a normal form with respect to a “cross section” θ of α , where by this we mean any function $\theta: \Sigma_n \rightarrow B_n$ such that $\alpha\theta = 1$, $\theta(P)$ is a fixed word in $\sigma_1, \dots, \sigma_{n-1}$ for each $P \in \Sigma_n$, and $\theta(1) = 1$.

NOTE. There is a cross section θ of $\alpha: B_n \rightarrow \Sigma_n$ such that if $b \in B_n$ and $\alpha(b)$ moves no symbol which is less than r or greater than s , then $\theta\alpha(b)$ can be expressed in terms of $\sigma_r, \sigma_{r+1}, \dots, \sigma_{s-1}$. To see this, we will decompose $\alpha(b)$ into disjoint cycles of length > 1 , where a particular cycle contains the least symbol which is moved by $\alpha(b)$ and is involved in no preceding cycle. Each of these cycles can be expressed as a product of transpositions involving the smallest symbol in that cycle and finally each such transposition (ij) , where $i < j$, can in turn be expressed in the form

$$(i \ i+1)(i+1 \ i+2) \cdots (j-2 \ j-1)(j-1 \ j)(j-2 \ j-1) \cdots (i+1 \ i+2)(i \ i+1).$$

For example, if $b = \sigma_4 \sigma_3^{-1} \sigma_3^3 \sigma_4 \sigma_7^{-1} \sigma_5^{-1} \sigma_6^2 \sigma_8$, then

$$\begin{aligned} \alpha(b) &= (3645)(798) = (36)(34)(35)(79)(78) \\ &= (34)(45)(56)(45)(34) \cdot (34) \cdot (34)(45)(34) \cdot (78)(89)(78) \cdot (78). \end{aligned}$$

This procedure gives $\alpha(b)$ canonically as a product of transpositions involving adjacent symbols, where we choose not to allow cancellations. We define $\theta_\alpha(b)$ to be the braid obtained from this expression for $\alpha(b)$ by simply replacing the transposition $(i \ i + 1)$ by σ_i . Thus for our example, $\theta_\alpha(b) = \sigma_3 \sigma_4 \sigma_5 \sigma_4 \sigma_3^3 \sigma_4 \sigma_3 \sigma_7 \sigma_8 \sigma_7^2$.

Let b be a braid in B_n which is expressed in the σ_i 's and define m and k by $m = \theta_\alpha(b)$ and $k = m^{-1}b$. Then since $\alpha(m) = \alpha(b)$, $k \in K_n$ and $\theta_\alpha(k) = 1$. Applying the Artin "combing" procedure to k , as written in $\sigma_1, \dots, \sigma_{n-1}$, we obtain $k = \mathcal{A}'_n \mathcal{A}''_{n-1} \dots \mathcal{A}'_2$ where \mathcal{A}'_i is a power product of $a_{n-i+1, n-i+2}, \dots, a_{n-i+1, n}$. Using Lemma 2.2 (v), we can replace each occurrence of $a_{i, n}$ in \mathcal{A}''_{n-i+1} by a word in the a_{ij} 's, where $1 \leq j < n, i \neq j$, and then rewrite the resulting expression in the σ_i 's, $1 \leq i \leq n-2$. We speak of this operation as "freeing the last string" of k . We next apply Artin's "combing" procedure to k , here considered as a plane braid on its first $n-1$ strings so as to insure that its n th string remains free, and obtain $k = \mathcal{A}'_n \mathcal{A}'_{n-1} \dots \mathcal{A}'_3$, where \mathcal{A}'_i is a power product of $a_{n-i+1, n-i+2}, \dots, a_{n-i+1, n-1}$ and $\mathcal{A}'_3 = a_n^{-q, n-1}$ for some integer q .

Recall, from Lemma 2.1, that $(\sigma_i \dots \sigma_{n-2})^{n-i}$ commutes with each $\sigma_j, i \leq j \leq n-2$, and hence with \mathcal{A}'_{n-i+1} and that $(\sigma_i \dots \sigma_{n-2})^{n-i} = a_{i, i+1} \dots a_{i, n-1} (\sigma_{i+1} \dots \sigma_{n-2})^{n-i-1}$, so that $(\sigma_i \dots \sigma_{n-2})^{(n-i)q} \mathcal{A}'_{n-i+1} = \mathcal{A}'_{n-i+1} (a_{i, i+1} \dots a_{i, n-1})^q (\sigma_{i+1} \dots \sigma_{n-2})^{(n-i-1)q}$. Using this and the fact that $\Delta_n = (\sigma_1 \dots \sigma_{n-2})^{n-1}$, we see that

$$\Delta_n^q \cdot k = (\sigma_1 \dots \sigma_{n-2})^{(n-1)q} \mathcal{A}'_n \dots \mathcal{A}'_3 = \mathcal{A}'_n \dots \mathcal{A}'_4,$$

where $a_n^{q, n-1} \mathcal{A}'_3 = 1$ and

$$\mathcal{A}'_{n-i+1} = \mathcal{A}'_{n-i+1} (a_{i, i+1} \dots a_{i, n-1})^q.$$

But then $b = mk = m \mathcal{A}'_n \dots \mathcal{A}'_4 \Delta_n^q = \mathcal{A}'_n \dots \mathcal{A}'_4 \Delta_n^e$, where $e = 0$ or 1 according as q is even or odd.

To see that each step in the described process is effectively computable, we consider a braid k in K_n expressed as a word in $\sigma_1, \dots, \sigma_{n-1}$ and enumerate the steps as follows: "Comb" k into Artin form; free the last string; "comb" the resulting braid (the last string remaining free); and finally, in order to eliminate the occurrences of $a_{n-2, n-1}$, multiply by a suitable even power of Δ_n , say $\Delta_n^{2r} = 1$, by postmultiplying by Δ_n^r and then premultiplying by Δ_n^r so as to move the "loop" $(a_{i, i+1} \dots a_{i, n-1})^r$ of Δ_n^r into juxtaposition with \mathcal{A}'_{n-i+1} . As pointed out in §2, "combing" is an effective procedure, while the other two steps are obviously effective.

The proof of uniqueness involves an induction on the number of strings. Of course it suffices to consider braids in K_n .

Since B_3 has order 12 [8, p. 248] and $B_3/K_3 = \Sigma_3$ has order 6, K_3 has order 2. Therefore $K_3 = \langle \Delta_3 : \Delta_3^2 = 1 \rangle$ because $\Delta_3 \in K_3$. This settles the question of uniqueness in case $n = 3$.

Before discussing the general case we note some properties of the homomorphism $j: D_n \rightarrow B_{n-1}$. For this purpose take B_{n-1} to be generated by $\sigma_2, \dots, \sigma_{n-1}$, so that if $1 < i < k \leq n$, then σ_i and a_{ik} ambiguously denote braids in D_n or B_{n-1} . As a result of the formulas given in §2, $j(a_{1i}) = 1$ if $1 < i \leq n$, and $j(a_{ik}) = a_{ik}$ if $1 < i < k \leq n$. By Lemma 2.2 (vii), $a_{i,i+1}, \dots, a_{i,n-1}$ freely generate a subgroup of K_n , and thus, if $i > 1$, j carries this group isomorphically onto the subgroup of K_{n-1} freely generated by the same named elements $a_{i,i+1}, \dots, a_{i,n-1}$ of K_{n-1} . Furthermore, by Lemma 2.1 (ii) $j(\Delta_n) = (a_{23} \cdots a_{2,n-1}) \cdots (a_{n-2,n-1}) = \Delta_{n-1}$.

If $b \in D_n$ then $\theta_\alpha(b)$ is expressible in terms of $\sigma_2, \dots, \sigma_{n-1}$ and hence $j\theta_\alpha(b) = \theta_\alpha(jb)$. Therefore if one applies j to a braid $b \in D_n$ expressed in the normal form $b = \theta_\alpha(b) \mathcal{A}_n \mathcal{A}_{n-1} \cdots \mathcal{A}_4 \Delta_n^\epsilon$ one obtains the normal form $j(b) = \theta_\alpha(jb) j(\mathcal{A}_{n-1}) \cdots j(\mathcal{A}_4) \Delta_{n-1}^\epsilon$ in K_{n-1} .

Suppose now that $n \geq 4$ and assume the conclusion of the theorem for K_{n-1} . Let $\mathcal{A}_n \cdots \mathcal{A}_4 \Delta_n^\epsilon$ and $\mathcal{A}'_n \cdots \mathcal{A}'_4 \Delta_n^\delta$ be normal forms for a braid k in K_n . Since these normal forms are carried by j onto normal forms for the braid $j(k)$ in K_{n-1} , then by the inductive assumption, $\Delta_{n-1}^\epsilon = \Delta_{n-1}^\delta$ and $j(\mathcal{A}_{n-i+1}) = j(\mathcal{A}'_{n-i+1})$ for $2 \leq i \leq n-3$. Now \mathcal{A}_{n-i+1} and \mathcal{A}'_{n-i+1} are elements of the subgroup of K_n freely generated by $a_{i,i+1}, \dots, a_{i,n-1}$ and, as was shown above, the restriction of j to this free group is an isomorphism. Therefore, $\mathcal{A}_{n-i+1} = \mathcal{A}'_{n-i+1}$ for $2 \leq i \leq n-3$ and $\Delta_n^\epsilon = \Delta_n^\delta$. But then $\mathcal{A}_n = \mathcal{A}'_n$ and the proof is complete.

LEMMA 3.2. *If $n \geq 3$, the only braid of finite order > 1 in the group K_n is the Dirac braid Δ_n .*

Proof. This lemma will be proved by induction on the number of strings. If $n = 3$, the conclusion follows from the fact that K_3 is generated by Δ_3 (see the proof of Theorem 3.1). Suppose now that $n \geq 4$ and assume the conclusion of the lemma for K_{n-1} . Let b be a braid of finite order in K_n , where b has normal form $\mathcal{A}_n \mathcal{A}_{n-1} \cdots \mathcal{A}_4 \Delta_n^\epsilon$. The homomorphism $j: K_n \rightarrow K_{n-1}$ which kills the first string carries b to a braid $j(b)$ which also has finite order, where K_{n-1} consists of braids on the last $n-1$ strings. As observed in the proof of Theorem 3.1, $j(b) = \mathcal{A}_{n-1} \cdots \mathcal{A}_4 \Delta_{n-1}^\epsilon$ is in normal form in K_{n-1} . But then by the inductive assumption, $j(b)$ is 1 or Δ_{n-1} , and hence $b = \mathcal{A}_n \Delta_n^\epsilon$. Since Δ_n is central and has order 2, $b \Delta_n^\epsilon$ must have finite order and, since $b \Delta_n^\epsilon = \mathcal{A}_n$ is a braid in the free group A_n ($n \geq 4$), we see that $b \Delta_n^\epsilon = 1$.

We shall see in Lemma 4.8 that Δ_n is the only finite order n -string braid whose associated permutation fixes as many as three symbols.

COROLLARY. *There is an effective procedure for determining if a given braid b in B_n has finite order.*

Proof. One simply computes the normal form for b^r , where r is the order of the permutation $\alpha(b)$. Notice that this procedure does not depend on the fact (proved topologically in [9]) that $B_n(E^2)$ has no elements of finite order. Since the proof of Lemma 3.2 can be modified to show that $K_n(E^2)$ has no elements of finite order, there is an algebraic procedure for showing that a given braid in $B_n(E^2)$ does not have finite order. This raises the

QUESTION. Can the nonexistence of braids of finite order in $B_n(E^2)$ be derived algebraically from the presentation of $B_n(E^2)$? (It can be shown that it suffices to prove algebraically that if p is an odd prime, then $B_p(E^2)$ has no elements of order p .)

The next result follows from the fact that two permutations in $\Sigma_n = \alpha(B_n)$ which have the same number of cycles of each length are conjugate.

LEMMA 3.3. *If P is a permutation on n symbols having the same number of cycles of each length as the permutation $\alpha(b)$ associated with an n -string braid b of finite order m , then there exists an n -string braid of order m with associated permutation P . If there are only a finite number r of n -string braids having finite order m and associated permutation P , then there are precisely rs n -string braids of order m having associated permutations with the same number of cycles of each length as P , where s is the number of conjugates of P in Σ_n .*

The n -string braids having associated permutations which leave each of the symbols $1, 2, \dots, k$ fixed form a subgroup, say D_n^k , of D_n since the associated permutations form a subgroup of $\alpha(D_n)$, the group of permutations on n symbols which leave the symbol 1 fixed. We will next show that deletion of the first k strings of the braids of D_n^k yields a natural homomorphism of D_n^k onto B_{n-k} .

LEMMA 3.4. *The function ρ_k defined by*

$$\rho_k(\sigma_i) = \sigma_i, \quad \text{if } i > k,$$

and, for $i < j$,

$$\begin{aligned} \rho_k(a_{ij}) &= 1 && \text{if } i \leq k, \\ &= a_{ij} && \text{if } i > k \end{aligned}$$

is a homomorphism of D_n^k onto B_{n-k} , where B_{n-k} is here generated by $\sigma_{k+1}, \dots, \sigma_{n-1}$.

Proof. If $b \in D_n^k$, then $\alpha(b)$ fixes each of the symbols $1, 2, \dots, k$ and b can be written in normal form without using $\sigma_1, \sigma_2, \dots, \sigma_k$, as was noted in the proof of Theorem 3.1. According to Lemma 2.1 (ii), Δ_n can be expressed in terms of the $a_{j,m}$, $1 \leq j < m < n$. But then, if $k < j < m < n$, the $a_{j,m}$ can in turn be expressed in terms of $\sigma_{k+1}, \dots, \sigma_{n-2}$. We thus see, by considering the normal form for an arbitrary braid in D_n^k , that D_n^k is generated by $\sigma_{k+1}, \dots, \sigma_{n-1}$ and the a_{ij} , where $1 < i \leq k$ and $i < j < n$.

The exact sequence

$$1 \longrightarrow A_n \longrightarrow D_n \xrightarrow{j_n} B_{n-1} \longrightarrow 1,$$

(cf. §2) induces the exact sequence

$$1 \longrightarrow A_n \longrightarrow D_n^k \xrightarrow{j_n} D_{n-1}^{k-1} \longrightarrow 1,$$

where D_{n-1}^{k-1} is the group of all braids on the last $n-1$ strings having permutations which leave each of the symbols $2, 3, \dots, k$ fixed. Inductively we obtain the exact sequences

$$1 \longrightarrow A_{n-i} \longrightarrow D_{n-i}^{k-i} \xrightarrow{j_{n-i}} D_{n-i-1}^{k-i-1} \longrightarrow 1$$

for $i=0, 1, \dots, k-2$ and finally the exact sequence

$$1 \longrightarrow A_{n-k+1} \longrightarrow D_{n-k+1}^1 \xrightarrow{j_{n-k+1}} B_{n-k} \longrightarrow 1.$$

The composition $\rho_k = j_{n-k+1} \cdots j_{n-1} j_n$ is thus seen to be the desired homomorphism of D_n^k onto B_{n-k} , and ρ_k is given, as in the statement of the theorem, by specifying its action on a set of generators of D_n^k .

NOTE. In the proof of the following theorem and in the sequel, we utilize the normal form of Theorem 3.1 with respect to various convenient choices of the cross section θ .

THEOREM 3.5. *The only braid of order 2 in $B_n, n \geq 3$, is the Dirac braid Δ_n .*

Proof. If $n=3$, then B_3 , as a *ZS*-metacyclic group of order 12 [8, p. 248], contains a single element of order 2. We will proceed by induction on n . Suppose that there exists a braid b , other than Δ_4 , of order 2 in B_4 . Then by Lemma 3.2, $\alpha(b)$ has order 2 and hence we may assume that $\alpha(b)=(34)$ or $(13)(24)$ according to Lemma 3.3. The former possibility will be ruled out when we verify the inductive step so we assume $\alpha(b)=(13)(24)$. Now if $b \neq \Delta_4$ has order 2, then so does $b\Delta_4$, hence we may assume that $b = (\sigma_1\sigma_2\sigma_3)^2 \mathcal{A}_4$, where $\mathcal{A}_4 = w(a_{12}, a_{14})$, a word in the free generators a_{12} and a_{14} of A_4 . By the corollary to Lemma 2.2, $(\sigma_1\sigma_2\sigma_3)^2 a_{1,i} (\sigma_1\sigma_2\sigma_3)^{-2} = a_{1,i} \Delta_4$ when $i=2$ or 4 , and therefore $b^2 = (\sigma_1\sigma_2\sigma_3)^4 [w(a_{12}, a_{14})]^2 \Delta_4^r$, where r is the length of the word w . But then, since $(\sigma_1\sigma_2\sigma_3)^4 = \Delta_4$, it follows that $[w(a_{12}, a_{14})]^2 = \Delta_4^{r+1}$ and that $w(a_{12}, a_{14})$, as a finite order braid in the free group A_4 , is the identity braid. Thus $b^2 = (\sigma_1\sigma_2\sigma_3)^4 = \Delta_4 \neq 1$, a contradiction.

Suppose that $n \geq 5$, that the result is known for braids on one or two less strings, and that there exists a braid b , other than Δ_n , of order 2 in B_n . Then $\alpha(b)$ has order 2 in Σ_n .

Case 1. If n is odd, or if n is even but $\alpha(b)$ is not the product of $n/2$ disjoint transpositions, then we may assume that $\alpha(b)$ leaves the symbol 1 fixed. Consequently $b = m \mathcal{A}_n \cdots \mathcal{A}_4 \Delta_n^\epsilon$ where m is a word in $\sigma_2, \dots, \sigma_{n-1}$ such that $\alpha(b) = \alpha(m)$. But then, on killing the first string of b , we obtain $j(b) = m \mathcal{A}_{n-1} \cdots \mathcal{A}_4 \Delta_{n-1}^\epsilon$ according to Lemma 2.1, where $j(b)$ is a braid in B_{n-1} which is here generated by $\sigma_2, \dots, \sigma_{n-1}$. Hence by the inductive assumption $m = \mathcal{A}_{n-1} = \cdots = \mathcal{A}_4 = 1$ and

consequently $b = \mathcal{A}_n \Delta_n^\epsilon$. Thus $\mathcal{A}_n^2 = \Delta_n^{2\epsilon}$, and \mathcal{A}_n , as a braid of finite order in A_n , is the identity braid. But then $b = \Delta_n^\epsilon$, a contradiction.

Case 2. If n is even and $\alpha(b)$ is the product of $n/2$ disjoint transpositions, then we may assume that $\alpha(b) = (12)(34) \cdots (n-2 \ n-1)$ and hence that $b = \sigma_1^{-1} \sigma_3 \cdots \sigma_{n-2} \mathcal{A}_n \cdots \mathcal{A}_4 \Delta_n^\epsilon$. Since σ_1 commutes with $\mathcal{A}_{n-2} \cdots \mathcal{A}_4$, we see that

$$b^2 = \sigma_1^{-2} \sigma_3 \sigma_5 \cdots \sigma_{n-1} \mathcal{A}'_{n-1} \mathcal{A}'_n \mathcal{A}_{n-2} \cdots \mathcal{A}_4 \sigma_3 \sigma_5 \cdots \sigma_{n-1} \mathcal{A}_n \cdots \mathcal{A}_4,$$

where $\mathcal{A}'_{n-1} = \sigma_1 \mathcal{A}_n \sigma_1^{-1}$, $\mathcal{A}'_n = \sigma_1 \mathcal{A}_{n-1} \sigma_1^{-1}$, and hence, by Lemma 3.4, that

$$\rho_2(b^2) = (\sigma_3 \sigma_5 \cdots \sigma_{n-1} \mathcal{A}_{n-2} \cdots \mathcal{A}_4)^2 = 1.$$

But then, by the inductive assumption,

$$\sigma_3 \sigma_5 \cdots \sigma_{n-1} \mathcal{A}_{n-2} \cdots \mathcal{A}_4 = \Delta_{n-2} \quad \text{or} \quad 1,$$

contrary to $\alpha \rho_2(b) \neq 1$.

LEMMA 3.6. *If an n -string braid b has finite order m , then its associated permutation $\alpha(b)$ has order m or $m/2$ according as m is odd or even. Further, for each odd k , if b has order $2k$, then $b \Delta_n$ has order k ; while if b has odd order k , then $b \Delta_n$ has order $2k$.*

Proof. If b has finite order m and $\alpha(b)$ has order k , then b^k , as a finite order braid in K_n , is Δ_n or 1 according to Lemma 3.2. Thus $m = 2k$ or k and hence, if m is odd, $m = k$. However if m is even, then $m = 2k$, since the assumption that $m = k$ implies, by Theorem 3.5, that $b^{k/2} = \Delta_n$, contrary to $\alpha(b)$ having order k . The second conclusion follows from the fact that Δ_n is central and is the unique order 2 braid in B_n .

We recall, from the discussion preceding Lemma 2.1, that if $n \geq 3$, then the cyclic group of order 2 generated by the Dirac braid Δ_n is contained in the center $Z(B_n)$ of B_n .

THEOREM 3.7. *If $n \geq 3$, then the center of B_n is the cyclic subgroup of order 2 generated by the Dirac braid. If $n \geq 2$, then the center of B_n is the unique subgroup of order 2 in B_n .*

Proof. We first note that, for $n \geq 3$, the symmetric group Σ_n of degree n is centerless and hence that $Z(B_n)$ is contained in the kernel K_n of the homomorphism $\alpha: B_n \rightarrow \Sigma_n$.

In particular, since K_3 was determined in proving Theorem 3.1, we have $\langle \Delta_3: \Delta_3^2 = 1 \rangle \subseteq Z(B_3) \subseteq K_3 = \langle \Delta_3: \Delta_3^2 = 1 \rangle$.

We now assume the result holds for B_{n-1} , $n \geq 4$, and consider an arbitrary braid $z \neq 1$ in $Z(B_n)$, where z has normal form $\mathcal{A}_n \mathcal{A}_{n-1} \cdots \mathcal{A}_4 \Delta_n^\epsilon$, since $z \in K_n$. The homomorphism $j: K_n \rightarrow K_{n-1}$ with kernel A_n maps $z \Delta_n^\epsilon$ onto $\mathcal{A}_{n-1} \cdots \mathcal{A}_4 \in Z(B_{n-1})$, where B_{n-1} is here generated by $\sigma_2, \dots, \sigma_{n-1}$. By the inductive assumption, $\mathcal{A}_{n-1} \cdots \mathcal{A}_4 = \Delta_{n-1}^\eta$, where $\eta = 0$ or 1. But then $\mathcal{A}_{n-1} \cdots \mathcal{A}_4 \Delta_{n-1}^\eta = 1$ and is in

normal form (in B_{n-1}), so that $\mathcal{A}_{n-1} = \dots = \mathcal{A}_4 = 1$ and $\eta = 0$. Since the homomorphism j restricted to A_i is, for $i = 4, \dots, n-1$, an isomorphism between free groups of rank $i-2$, $z\Delta_n^e = \mathcal{A}_n$ and is in the center of B_n , which is contrary to A_n being free of rank $n-2 \geq 2$ unless $\mathcal{A}_n = 1$. Thus $z = \Delta_n$. Finally, since B_2 is a cyclic group of order 2 in which $\Delta_2 = \sigma_1^2$ is the identity and $B_1 = \pi_1(S^2) = 1$, we completely determine the center of B_n and obtain, in view of Theorem 3.5, the second assertion.

Just as Artin's combing procedure can be applied to a braid of B_n , our procedure for putting a braid of B_n in normal form, when applied to a mapping class c in M_n , yields an equivalent mapping class, since each relation in the generators of B_n holds for the corresponding generators of M_n .

THEOREM 3.8 (NORMAL FORM). *If c is a mapping class of the n -punctured 2-sphere, $n \geq 3$, then c is uniquely expressible in the form $c = m\mathcal{A}_n \dots \mathcal{A}_4$, where m depends only on the permutation $\alpha(c)$ and \mathcal{A}_i is in the group freely generated by $a_{n-i+1, n-i+2}, \dots, a_{n-i+1, n-1}$. If c is expressed in terms of $\sigma_1, \dots, \sigma_{n-1}$, then there is an effective procedure for putting b into this form with m expressed in terms of $\sigma_1, \dots, \sigma_{n-1}$.*

Proof. Let c be a mapping class in M_n which is expressed in the σ_i 's. Then putting c into the normal form of a braid in B_n , we obtain $c = m\mathcal{A}_n \dots \mathcal{A}_4 \Delta_n^e$ and, since Δ_n is trivial in M_n , $c = m\mathcal{A}_n \dots \mathcal{A}_4$. This form is effectively computable, hence we have only to establish uniqueness. Assume $m\mathcal{A}_n \dots \mathcal{A}_4 = m'\mathcal{A}'_n \dots \mathcal{A}'_4$ in M_n and hence that $\mathcal{A}_n \dots \mathcal{A}_4 = \mathcal{A}'_n \dots \mathcal{A}'_4$. But then, since the n -string braids $\mathcal{A}_n \dots \mathcal{A}_4$ and $\mathcal{A}'_n \dots \mathcal{A}'_4$ have the same image under δ_n , $\mathcal{A}_n \dots \mathcal{A}_4 = \mathcal{A}'_n \dots \mathcal{A}'_4 \Delta_n^e$ in B_n and thus, by Theorem 3.1, $\mathcal{A}_i = \mathcal{A}'_i$ in B_n and hence in M_n , for $i = 4, \dots, n$.

By combining Theorem 4.6 of [12] with the representation of M_n as a factor group of $B_n(E^2)$ [10], we could obtain a different normal form for M_n , but we will not pursue this here.

THEOREM 3.9. *If $n \geq 3$, then M_n is centerless.*

Proof. As in proving Theorem 3.7, we note that the center of M_n is contained in the kernel of $\alpha: M_n \rightarrow \Sigma_n$, and proceed by induction on n . The basis for the induction follows from the fact that

$$\begin{aligned} M_3 &= \langle \sigma_1, \sigma_2 : \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \sigma_1\sigma_2^2\sigma_1 = 1, (\sigma_1\sigma_2)^3 = 1 \rangle \\ &= \langle \sigma_1, \sigma_2 : (\sigma_1\sigma_2)^3 = \sigma_1^2 = \sigma_2^2 = 1 \rangle, \end{aligned}$$

is the centerless group Σ_3 .

4. Finite orders. In this section we determine all possible orders of elements in B_n and M_n and indicate when there are only finitely many elements of a given order. The fact that there are no finite order braids in the plane will be used to show that there cannot exist braids of certain finite orders in B_n [9, p. 126]. To exhibit braids of a given order we will use a suitable factorization of the Dirac

braids. We will then use the normal form to determine when there are only finitely many braids of a given order.

LEMMA 4.1. *If $n \geq 5$ and d is a divisor of $2n$ which is odd and at least $n/(n-4)$ or even and at least $2n/(n-4)$, then there are infinitely many n -string braids of order d .*

Proof. Suppose $q < n-3$ divides n . Then the n -string braids $a_{12}^{-m} a_{q+1, q+2}^m$ are in normal form and hence the braids

$$b(m, q) = (\sigma_1 \cdots \sigma_{n-1})^q a_{12}^{-m} a_{q+1, q+2}^m = a_{q+1, q+2}^{-m} (\sigma_1 \cdots \sigma_{n-1})^q a_{q+1, q+2}^m$$

are, for given q , distinct for distinct m . We next note that the order of $b(m, q)$ is at least n/q , the order of its associated permutation $\alpha(b(m, q)) = (1 \ n \cdots 2)^q$, and that $[b(m, q)]^{n/q} = a_{q+1, q+2}^{-m} (\sigma_1 \cdots \sigma_{n-1})^n a_{q+1, q+2}^m = \Delta_n$. We thus see that if d is an odd divisor of $2n$ which is at least $n/(n-4)$, then, by Lemma 3.6, each of the infinitely many distinct $b(m, n/d) \Delta_n$ has order d ; while, if d is an even divisor of $2n$ which is at least $2n/(n-4)$, then each of the infinitely many distinct $b(m, 2n/d)$ has order d .

LEMMA 4.2. *If d divides $2n-2$, where $n \geq 4$ and $d \geq 3$, then there are infinitely many n -string braids of order d .*

Proof. Suppose $q < n-2$ divides n . Then, for given q , the n -string braids $b(m, q) = (\sigma_2 \cdots \sigma_{n-1})^q a_{12}^{-m} a_{1, q+2}^m = a_{1, q+2}^m (\sigma_2 \cdots \sigma_{n-1})^q a_{1, q+2}^{-m}$ are distinct for distinct m , each has order $\geq (n-1)/q$ and $[b(m, q)]^{n/q} = \Delta_n$. Thus if d is an odd divisor of $2n-2$ which is at least $(n-1)/(n-3)$, then each of the infinitely many distinct $b(m, (n-1)/d) \Delta_n$ has order d ; while, if $d \geq (2n-2)/(n-3)$ is an even divisor of $2n-2$, then each of the infinitely many distinct $b(m, (2n-2)/d)$ has order d . We next note that $(n-1)/(n-3) \leq 3$ for each $n \geq 4$, and that $(2n-2)/(n-3) \leq 4$ for each $n \geq 5$. Finally, if $n=4$, then the only divisor of $2n-2$ which is even and at least 4 is $6 = (2n-2)/(n-3)$.

LEMMA 4.3. *There are infinitely many 4-string braids of order 8.*

Proof. Since the 4-string braids $a_{12}^{-m} (a_{13}^{-1} a_{12}^{-1})^m$ are words in the free generators a_{12} and a_{13} of A_4 , the braids $b(m) = \sigma_1 \sigma_2 \sigma_3 a_{12}^{-m} (a_{13}^{-1} a_{12}^{-1})^m = a_{23}^{-m} \sigma_1 \sigma_2 \sigma_3 (a_{23} \Delta_4)^m$ are distinct for distinct m and each has order at least 4, the order of its associated permutation (4321). But then, since $[b(m)]^4 = a_{23}^{-m} (\sigma_1 \sigma_2 \sigma_3)^4 a_{23}^m \Delta_4^4 = \Delta_4^{4m+1}$, each of the infinitely many distinct $b(m)$ has order 8.

We next note that there are but two distinct braids of order 4 among the squares of the infinitely many 4-string braids of order 8 found above, since

$$[b(m)]^2 = a_{23}^{-m} (\sigma_1 \sigma_2 \sigma_3)^2 a_{23}^m = a_{23}^{-m} a_{14}^m (\sigma_1 \sigma_2 \sigma_3)^2 = (\sigma_1 \sigma_2 \sigma_3)^2 \Delta_4^m.$$

In proving the next result we will show that there are, in fact, no other 4-string braids of order 4 with associated permutation (13)(24).

THEOREM 4.4. *There are only six 4-string braids of order 4 having associated permutations which consist of two disjoint transpositions. However, if n is even and at*

least 6, then there are infinitely many n -string braids of order 4 with $n/2$ disjoint transpositions as associated permutation.

Proof. We first note that there are only three permutations on four symbols consisting of two disjoint transpositions and that if b is a 4-string braid of order 4 having associated permutation $\alpha(b)=(13)(24)$, then $b=(\sigma_1\sigma_2\sigma_3)^2w(a_{12}, a_{14})\Delta_4^\varepsilon$, where $w(a_{12}, a_{14})$ is a word in the free generators a_{12} and a_{14} of A_4 and $\varepsilon=0$ or 1. Since $(\sigma_1\sigma_2\sigma_3)^2a_{1,i}(\sigma_1\sigma_2\sigma_3)^{-2}=a_{1,i}\Delta_4$, for $i=2, 4$ (Corollary to Lemma 2.2), $\Delta_4=b^2=w(a_{12}\Delta_4, a_{14}\Delta_4)\Delta_4w(a_{12}, a_{14})=[w(a_{12}, a_{14})]^{2\Delta_4^{\eta+1}}$, where η is the length of $w(a_{12}, a_{14})$. But then $w(a_{12}, a_{14})$, as a finite order braid in the free group A_4 , is the identity braid and hence $b=(\sigma_1\sigma_2\sigma_3)^2\Delta_4^\varepsilon$, where $\varepsilon = 0$ or 1. The first conclusion now follows from Lemma 3.3.

In proving Lemma 4.1 we exhibited, for each even integer $n \geq 8$, infinitely many n -string braids with associated permutation $\alpha((\sigma_1\sigma_2 \cdots \sigma_{n-1})^{n/2})=(n \cdots 2 1)^{n/2}$, the product of $n/2$ disjoint cycles. The second conclusion now follows from the fact that the 6-string braids $b(m)=(\sigma_1\sigma_2 \cdots \sigma_5)^3(a_{12}a_{13}a_{14}a_{15})^m a_{34}^m$ have associated permutation $(14)(25)(36)$, are distinct for distinct m , and each has order 4 since, by (vi) of Lemma 2.2, $[b(m)]^2=(\sigma_1 \cdots \sigma_5)^3 a_{16}^{-m} a_{34}^m a_{34}^{-m} a_{16}^m (\sigma_1 \cdots \sigma_5)^3 = \Delta_6$.

The existence of infinitely many 4-string braids of order 4 depends, in view of the preceding theorem and Lemma 3.6, on our finding infinitely many such braids with a single transposition as associated permutation. This is established as a special case of the following

LEMMA 4.5. *If d divides $2n-4$, where $n \geq 4$ and $d \geq 3$, then there are infinitely many n -braids of order d .*

Proof. If $1 \leq q < n-2$, then the n -string braids $a_{12}^{m+1}a_{13}a_{14} \cdots a_{1,q+1}a_{1,q+2}^{-m}$ are reduced words in the free generators $a_{12}, \dots, a_{1,n-1}$ of A_n and hence the braids

$$\begin{aligned} b(m, q) &= a_{12}^{m+1}a_{13} \cdots a_{1,q+1}a_{1,q+2}^{-m}(\sigma_2 \cdots \sigma_{n-2})^q \\ &= a_{12}^m a_{12} a_{13} \cdots a_{1,q+1}(\sigma_2 \cdots \sigma_{n-2})^q a_{12}^{-m} \\ &= a_{12}^m \left[\prod_{i=0}^{q-1} (\sigma_2 \cdots \sigma_{n-2})^{-i} a_{12} (\sigma_2 \cdots \sigma_{n-2})^i \right] (\sigma_2 \cdots \sigma_{n-2})^q a_{12}^{-m} \\ &= a_{12}^m (a_{12} \sigma_2 \cdots \sigma_{n-2})^q a_{12}^{-m} \end{aligned}$$

are, for given q , distinct for distinct m . We next note that the order of $b(m, q)$ is at least $(n-2)/q$, the order of its associated permutation $((n-1) \cdots 3 2)^q$, and that $[b(m, q)]^{(n-2)/q} = a_{12}^m (a_{12} \sigma_2 \cdots \sigma_{n-2})^{n-2} a_{12}^{-m} = \Delta_n$, since $(a_{12} \sigma_2 \cdots \sigma_{n-2})^{n-2} = a_{12} \cdots a_{1,n-1} (\sigma_2 \cdots \sigma_{n-2})^{n-2} = (a_{12} \cdots a_{1,n-1})(a_{23} \cdots a_{2,n-1}) \cdots (a_{n-2,n-1}) = \Delta_n$. But then, if d is odd, each of the infinitely many distinct $b(m, (n-2)/d)\Delta_n$ has order d ; while, if d is even, each of the infinitely many distinct $b(m, (2n-4)/d)$ has order d .

We summarize our results in the following

LEMMA 4.6. *If $n \geq 3$, then there exist n -string braids of each order d dividing $2n, 2n-2$ or $2n-4$. Further, if $n \geq 4$ and $d > 2$, then there are infinitely many such braids.*

Proof. We first note that the Dirac braid Δ_n is, for $n \geq 3$, the unique n -string braid of order 2 and that, since B_3 is a ZS -metacyclic group of order 12 [8, p. 248], there are two 3-string braids of order 3, six of order 4 and two of order 6, in addition to Δ_3 and the identity. There exist, by Lemma 4.2, infinitely many 4-string braids of each of the orders 3 and 6 and, by Lemmas 4.5 and 4.1, infinitely many of each of the orders 4 and 8. It only remains for us to consider the second conclusion for $n \geq 5$.

If $n \geq 5$, then the second conclusion follows for divisors of $2n - 2$ and $2n - 4$ by Lemmas 4.2 and 4.5, and for divisors of $2n$ by Lemma 4.1 and the existence, by Lemma 4.5, of infinitely many 6-string braids of order 4.

We will use the next four lemmas to show that if m is any positive integer which does not divide $2n$, $2n - 2$ or $2n - 4$ then no n -string braid has order m .

LEMMA 4.7. *The subgroup of B_n generated by $\sigma_1, \sigma_2, \dots, \sigma_{n-3}$ is isomorphic to the corresponding subgroup of $B_n(E^2)$. Therefore no braid on S^2 having its last two strings free can have finite order > 1 .*

Proof. Primes will be used to distinguish braids on the plane from corresponding braids on the 2-sphere. Let $B_{n-3}(E^2)$ be the subgroup of $B_n(E^2)$ generated by $\sigma'_1, \sigma'_2, \dots, \sigma'_{n-3}$ and h the restriction of the epimorphism $\beta: B_n(E^2) \rightarrow B_n$ given by $h(\sigma'_i) = \sigma_i$ for $1 \leq i \leq n - 3$. The kernel of h is contained in $K_n(E^2)$, the kernel of $\alpha': B_n(E^2) \rightarrow \Sigma_n$, since h preserves permutations. Because $\sigma'_1, \dots, \sigma'_{n-3}$ generate $B_{n-3}(E^2)$, any braid b' in the kernel of h has Artin normal form

$$b' = w_n(a'_{12}, \dots, a'_{1,n-2}) \cdots w_4(a'_{n-3,n-2}),$$

a word in the a'_{ij} where $i < j \leq n - 2$. Now

$$h(b') = w_n(a_{12}, \dots, a_{1,n-2}) \cdots w_4(a_{n-3,n-2}),$$

which we recognize as the normal form for a braid in B_n . But then, since $h(b')$ is the identity braid in B_n , it follows that each of the words w_n, \dots, w_4 is the identity. Thus $b' = 1$ and hence h is an isomorphism.

There is however, for each $n \geq 3$, an n -string braid of finite order > 1 having its last string free. For example, each of the finite order n -string braids, $n \geq 4$, exhibited in Lemma 4.5 has its last string free, as does the order four 3-string braid σ_1 . We thus see that, although S^2 is the one point compactification of E^2 , the n -string braid group of E^2 is not just the subgroup of B_{n+1} consisting of those $(n + 1)$ -string braids which have a free "string at infinity".

LEMMA 4.8. *If the permutation associated with an n -string braid leaves at least three, but not all n , symbols fixed, then the braid does not have finite order.*

Proof. According to Lemma 3.3 it suffices to consider the subgroup G_n of B_n consisting of all braids having permutations which fix each of the symbols $n - 2$, $n - 1$ and n . If $b \in G_n$ then b has normal form $m\mathcal{A}_n \cdots \mathcal{A}_4 \Delta_n^e$ where m is (as was

noted in the proof of Theorem 3.1) expressible in terms of $\sigma_1, \dots, \sigma_{n-4}$. Let H_n consist of all braids in G_n which have normal form $m\mathcal{A}_n \cdots \mathcal{A}_4$. We will first show that H_n is a subgroup of G_n , and hence that G_n is the direct product of H_n and the center $\langle \Delta_n : \Delta_n^2 = 1 \rangle$ of B_n . We will next show that H_n is isomorphic to a subgroup of $B_n(E^2)$, and hence has no elements of finite order. It will then follow that a braid of finite order in G_n is a power of Δ_n and is thus associated with the identity permutation.

Each element in H_n is expressible in terms of the following braids:

$$(*) \quad \sigma_1, \sigma_2, \dots, \sigma_{n-4}; \quad a_{ij} \text{ for } 1 \leq i \leq n-3 \text{ and } i < j \leq n-1.$$

In fact, H_n is the group generated by the braids in (*) because the transformation formulas of Artin [2, Theorem 8, p. 120 and (29), p. 119] permit one to rearrange a word in the generators (*) until it is brought into the normal form of an element of H_n . Of course, these computations are valid in both B_n and $B_n(E^2)$ so that there is a subgroup $H_n(E^2)$ of $B_n(E^2)$ which corresponds isomorphically to H_n by associating a normal form $m\mathcal{A}_n \cdots \mathcal{A}_4$ in H_n with the same normal form considered as a braid in $B_n(E^2)$. In view of our choice of normal form, this isomorphism between H_n and $H_n(E^2)$ carries each of the generators in (*) onto the same named generators of $H_n(E^2)$.

By letting $G_n = K_n$ (i.e. $m = 1$) in the above proof, we obtain the following

COROLLARY. *Let A'_{n-i+1} be the subgroup of K_n freely generated by $a_{i,i+1}, \dots, a_{i,n-1}$, for $2 \leq i \leq n-3$. Then if $n \geq 3$, $A_n A'_{n-1} \cdots A'_4$ is a subgroup of K_n and $K_n = A_n A'_{n-1} \cdots A'_4 Z(B_n)$, where $Z(B_n) = \langle \Delta_n : \Delta_n^2 = 1 \rangle$ is a direct factor of K_n .*

LEMMA 4.9. *If the order m of a permutation P on n symbols does not divide $2n$, $2n-2$ or $2n-4$, then some power of P fixes each of at least three, but not all n , symbols.*

Proof. We first note that the hypothesis is not satisfied unless $n > 7$ and that if the length of some cycle of P is a multiple of each of the other cycle lengths, then $m \leq n$. Thus if $m > n$, then P has cycles Q and R such that the length q of Q is less than but does not divide the length r of R . But then, since R^r is the identity permutation, P^r fixes each of the r symbols moved by R , where $m > n \geq q + r > r > q > 1$.

We may thus assume that $m < n-2$ and that P fixes fewer than three symbols, then, since $m > 4$, P contains a cycle of length > 2 and hence a cycle of shortest length $s > 2$. If P has two transpositions or a transposition and a fixed symbol, then P^2 is not the identity permutation but fixes at least three symbols and we are finished. Thus we assume P , which fixes at most two symbols, either has no transposition and at most two fixed symbols, or has a single transposition and no fixed symbols. Now if some cycle of P has length $> s$, then P^s is not the identity permutation but fixes more than 3 symbols. If not, then P has, say, k cycles of length s and $ks = n, n-1$, or $n-2$. In any case, $m = s$ or $2s$ and is thus a divisor of $2n, 2n-2$, or $2n-4$, contrary to assumption.

SCHOLIUM. If P is a permutation of order m on n symbols such that P^r leaves at most two symbols fixed for each $r < m$, then all cycles of P have the same length s , except for the possible occurrence of at most two fixed symbols or, in case $s = 2^k > 2$, a single transposition.

Proof. It only remains for us to note that if all cycles of P , except for a single transposition, have length s which is not a positive power of 2, then either $s = 1$ and there is nothing to prove, since then $m = 2$, or an odd prime p divides s . But in that case, P^p fixes at least p symbols, where $2 < p \leq s < 2s = m$, contrary to assumption.

LEMMA 4.10. *If an n -string braid has finite order m , then m divides $2n$, $2n - 2$ or $2n - 4$.*

Proof. As noted in the discussion following Theorem 3.7, B_1 is trivial and B_2 is cyclic of order 2. We also note that the orders of 3-string braids, as listed in the proof of Lemma 4.6, satisfy the conclusion. Let b be an n -string braid of finite order m , where $n > 4$. Since each power of b has finite order, $(\alpha(b))^r = \alpha(b^r)$ fixes at most two symbols for each $r < m$, by Lemma 4.8, and, by the scholium to Lemma 4.9, all cycles of the permutation $\alpha(b)$ have the same length, say s , except possibly for at most two cycles of length 1 or, if $s = 2^k > 2$, a single transposition. In any case, s divides n , $n - 1$ or $n - 2$. But then, since by Lemma 3.6 $m = s$ or $2s$, it follows that m divides $2n$, $2n - 2$ or $2n - 4$.

Notice that we have not excluded the possibility that the permutation associated with some finite order braid might have cycles of equal length $2^k > 2$ except for a single transposition.

Summarizing the results of Lemmas 4.6 and 4.10 we obtain the following.

THEOREM 4.11. *If $n \geq 3$, then there exist n -string braids of order m if and only if m is a divisor of $2n$, $2n - 2$ or $2n - 4$. Further, there are infinitely many such braids if and only if $n \geq 4$ and $m > 2$.*

Recall that when X is E^2 the splitting of the exact sequence

$$(1) \quad 1 \rightarrow A_n(X) \rightarrow D_n(X) \rightarrow B_{n-1}(X) \rightarrow 1$$

induces a splitting of the fundamental exact sequence

$$(2) \quad 1 \rightarrow A_n(X) \rightarrow K_n(X) \rightarrow K_{n-1}(X) \rightarrow 1.$$

When X is S^2 , the following result implies that (1) does not generally split although, by a geometric argument [8, p. 256], or by arguments used in proving Lemma 4.8, (2) does.

THEOREM 4.12. *If $n \geq 6$ then B_n is not isomorphically embeddable in B_{n+1} .*

Proof. It suffices for us to note that if $n \geq 4$, then, by Theorem 4.11, B_n has a braid of order $2n - 4$ yet if $n \geq 6$, then B_{n+1} does not. For if there were an

$(n + 1)$ -string braid of order $2n - 4$, then, by Theorem 4.11, $2n - 4$ would divide $2n + 2$, $2n$ or $2n - 2$ with quotient ≥ 2 . But then $2(2n - 4) \leq 2n + 2$ and hence $n \leq 5$.

Assume that $n \geq 3$ and note that the exact sequence

$$1 \longrightarrow Z(B_n) \longrightarrow B_n \xrightarrow{\delta} M_n \longrightarrow 1$$

induces the exact sequence

$$(3) \quad 1 \rightarrow Z(B_n) \rightarrow K_n \rightarrow L_n \rightarrow 1$$

where L_n is the subgroup of all mapping classes in M_n having identity permutation. Using Theorem 3.8 and the corollary to Lemma 4.8, $K_n = L_n \times Z(B_n)$. Using this and the fact that $M_2 = \langle \sigma_1 : \sigma_1^2 = 1 \rangle$ we obtain the following

LEMMA 4.13. *If $n \geq 3$, then the exact sequence (3) splits and, if $n \geq 4$, $L_n = A_n A'_{n-1} \cdots A'_4$. Furthermore, if c is a mapping class in M_n of finite period m , then its associated permutation $\alpha(c)$ has order m .*

Suppose c is a mapping class of finite period k in M_n which is expressed as a power product of $\sigma_1, \dots, \sigma_{n-1}$. Putting c into the normal form of a braid in B_n , we obtain $c = m \mathcal{A}_n \cdots \mathcal{A}_4 \Delta_n^\varepsilon$. Thus $c = m \mathcal{A}_n \cdots \mathcal{A}_4$ is in normal form and hence, by the previous lemma $\alpha(m)$ is a permutation of order k . We consider the n -string braid $b = m \mathcal{A}_n \cdots \mathcal{A}_4 \Delta_n^\varepsilon$ and observe that if k is even, then $b^k = (m \mathcal{A}_n \cdots \mathcal{A}_4)^k \Delta_n^{k\varepsilon} = \Delta_n^n$ and, in view of Lemma 3.6, b has order $2k$; whereas if k is odd, then $(b \Delta_n)^k = \Delta_n^k$ and, in view of Lemma 3.6, $b \Delta_n$ has order $2k$. Thus a mapping class in M_n , $n \geq 3$, cannot have finite period k unless there is a braid in B_n of order $2k$, in which case k divides n , $n - 1$, or $n - 2$, by Lemma 4.10.

Conversely, suppose k is a divisor of n , $n - 1$ or $n - 2$. There are, by Lemma 4.6, n -string braids in B_n of order k or $2k$, according as k is odd or even, having permutations of order k . And, if $k > 1$ and $n \geq 4$, there are infinitely many such braids. Now δ_n maps these braids onto mapping classes in M_n having finite periods and hence period k , the order of their permutations. But δ_n is one or two to one according as k is odd or even. We have thus establishing the following

THEOREM 4.14. *If $n \geq 3$, then there exists a mapping class of the n -punctured 2-sphere having period m if and only if m is a divisor of n , $n - 1$ or $n - 2$. Further, there are infinitely many such mapping classes if and only if $n \geq 4$ and $m > 1$.*

The next result follows from the fact that, for $n \geq 4$, B_n has infinitely many elements of each of the orders 3 and 4, and a unique element of order 2, while M_n has infinitely many elements of each of the orders 2 and 3.

THEOREM 4.15. *If $n \geq 4$, then M_n is not isomorphically embeddable in B_n and hence the exact sequence*

$$1 \rightarrow \langle \Delta_n : \Delta_n^2 = 1 \rangle \rightarrow B_n \rightarrow M_n \rightarrow 1$$

does not split.

In view of Lemma 3.2, Theorem 4.4 and the following two results we see, by Lemma 3.3, that the braids of orders 2, 4 and 8 in B_4 can be completely classified by their normal forms.

THEOREM 4.16. *Each of the infinitely many 4-string braids of order 8 is a conjugate of $(\sigma_1\sigma_2\sigma_3)\Delta_4^\varepsilon$, where $\varepsilon=0$ or 1. In particular, if b has order 8 in B_4 and $\alpha(b)=(4321)$, then in normal form*

$$b = \sigma_1\sigma_2\sigma_3 a_{12}^{p_1} a_{14}^{q_1} a_{12}^{p_2} a_{14}^{q_2} \cdots a_{12}^{p_m} a_{14}^{q_m} \Delta_4^\varepsilon$$

where $p_i = -q_{m-i+1}$ ($i=1, 2, \dots, m$) and $p_i \neq 0$ ($i=2, \dots, m$).

Proof. By Lemma 3.3 the first statement in the theorem follows from the second. Since b^2 has order 4 and permutation (13)(24), $b^2 = (\sigma_1\sigma_2\sigma_3)^2 \Delta_4^\mu$ according to Theorem 4.4. In normal form, $b = \sigma_1\sigma_2\sigma_3 w(a_{12}, a_{14}) \Delta_4^\varepsilon$ and, by the corollary to Lemma 2.2,

$$b^2 = \sigma_1\sigma_2\sigma_3 w(a_{12}, a_{14}) w(a_{14}, a_{12}) \sigma_1\sigma_2\sigma_3 \Delta_4^\mu$$

where μ is 0 or 1 according as the exponent sum of the a_{12} 's in $w(a_{12}, a_{14})$ is even or odd. Thus $w(a_{12}, a_{14}) w(a_{14}, a_{12}) = \Delta_4^{-\mu} = 1$, since A_4 is free. We next note that if

$$w(a_{12}, a_{14}) = a_{12}^{p_1} a_{14}^{q_1} a_{12}^{p_2} a_{14}^{q_2} \cdots a_{12}^{p_m} a_{14}^{q_m}$$

then $p_i + q_{m-i+1} = 0$, since $w(a_{12}, a_{14})$ and $w(a_{14}, a_{12})$ are inverses, and set

$$\begin{aligned} v(a_{12}, a_{14}) &= a_{12}^{p_1} a_{14}^{q_1} \cdots a_{14}^{q_r} & \text{if } m = 2r, \\ &= a_{12}^{p_1} a_{14}^{q_1} \cdots a_{12}^{p_r} & \text{if } m = 2r - 1. \end{aligned}$$

But then, since $w(a_{12}, a_{14}) = v(a_{12}, a_{14}) [v(a_{14}, a_{12})]^{-1}$ and

$$\sigma_1\sigma_2\sigma_3 v(a_{12}, a_{14}) = v(a_{14}, a_{12}) \sigma_1\sigma_2\sigma_3 \Delta_4^m,$$

we see that

$$b = v(a_{14}, a_{12}) \sigma_1\sigma_2\sigma_3 [v(a_{14}, a_{12})]^{-1} \Delta_4^{m+\varepsilon}.$$

Considerations similar to those in the preceding proof give us the following

THEOREM 4.17. *Each of the infinitely many 4-string braids of order 4 with a single transposition as associated permutation is a conjugate of $\sigma_2 a_{12} \Delta_4^\varepsilon$, where $\varepsilon=0$ or 1. In particular, if b has order 4 in B_4 and $\alpha(b)=(23)$, then in normal form*

$$b = \sigma_2 a_{12} a_{13}^{p_1} a_{12}^{q_1} a_{13}^{p_2} a_{12}^{q_2} \cdots a_{13}^{p_m} a_{12}^{q_m} \Delta_4^\varepsilon$$

where $p_i = -q_{m-i+1}$ ($i=1, 2, \dots, m$) and $p_i \neq 0$ ($i=2, \dots, m$).

QUESTION. Can the braids of finite order in B_n be classified by their normal forms?

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