ON THE CORONA THEOREM AND ITS APPLICATION TO SPECTRAL PROBLEMS IN HILBERT SPACE(1)

BY

PAUL A. FUHRMANN

1. **Introduction.** Let T be a completely nonunitary contraction in a Hilbert space H and let H^∞ be the Banach algebra of all bounded analytic functions in the unit disc D. In [15] B. Sz.-Nagy and C. Foias constructed a functional calculus in this setting. For each T, a map $u \to u(T)$ was defined, the map being an algebra homomorphism of H^∞ into the algebra of bounded operators in H with the following properties:

(i) $\|u(T)\| \leqq \|u\|_\infty$.

(ii) If $\{u_n\}$ is a uniformly bounded set of functions satisfying $u_n(e^{it}) \to u(e^{it})$ a.e., then $u_n(T)$ converges to $u(T)$ in the strong operator topology.

(iii) $u(T)^* = \tilde{u}(T^*)$ where $\tilde{u}(z) = \overline{u(\bar{z})}$.

The spectral properties of this functional calculus, i.e. the determination of $\sigma(u(T))$ in terms of the operator T and the analytic behavior of u, have not been derived. Some progress in this direction has been made by C. Foias and W. Mlak [4], who proved that the classical spectral mapping theorem $\sigma(u(T)) = u(\sigma(T))$ holds if u can be extended continuously to those points on the unit circle that are in $\sigma(T)$.

The above spectral problem is the subject of this paper. At the sacrifice of the generality in which the Sz.-Nagy-Foias calculus was formulated, this problem has been completely solved. Instead of dealing with arbitrary, completely nonunitary contractions, we restrict ourselves to those contractions T whose adjoints can be represented as restrictions of the left shift to left invariant subspaces of $H^2(N)$ (see §2 for definitions). These operators comprise a fairly general class of operators since any contraction A in a Hilbert space H for which A^{*n} converges to zero in the strong operator topology is unitarily equivalent to an operator of this form. This is a refinement of a theorem of Rota [13] due to Sz.-Nagy and Foias [16] and Branges and Rovnyak [2]. To get our results we will have to assume that dim N is finite.

Let N be a separable Hilbert space. $H^2(N)$ is the Hardy class of order 2, i.e. the set of all N-valued square integrable functions on the unit circle whose Fourier

Presented to the Society, November 8, 1966 under the title *Some results in spectral theory in Hilbert space and the corona theorem*; received by the editors February 20, 1967.

(1) This paper was partially supported by the National Science Foundation under contract NSF GP-5455.

coefficients vanish for all negative indices. (For details we refer to [7].) The $H^2(N)$ norm is defined by

$$\|F\| = \left\{\frac{1}{2\pi}\int_0^{2\pi} \|F(e^{it})\|^2\, dt\right\}^{1/2}.$$

All functions in $H^2(N)$ have analytic continuations into the disc, and whenever convenient we will assume that the functions have been continued.

As usual a subspace of $H^2(N)$ is called (right) invariant if it is invariant under multiplication by z. The operator of multiplication by z in $H^2(N)$ is called the right shift. Its adjoint, the left shift, is the operator that sends $F(z)$ into $(F(z)-F(0))/z$. A subspace of $H^2(N)$ is called left invariant if it is invariant under the left shift. The orthogonal complement of a left invariant subspace is right invariant.

Now let K be a left invariant subspace of $H^2(N)$ and P the orthogonal projection of $H^2(N)$ onto K. If we embed $H^2(N)$ naturally in $L^2(N)$, then we consider P to be the orthogonal projection of $L^2(N)$ onto K. For each F in K and u in H^∞ we define

$$u(T)F = P(uF). \tag{1.1}$$

Clearly T^* is the left shift restricted to K. The definition above is a special case of the Sz.-Nagy-Foias calculus.

A basic tool in our approach is the Beurling-Lax representation theorem for invariant subspaces. Following Halmos [6], we define a rigid analytic function to be an N-contraction valued analytic function in the unit disc having a.e. partial isometries with a fixed initial space as boundary values on the unit circle. A rigid function is called inner if its boundary values are a.e. unitary operators.

We quote now the Beurling-Lax theorem [1], [9], [6].

THEOREM. *All invariant subspaces of $H^2(N)$ are of the form $SH^2(N)$ where S is a rigid analytic function.*

As was stated, the objects of our study are the operators $u(T)$ defined by (1.1). Now the orthogonal complement of K in $H^2(N)$ is an invariant subspace and thus there corresponds to it a rigid analytic function S. S of course is not unique. Our aim is to study $u(T)$ in terms of u and S. This idea of course is not new. In the scalar case ($\dim N=1$) the spectrum of T was determined in such fashion by Moeller [12], and his work has been generalized by Lax and Phillips [10], Helson [7] and Sz.-Nagy and Foias [16]. The analysis of $\sigma(u(T))$ comprises §2 of this paper. The Carleson corona theorem [3] is fundamentally used. In fact part of the proof of Theorem 2.3 taken in conjunction with [7, Theorem 11] suggests a possible generalization of the corona theorem to the case of matrix valued functions. This is taken up in §3.

One last remark is in order. Except for the H^∞ norm, no attempt has been made to distinguish in our notation the other norms appearing. We believe that it is clear from the context which norm is meant. We follow Helson however by always taking $\|F(e^{it})\|$ to be the norm of $F(e^{it})$ as a vector in N while reserving $\|F\|$ to the integrated $H^2(N)$ norm.

2. **Spectral analysis.** As will be seen later there is a marked difference in the spectral properties of T between the case that a rigid noninner function corresponds to $K^\perp$ and that to which the corresponding function is inner. In the case that S is inner there is a certain symmetry in the roles of T and T^*. The exact statement is given by Theorem 2.1. It is closely related to a theorem in [16, p. 42]. Throughout this paper, for any operator (scalar) valued analytic function A, we define

$$\tilde{A}(z) = A(\bar{z})^*.$$

THEOREM 2.1. *Let $K=H^2(N)\ominus SH^2(N)$ with S inner and T defined in K by* (1.1). *T is unitarily equivalent to the left shift in*

$$\tilde{K} = H^2(N)\ominus \tilde{S}H^2(N).$$

Proof. The space $L^2(N)$ has the following decomposition:

$$L^2(N) = K^2(N)\oplus K\oplus SH^2(N),$$

where $K^2(N)$ is the space of all conjugate, analytic, square, integrable, N-vector valued functions with mean values zero. For each F in $L^2(N)$ we define

$$(\tau F)(e^{it}) = e^{-it}S(e^{-it})^*F(e^{-it}).$$

τ is a unitary map in $L^2(N)$. Clearly τ maps $SH^2(N)$ onto $K^2(N)$, $K^2(N)$ onto $\tilde{S}H^2(N)$, and thus K onto $\tilde{K}$. The following is obvious:

$$\tau(e^{it}F(e^{it})) = e^{-it}S(e^{-it})^*(e^{-it}F(e^{-it})) = e^{-it}(\tau F)(e^{it}).$$

Moreover, if P and $\tilde{P}$ are the orthogonal projections on K and $\tilde{K}$ respectively, then $\tau P=\tilde{P}\tau$. It follows that $\tau PU=\tilde{P}U^*\tau$ where U is the right shift in $L^2(N)$. Clearly restricted to $\tilde{K}$, the operator $\tilde{P}U^*$ is the left shift; and thus the theorem is proved.

A theorem generalizing this one is stated in [5]. The proof is essentially the same.

LEMMA 2.1. *Let N be finite-dimensional, S inner, $d=\det S$; then $\det \tilde{S}(z)=\tilde{d}(z)$.*

Proof. Choose an orthonormal basis in N. In terms of the matrix representation of S, this is trivial.

In the following much use will be made of the eigenfunctions of T and T^*. The next theorem summarizes the needed information concerning them. In this connection we refer to [11].

THEOREM 2.2. *Let $K=H^2(N)\ominus SH^2(N)$ with S rigid and let λ be a complex number of modulus less than one.*

(a) *$\lambda\in\sigma_p(T^*)$ if and only if $S(\lambda)^*$ has a nontrivial null space. The normalized eigenfunctions of T^* have the form $(1-|\lambda|^2)^{1/2}x/(1-\bar{\lambda}z)$ where x is a unit vector in N for which $S(\lambda)^*x=0$.*

(b) *If S is inner, then $\lambda \in \sigma_p(T)$ if and only if $S(\lambda)$ has a nontrivial null space. The normalized eigenfunctions of T are of the form $(1-|\lambda|^2)^{1/2}(S(z)x/(z-\lambda))$ where x is a unit vector in N for which $S(\lambda)x=0$.*

Proof. (a) An eigenfunction of T^* with eigenvalue $\bar\lambda$ satisfies $T^*F=\bar\lambda F$ or $(F(z)-F(0))/z=\bar\lambda F(z)$, which means that $F(z)=F(0)/(1-\bar\lambda z)$. Thus $T^*-\bar\lambda$ has a null function if and only if for some x in N, $x/(1-\bar\lambda z)$ is orthogonal to $SH^2(N)$. The orthogonality condition is

$$0 = \frac{1}{2\pi}\int_0^{2\pi}\left(S(e^{it})G(e^{it}), \frac{x}{1-\bar\lambda e^{it}}\right)dt = \frac{1}{2\pi}\int_0^{2\pi}(S(e^{it})G(e^{it}), x)\frac{e^{it}}{e^{it}-\lambda}\,dt$$

$$= (S(\lambda)G(\lambda), x) = (G(\lambda), S^*(\lambda)x).$$

Since this is true for all G in $H^2(N)$, we infer that $x/(1-\bar\lambda z)$ is in K if and only if $S(\lambda)^*x=0$.

(b) While the statement can be easily proven directly, it is a corollary of Theorem 2.2(a) and Theorem 2.1. In fact, $\lambda-T$ has a null function if and only if $\tilde S(\bar\lambda)^*x=0$ for a nonnull x in N. But $\tilde S(\bar\lambda)^*=S(\lambda)$, so (b) is proved. Under the inverse of τ, $\tilde\tau$ (see Theorem 2.1), $\tilde\tau(x/(1-\bar\lambda e^{it}))=e^{-it}S(e^{-it})^*(x/(1-\bar\lambda e^{-it}))=S(e^{it})x/(e^{it}-\lambda)$, which exhibits the structure of the eigenfunctions of T. In both cases the normalization is obtained by a simple computation.

From now on we deal strictly with the case that N is finite-dimensional.

LEMMA 2.2. *Let S be a rigid analytic function. If S is not inner, then $S(z)$ is nowhere invertible in the unit disc.*

Proof. If S is not inner, det $S(e^{it})$ vanishes on a set of positive measure, hence, being in H^∞, vanishes identically. But det $S(z)=0$ is equivalent to the noninvertibility of $S(z)$.

COROLLARY 2.1. *Let $K=H^2(N)\ominus SH^2(N)$ with S a noninner rigid function, then $\sigma_p(T^*)$ is the open unit disc D.*

Proof. In a finite-dimensional space an operator is not invertible if and only if it has a nontrivial null space. By Lemma 2.2, for each z in D, $S(z)$ is not invertible, and we use Theorem 2.2(a). No point of the unit circle is in $\sigma_p(T^*)$ as T^* is completely nonunitary.

The following lemma will be needed in the proof of Theorem 2.3.

LEMMA 2.3. *Let A be a linear operator in an n-dimensional Hilbert space, then*

$$|\det A| \geqq \|A^{-1}\|^{-n} \tag{2.1}$$

where $\|A^{-1}\|^{-1}=0$ if A is not invertible.

Proof. If A is not invertible, (2.1) reduces to a triviality. Otherwise,

$$\det A = \prod_{i=1}^n a_i$$

where a_i are the eigenvalues of A. a_i^{-1} are the eigenvalues of A^{-1} and clearly $|a_i^{-1}| \leqq \|A^{-1}\|$. Thus $|a_i| \geqq \|A^{-1}\|^{-1}$, and by taking the product of all inequalities we get (2.1).

LEMMA 2.4. *Let S be inner and $d = \det S$, then $dH^2(N) \subset SH^2(N)$. In the terminology of Helson d is stronger than S.*

Proof. See Helson [7, Theorem 11]. It follows that $d(T)=0$.

The central result of this section is Theorem 2.3.

Let N be a finite-dimensional Hilbert space $K = H^2(N) \ominus SH^2(N)$ with S a rigid analytic function and $u(T)$ defined by (1.1) $d = \det S(z)$.

THEOREM 2.3. *$u(T)$ has a bounded inverse if and only if there exists a $\delta > 0$ such that*

$$|u(z)| + \|S(z)^{-1}\|^{-1} \geqq \delta \tag{2.2}$$

for all z in D.

Proof. If $|u(z)| + \|S(z)^{-1}\|^{-1} \geqq \delta$ for a $\delta > 0$, then by Lemma 2.3 there exists a $\delta^1 > 0$ such that $|u(z)| + |d(z)| \geqq \delta^1$. By the corona theorem [3] there exist two functions a, b in H^∞ such that $au + bd = 1$. In terms of operators this becomes

$$a(T)u(T) + b(T)d(T) = 1,$$

and as by Lemma 2.4 $d(T)=0$, $a(T)u(T) = u(T)a(T) = 1$. Thus $u(T)$ has a bounded inverse. Moreover the inverse of $u(T)$ is again of this form. This, if $\dim N = 1$, is a special case of a theorem of Sarason. (See Remark 2.1.)

To prove the converse assume there is no $\delta > 0$ for which (2.2) holds. Thus there are points λ_n in D for which

$$\lim \{|u(\lambda_n)| + \|S(\lambda_n)^{-1}\|^{-1}\} = 0.$$

We will show that there are functions F_n in K such that $\lim \|F_n\| = 1$ and $\lim \|u(T)^* F_n\| = 0$, which means that $u(T)$ cannot have a bounded inverse. The method is an elaboration of the one used by Moeller [12].

We assume first that S is inner. Since

$$\|S(\lambda)^{-1}\|^{-1} = \|S(\lambda)^{*-1}\|^{-1} = \min \{\|S(\lambda)^* x\| \mid x = 1\},$$

let x_n be such that $\|S(\lambda_n)^* x_n\| = \|S(\lambda_n)^{-1}\|^{-1}$, and let

$$E_n(z) = (1 - |\lambda_n|^2)^{1/2} x_n / (1 - \bar{\lambda}_n z).$$

E_n is a normalized eigenfunction of the left shift with $\bar{\lambda}_n$ as the respective eigenvalue. By Theorem 2.2(a) E_n is in K if and only if $S(\lambda_n)^* x_n = 0$. We decompose E_n into its components F_n in K and G_n in $K^\perp$. It is elementary to check that

$$F_n(z) = (1 - |\lambda_n|^2)^{1/2} (1 - S(z) S(\lambda_n)^*) x_n / (1 - \bar{\lambda} z)$$

and

$$G_n(z) = (1 - |\lambda_n|^2)^{1/2} S(z) S(\lambda_n)^* x_n / (1 - \bar{\lambda}_n z).$$

In the $H^2(N)$ norm $\|G_n\|=\|S(\lambda_n)^*x_n\|$, and since $\|F_n\|^2+\|G_n\|^2=1$, we have $\lim \|F_n\|=1$. Now $u(T)^*=\tilde{u}(T^*)$. If we denote by S the right shift in $H^2(N)$, it follows by a trivial computation that

$$u(S)^*E_n = \tilde{u}(S^*)E_n = \tilde{u}(\bar{\lambda}_n)E_n = \overline{u(\lambda_n)}E_n.$$

Hence

$$u(T)^*F_n-\overline{u(\lambda_n)}F_n = u(S)^*F_n-\overline{u(\lambda_n)}F_n = -(u(S)^*G_n-\overline{u(\lambda_n)}G_n).$$

Thus we get the following estimate,

$$\|u(T)^*F_n\| \leqq 2|u(\lambda_n)|+\|u\|_\infty\cdot\|G_n\|;$$

and as the right side tends to zero with increasing n the theorem is proved.

In the case that S is not inner, by Corollary 2.1 there is at least one eigenfunction of T^* for each point in the open unit disc; and no decomposition of the eigenfunction, as carried out above, is necessary.

It should be noted that the rigid function in the Beurling-Lax theorem is unique only up to multiplication on the right by a constant rigid function (by a constant unitary operator if S is inner). It is evident that the formulation of Theorem 2.3 is independent of the rigid function used.

Remark 2.1. The appearance of the corona theorem in the proof of Theorem 2.3 becomes more transparent by the following argument due to Ralph Gellar. Let K be a proper left invariant subspace of H^2, its orthogonal complement given by qH^2 where q is an inner function. Let T be defined by (1.1). The following theorem has been proved by D. Sarason [14].

Theorem. *Every bounded operator in K commuting with T is of the form $u(T)$ with u in H^∞.*

Since the inverse of an operator commuting with T also commutes with T, the inverse of $u(T)$, if it exists, is of the same form. Thus the operator $u(T)$ has an inverse if and only if, under the natural homomorphism of H^∞ onto H^∞/qH^∞, u is mapped into an invertible element. This is clearly the case if and only if u and q are not contained in a proper ideal of H^∞, i.e. if and only if there exist $a, b \in H^\infty$ such that $au+bq=1$. By the corona theorem this is equivalent to the existence of a $\delta>0$ such that, for all z in D, $|u(z)|+|q(z)|\geqq\delta$.

A few corollaries follow immediately from Theorem 2.3.

Corollary 2.2. *$\lambda\in\rho(u(T))$ if and only if there exists a $\delta>0$ for which $|\lambda-u(z)| + \|S(z)^{-1}\|^{-1}\geqq\delta$ for all z in D.*

Hence the resolvent set and the spectrum of $u(T)$ have been completely determined.

Corollary 2.3. *$u\in H^\infty$. $u(T^*)$ has a bounded inverse if and only if there exists a $\delta>0$ such that $|u(\bar{z})|+\|S(z)^{-1}\|^{-1}\geqq\delta$ for all z in D.*

Proof. $u(T^*)$ has a bounded inverse if and only if $u(T^*)^* = \tilde{u}(T)$ has. For $\tilde{u}(T)$ we apply Theorem 2.3.

The next corollary is the generalization of Moeller's [12] theorem in our context.

COROLLARY 2.4. *Assume S is inner, λ a complex number.*
(a) $|\lambda| < 1$, *then* $\lambda \in \sigma(T)$ *if and only if* $S(\lambda)$ *is not invertible*;
(b) $|\lambda| = 1$, *then* $\lambda \in \sigma(T)$ *if and only if* $S(z)$ *has no analytic continuation at* λ.

Proof. From [7, Theorem 11] it follows that there exists an $m > 0$ for which $\|S(z)^{-1}\|^{-1} \geqq m|d(z)|$.

From the inequality above and Lemma 2.3 it follows that $\|S(z)^{-1}\|^{-1}$ is bounded away from zero for all z in D sufficiently close to a point λ on the unit circle if and only if $|d(z)|$ is bounded away from zero there. For $d(z)$ being a scalar inner function, this condition is equivalent to the possibility of extending d analytically at the point λ. By Theorem 12 in [7] d has an analytic continuation at λ if and only if S has, which proves statement (b). (a) of course is trivial.

COROLLARY 2.5. $\lambda \in \sigma(T)$, $u \in H^\infty$.
(a) $|\lambda| < 1$, *then* $u(\lambda) \in \sigma(u(T))$;
(b) $|\lambda| = 1$ *and u extends continuously to* λ, *then* $u(\lambda) \in \sigma(u(T))$.

Let A be the subalgebra of H^∞ of functions having continuous extensions to the closed unit disc. The theorem that follows is not new (see [4]) but it is an easy corollary of Theorem 2.3. It is of course a form of the classical spectral mapping theorem.

THEOREM 2.4. $u \in A$, *then* $\sigma(u(T)) = u(\sigma(T))$.

Proof. Assume $\lambda \in \sigma(T)$. By Theorem 2.3 there are points λ_n in D such that $\lambda_n \to \lambda$ and $\|S(\lambda_n)^{-1}\|^{-1} \to 0$. It follows that

$$\lim \{|u(\lambda_n) - u(\lambda)| + \|S(\lambda_n)^{-1}\|^{-1}\} = 0$$

or that

$$u(\lambda) \in \sigma(u(T)).$$

Conversely, assume without loss of generality that $0 \in \sigma(u(T))$. There exist therefore points λ_n in D for which

$$|u(\lambda_n)| + \|S(\lambda_n)^{-1}\|^{-1} \to 0.$$

Let λ be a point of accumulation of the λ_n. Clearly $\lambda \in \sigma(T)$ and $u(\lambda) = 0$.

REMARK 2.2. By the same argument, $\sigma(u(T)) = u(\sigma(T))$ holds true even if u has a continuous extension only to the intersection of $\sigma(T)$ with the unit circle. Again this is proved in a more general context in [4].

In the rest of this section we will attempt to study the finer properties of the spectrum. We will assume now that $K^\perp = SH^2(N)$ is given by an inner function S, N is finite-dimensional, $u \in H^\infty$ and $d = \det S$.

THEOREM 2.5. $0 \in \sigma_p(u(T))$ *if and only if u and d have a nontrivial common inner factor.*

Proof. Assume $0 \in \sigma_p(u(T))$. Hence there exists a nonnull F in K for which $u(z)F(z) = S(z)G(z)$ for some G in $H^2(N)$. It follows that

$$u(z)M(z)F(z) = d(z)G(z),$$

$M(z)$ being defined by $M(z) = d(z)S(z)^{-1}$. Now if d and u have no common inner factor, we must have $M(z)F(z) = d(z)H(z)$ for some H in $H^2(N)$. In other words $F(z) = S(z)H(z)$ which means that $F \in K^\perp$ contrary to our assumption. Thus d and u must have a common inner factor.

Conversely, let us assume that u and d have a nontrivial common inner factor, then also u and q have a nontrivial inner factor, where q is the characteristic scalar inner function of S. (See [7, p. 81].) Let $q = a\psi$ and $u = b\psi$ where ψ is the common inner factor of q and u. Since q is the characteristic scalar inner function of S we cannot have $aH^2(N) \perp K$. Hence there exists a G in $H^2(N)$, for which the decomposition $aG = F + SH$ into components in K and $K^\perp$ respectively, gives a nonnull F. Obviously F is a null vector of $u(T)$ for $uF = b\psi F = ba\psi G - SuH = q(bG) - SuH$. Hence uF is in $K^\perp$ and $u(T)F = 0$.

As in Theorem 2.3 the foundation of this theorem is independent of the inner function used in the Beurling-Lax representation of $K^\perp$. The determinants of two inner functions corresponding to $K^\perp$ differ by a constant of modulus one.

COROLLARY 2.6. *Under the same assumptions,* $0 \in \sigma_p(u(T^*))$ *if and only if u and* $\tilde{d}$ *have a nontrivial common inner factor.*

Proof. By Theorem 2.1 T^* is unitarily equivalent to the operator in $\tilde{K} = H^2(N) \ominus \tilde{S}H^2(N)$ of multiplication by z followed by orthogonal projection into $\tilde{K}$. We apply Theorem 2.5 and note that by Lemma 2.1 det $\tilde{S} = \tilde{d}$.

COROLLARY 2.7. *u outer, then* $0 \notin \sigma_p(u(T))$.

The same is true in greater generality [15, Theorem 2].

COROLLARY 2.8. $\lambda \in \sigma_p(T)$ *implies* $u(\lambda) \in \sigma_p(u(T))$.

Proof. This follows from Property (ii) of the Sz.-Nagy-Foias calculus but also trivially from Theorem 2.5.

COROLLARY 2.9. *u inner,* λ *a complex number of modulus one, then*

$$\lambda \notin \sigma_p(u(T)).$$

Proof. For $\lambda - u$ is outer [8, p. 142]. Of course it follows that T itself has no eigenvalues on the unit circle. This is true of any completely nonunitary contraction.

COROLLARY 2.10. *Under the assumption that S is inner, the residual spectrum of* $u(T)$ *is empty for all u in* H^∞.

Proof. Without loss of generality we will show that $0 \notin \sigma_r(u(T))$. Now $0 \in \sigma_r(u(T))$ if and only if $u(T)$ is one-to-one and its range is not dense in K. Equivalently, $0 \notin \sigma_p(u(T))$ and $0 \in \sigma_p(u(T)^*) = \sigma_p(\tilde{u}(T^*))$. By Theorem 2.5 and Corollary 2.6, this means that u and d have no common nontrivial inner factor while $\tilde{u}$ and $\tilde{d}$ have, which is impossible.

REMARK 2.3. It should be noted that Corollary 2.10 is true only in the case of N being finite-dimensional and does not generalize. The above corollary points out the difference of dealing with inner functions as distinguished from noninner rigid functions. In general, for a noninner rigid function S, $\sigma_p(T)$ is at most the set of zeros of a Blaschke product, while $\sigma_p(T^*)$ is, by Corollary 2.1, the whole open unit disc; thus the residual spectrum of T is fairly large.

In the extreme case where $K = H^2(N)$ we have $\sigma_p(T^*) = \sigma_r(T) = D$. In this case the following is also true.

THEOREM 2.6. Let u be a nonnull function in H^∞, T the right shift in $H^2(N)$.
(a) $0 \notin \sigma_p(u(T))$,
(b) $0 \in \sigma_p(u(T^*))$ if and only if u is not outer,
(c) $0 \in \sigma_r(u(T))$ if and only if u is not outer.

Proof. (a) Obvious.

(b) $0 \in \sigma_p(u(T^*)) = \sigma_r(\tilde{u}(T)^*)$ if and only if the range of $\tilde{u}(T)$ is not dense in $H^2(N)$, which by the well-known theorems in [8, p. 101] is the case only if $\tilde{u}$ is not outer. But $\tilde{u}$ is outer if and only if u is.

(c) Follows from (a) and (b).

3. **The matrix corona theorem.** We outline here a generalization of the Carleson corona theorem [3] to the case of matrix valued analytic functions. The proof is by reduction to the scalar case via determinants. The availability of this version of the theorem enables us to tackle spectral problems of greater generality than those in §2. For a statement of some result in this direction see [5]. A more detailed exposition will be published separately.

Let N be an n-dimensional Hilbert space. Let A_i, $i = 1, \ldots, p$, be bounded N-operator valued analytic functions in the unit disc.

THEOREM 3.1. (a) *A necessary and sufficient condition for the existence of bounded N-operator valued analytic functions B_i such that $\sum_{i=1}^p B_i(z)A_i(z) = 1$ is the existence of a $\delta > 0$ for which*

$$\inf\left\{\sum_{i=1}^{p} \|A_i(z)x\| \mid x \in N, \|x\| = 1\right\} \geqq \delta \tag{3.1}$$

for all z in D.

(b) *A necessary and sufficient condition for the existence of bounded N-operator valued analytic functions B_i such that $\sum_{i=1}^p A_i(z)B_i(z) = 1$ is the existence of a $\delta > 0$ for which*

$$\inf\left\{\sum_{i=1}^{p} \|A_i(z)^*x\| \mid x \in N, \|x\| = 1\right\} \geqq \delta \tag{3.2}$$

for all z in D.

Proof. (a) The necessity part is simple. If no $\delta > 0$ exists for which (3.1) is true for all z in D, then there exist unit vectors x_n and points z_n in the disc such that

$$\lim_{n\to\infty} \sum_{i=1}^{p} \|A_i(z_n)x_n\| = 0.$$

Now $\sum_{i=1}^{p} B_i(z)A_i(z) = 1$ is impossible, for from $\sum_{i=1}^{p} B_i(z_n)A_i(z_n)x_n = x_n$ we get the following estimate,

$$1 \leqq \max_i \sup \{\|B_i(z)\| \mid z \in D\} \sum_{i=1}^{p} \|A_i(z_n)x_n\|;$$

and the right hand side tends to zero.

To prove the sufficiency we choose a fixed orthonormal basis in N and express the A_i in matrix form. We retain the letter A_i for the corresponding matrix and denote its elements by $A_{jk}^{(i)}$. Let W be the $pn \times n$ matrix composed of the rows of all the A_i. Let $W_{i_1 \cdots i_n}$ be the $n \times n$ matrix whose rows are the $i_1, \ldots, i_n$ rows of W.

We claim that if (3.1) holds for a $\delta > 0$, there exists a $\delta^1 > 0$ such that

$$\sum |\det W_{i_1 \cdots i_n}(z)| \geqq \delta^1 \tag{3.3}$$

for all z in D, the summation being on all $1 \leqq i_1 < \cdots < i_n \leqq pn$.

The basic idea is that if $\sum |\det W_{i_1 \cdots i_n}(z)| = 0$; then the vectors represented by the rows of the $A_i(z)$ lie all in a subspace of N of dimension at most $n-1$. Therefore there exists a vector in N orthogonal to all of them, and this implies that for some unit vector x in N

$$\sum_{i=1}^{p} \|A_i(z)x\| = 0.$$

In general if no $\delta^1 > 0$ exists for which (3.3) holds, we have a sequence of points z_ν in D satisfying

$$\lim_{\nu} \sum |\det W_{i_1 \cdots i_n}(z_\nu)| = 0. \tag{3.4}$$

Thus it is enough to show that (3.4) implies

$$\liminf_{\nu} \left\{ \sum_{i=1}^{p} \|A_i(z_\nu)x\| \mid x \in N, \|x\| = 1 \right\} = 0. \tag{3.5}$$

Let $\varepsilon_\nu = \sum |\det W_{i_1 \cdots i_n}(z_\nu)|$. Let $y_i^{(\nu)}$ be the vector in N represented by the ith row of $W(z_\nu)$. There is one set of indices $i_1 \cdots i_n$ such that for all $j_1 \cdots j_n$

$$|\det W_{j_1 \cdots j_n}(z_\nu)| \leqq |\det W_{i_1 \cdots i_n}(z_\nu)|. \tag{3.6}$$

By Lemma 2.3 there exists a unit vector x in N for which $|(x, y_{i_j}^{(\nu)})| \leqq \varepsilon_\nu^{1/n}$. If $\varepsilon_\nu > 0$, then $y_{i_j}^{(\nu)}$, $j = 1, \ldots, n$, are a basis for N. Thus each $y_k^{(\nu)}$ has a representation $y_k^{(\nu)} = \sum_j \beta_{kj} y_{i_j}^{(\nu)}$. From (3.6) it is clear that $|\beta_{kj}| \leqq 1$, and thus for each y_k, $|(x, y_k^{(\nu)})| \leqq n\varepsilon_\nu^{1/n}$. This estimate shows that (3.4) implies (3.5).

We have seen therefore that if (3.1) is true for a $\delta > 0$, there exists a $\delta^1 > 0$ for which (3.2) is true. We invoke now the scalar corona theorem to get the existence of $a_{i_1 \cdots i_n}$ in H^∞ for which

$$\sum a_{i_1 \cdots i_n}(z) \det W_{i_1 \cdots i_n}(z) = 1. \tag{3.7}$$

To finish the proof we have to show the existence of $B_{ik}^{(l)}$ in H^∞ which satisfy

$$\sum_{l=1}^{p} \sum_{k=1}^{n} B_{ik}^{(l)}(z) A_{jk}^{(l)}(z) = \delta_{ij} \tag{3.8}$$

for all z in D. Now for a fixed j (3.7) can, after determinant expansion, be rewritten in the form

$$\sum_{l=1}^{p} \sum_{k=1}^{n} B_{jk}^{(l)}(z) A_{kj}^{(l)}(z) = 1$$

where obviously the $B_{jk}^{(l)}$ are in H^∞. Thus (3.8) is automatically satisfied whenever $i=j$. If $i \neq j$, then

$$\sum_{l=1}^{p} \sum_{k=1}^{n} B_{ik}^{(l)}(z) A_{kj}^{(l)}(z) = \sum a_{i_1 \cdots i_n} \det W_{i_1 \cdots i_n}^1(z)$$

where $W_{i_1 \cdots i_n}^1$ is equal to $W_{i_1 \cdots i_n}$ with the ith column replaced by the jth and thus $\det W_{i_1 \cdots i_n}^1(z) \equiv 0$ for all $1 \leqq i_1 < \cdots < i_n \leqq pn$. Thus (3.8) holds for all $i, j = 1, \ldots, n$.

(b) $\sum_{i=1}^{p} A_i(z) B_i(z) = 1$ if and only if $\sum_{i=1}^{p} \tilde{B}_i(z) \tilde{A}_i(z) = 1$, and we apply part (a) of the theorem.

ACKNOWLEDGEMENT. My warm thanks to Professor P. D. Lax for directing me to the problems of this paper.

REFERENCES

1. A. Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta Math. **81** (1949), 239–255.

2. L. de Branges and J. Rovnyak, *The existence of invariant subspaces*, Bull. Amer. Math. Soc. **70** (1964), 718–721.

3. L. Carleson, *Interpolation by bounded analytic functions and the corona problem*, Ann. of Math. (2) **76** (1962), 547–559.

4. C. Foias and W. Mlak, *The extended spectrum of complementary nonunitary contractions and the spectral mapping theorem*, Studia Math. **26** (1966), 239–245.

5. P. A. Fuhrmann, *A functional calculus in Hilbert space based on operator valued analytic functions*, Notices Amer. Math. Soc. **14** (1967), 247.

6. P. R. Halmos, *Shifts on Hilbert spaces*, J. Reine Angew. Math. **208** (1961), 102–112.

7. H. Helson, *Lectures on invariant subspaces*, Academic Press, New York, 1964.

8. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N. J., 1962.

9. P. D. Lax, *Translation invariant subspaces*, Acta Math. **101** (1959), 163–178.

10. P. D. Lax and R. S. Phillips, *Scattering theory*, Bull. Amer. Math. Soc. **70** (1964), 130–142.

11. ———, *Scattering theory*, Academic Press, New York, 1967.

12. J. W. Moeller, *On the spectra of some translation invariant spaces*, J. Math. Anal. Appl. **4** (1962), 276–296.

13. G. C. Rota, *On models for linear operators*, Comm. Pure Appl. Math. **13** (1960), 469–472.

14. D. Sarason, *Generalized interpolation in H^∞* (to appear).

15. B. Sz.-Nagy and C. Foias, *Sur les contractions de l'espace de Hilbert*. VI: *Calcul fonctionel*, Acta Sci. Math. (Szeged) **23** (1962), 130–167.

16. ———, *Sur les contractions de l'espace de Hilbert*. VIII: *Fonctions caractéristique, modèles fonctionels*, Acta Sci. Math. (Szeged) **25** (1964), 38–71.

COLUMBIA UNIVERSITY,
NEW YORK, NEW YORK