# EXTENSIONS OF UNIFORMLY CONTINUOUS PSEUDOMETRICS(¹)

BY
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1. **Introduction.** Consider a uniform subspace $S$ of a uniform space $X$. In [10], M. Katětov proved that every bounded uniformly continuous real-valued function on $S$ has a bounded uniformly continuous real-valued extension to $X$. A similar theorem, due to J. R. Isbell [8], states that every bounded uniformly continuous pseudometric on $S$ has a bounded uniformly continuous pseudometric extension to $X$. In this paper, we present a unified exposition of the above two theorems along with other related results. In particular, using a construction due to H. L. Shapiro [12], we prove a more general theorem from which the theorems of Katětov and Isbell may be derived.

Following Shapiro [12], if $\gamma$ is an infinite cardinal number, then a subset $S$ of a topological space $X$ is said to be *$P^{\gamma}$-embedded* in $X$ if every continuous $\gamma$-separable pseudometric on $S$ has a continuous $\gamma$-separable pseudometric extension to $X$. (A pseudometric $d$ on $X$ is called *$\gamma$-separable* if there exists a subset $A$ of $X$ such that $|A| \leqq \gamma$ and such that $A$ is dense in $(X, \mathscr{T}_d)$, where $\mathscr{T}_d$ is the topology on $X$ induced by $d$.)

In §2, we show that every pseudometric on a topological space $X$ that is induced by a continuous real-valued function on $X$ is $\aleph_0$-separable. This enables us to show that every $P^{\aleph_0}$-embedded subset of a topological space $X$ is $C$-embedded in $X$. The converse of this theorem was demonstrated by Shapiro [12]. We also give an example of a normal space $X$ having a closed $P^{\aleph_0}$-embedded subset that is not $P^{\aleph_1}$-embedded in $X$. This settles an open problem that was posed by R. Arens [1].

The notation and terminology, except for uniform spaces, will be consistent with that of [6]. For the general theory of uniform spaces, we refer the reader to [11].

If $d$ is a pseudometric on a set $X$, then we will denote by $\mathscr{T}_d$ the topology on $X$ whose base consists of the sets $S_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$, where $\varepsilon > 0$ and $x \in X$. We recall that the pseudometric $d$ is continuous on a topological space $(X, \mathscr{T})$ if and only if $\mathscr{T}_d \subset \mathscr{T}$. If $f$ is a real-valued function on a set $X$, then the pseudometric $\psi_f$ on $X$ defined by

$$\psi_f(x, y) = |f(x) - f(y)|,$$

for $x, y \in X$, will be called the *pseudometric on $X$ associated with $f$.*

Let $\mathfrak{U} = (U_\alpha)_{\alpha \in I}$ and $\mathfrak{V} = (V_\beta)_{\beta \in J}$ be two covers of a set $X$. We say that $\mathfrak{U}$ has

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*power at most* $\gamma$ ($\gamma$ an infinite cardinal number) if $|I| \leqq \gamma$, and we say that $\mathfrak{U}$ is a *refinement* of $\mathfrak{V}$ (written $\mathfrak{U} < \mathfrak{V}$) if, for each $\alpha \in I$, there exists $\beta \in J$ such that $U_\alpha \subset V_\beta$. Following J. W. Tukey [14], we say that $\mathfrak{U}$ is a *star-refinement* of $\mathfrak{V}$ (written $\mathfrak{U} <^* \mathfrak{V}$) if $(\text{st}\,(U_\alpha, \mathfrak{U}))_{\alpha \in I}$ is a refinement of $\mathfrak{V}$, where

$$\text{st}\,(U_\alpha, \mathfrak{U}) = \bigcup \{U_\beta : \beta \in I \text{ and } U_\beta \cap U_\alpha \neq \varnothing\}.$$

A cover $\mathfrak{U}$ of a topological space $X$ is *normal* if there is a sequence $(\mathfrak{U}_n)_{n \in N}$ of open covers of $X$ such that $\mathfrak{U}_1 < \mathfrak{U}$ and $\mathfrak{U}_{n+1} <^* \mathfrak{U}_n$ for each $n \in N$. If $(\mathfrak{U}_n)_{1 \leqq n \leqq m}$ is a finite sequence of covers of a set $X$ and if $\mathfrak{U}_n = (A_n(\alpha))_{\alpha \in J_n}$ for each $1 \leqq n \leqq m$, then by $\mathfrak{U}_1 \wedge \cdots \wedge \mathfrak{U}_m$ or by $\bigwedge_{n=1}^m \mathfrak{U}_n$ we mean the cover

$$(A_1(\alpha_1) \cap \cdots \cap A_m(\alpha_m))_{(\alpha_1, \ldots, \alpha_m) \in J_1 \times \cdots \times J_m}.$$

If $\mathfrak{U} = (U_\alpha)_{\alpha \in I}$ is a family of subsets of a set $X$ and if $S \subset X$, then by $\mathfrak{U}|S$ we mean the family $(S \cap U_\alpha)_{\alpha \in I}$.

Suppose that $(X, \mathscr{U})$ is a uniform space and that $\mu$ is a collection of covers of the set $X$. Then we say that $\mu$ *generates* the uniformity $\mathscr{U}$ in case the collection

$$\Big\{\bigcup_{\alpha \in I} (V_\alpha \times V_\alpha) : (V_\alpha)_{\alpha \in I} \in \mu\Big\}$$

is a base for $\mathscr{U}$. If $U \in \mathscr{U}$, then, as usual

$$U(x) = \{y \in X : (x, y) \in U\}.$$

If $X$ is nonempty and if $\mathfrak{V}$ is a cover of $X$, then we say that $\mathfrak{V}$ is a *uniform cover* of $X$ if there exists $U \in \mathscr{U}$ such that $(U(x))_{x \in X} < \mathfrak{V}$. If $X$ is empty, then, by definition, every cover of $X$ is a uniform cover of $X$.

Now suppose that $\mu$ denotes the collection of all uniform covers of a uniform space $(X, \mathscr{U})$. Then $\mu$ has the following properties:

(1) If $\mathfrak{V} \in \mu$ and if $\mathfrak{V} < \mathfrak{W}$, then $\mathfrak{W} \in \mu$.

(2) If $\mathfrak{V} \in \mu$ and if $\mathfrak{W} \in \mu$, then $\mathfrak{V} \wedge \mathfrak{W} \in \mu$.

(3) If $\mathfrak{W} \in \mu$, then there exists $\mathfrak{V} \in \mu$ such that $\mathfrak{V} <^* \mathfrak{W}$.

Moreover, if $\mu$ denotes the collection of all uniform covers of the uniform space $(X, \mathscr{U})$, then $\mu$ generates the uniformity $\mathscr{U}$. Finally, we note that every uniform cover of a uniform space $(X, \mathscr{U})$ has an open uniform refinement, and consequently every uniform cover of $(X, \mathscr{U})$ is normal, where open, of course, refers to the topology on $X$ induced by $\mathscr{U}$.

Most of the results in this paper are contained in the author's doctoral dissertation which was written at Purdue University under the direction of Professor Robert L. Blair, to whom the author wishes to express his appreciation.

2. **$\aleph_0$-separable pseudometrics.** Most of the results that will be needed concerning $\gamma$-separable pseudometrics can be found in Shapiro's paper [12]. The following theorem has a useful corollary.

2.1. THEOREM. *Suppose that $X$ is a topological space, that $\gamma$ is an infinite cardinal* *number, that $(Y, d)$ is a $\gamma$-separable pseudometric space, and that $f$ is a continuous map from $X$ into $Y$. Then $r=d\circ(f\times f)$ is a continuous $\gamma$-separable pseudometric on $X$.*

**Proof.** It is easy to verify that $r$ is a continuous pseudometric on $X$. We now prove that $r$ is $\gamma$-separable. Since $\gamma$-separability, like separability, is an hereditary property of pseudometric spaces, it follows that there exists a dense subset $A$ of $f(X)$ such that $|A|\leqq\gamma$. For each $a\in A$, we choose precisely one point $x_a\in f^{-1}(a)$, and we set $D=\{x_a : a\in A\}$, so that $|D|\leqq\gamma$. Now suppose that $x\in X$ and $\varepsilon>0$. Then the nonempty open subset $S_d(f(x),\varepsilon)\cap f(X)$ of $f(X)$ contains some point $a\in A$. But then $x_a\in D$ and $f(x_a)=a$, so we have $d(f(x),f(x_a))<\varepsilon$, and hence $r(x,x_a)<\varepsilon$. Therefore $x_a\in S_r(x,\varepsilon)$, so that $S_r(x,\varepsilon)\cap D\neq\varnothing$. It follows that $D$ is a dense subset of $(X,\mathcal{T}_r)$, and hence that $r$ is a $\gamma$-separable continuous pseudometric on $X$.

2.2. COROLLARY. *If $X$ is a topological space and if $f\in C(X)$, then the pseudometric $\psi_f$ on $X$ that is associated with $f$ is a continuous $\aleph_0$-separable pseudometric on $X$.*

**Proof.** By 2.1, since $\boldsymbol{R}$ is separable, the continuous pseudometric $r=d\circ(f\times f)$ is $\aleph_0$-separable, where $d$ is the usual metric on $\boldsymbol{R}$. But clearly we have $r=\psi_f$.

Let us recall that if $f$ is a real-valued function defined on a set $X$, then its *zero-set* is the set $Z_X(f)=\{x\in X : f(x)=0\}$.

2.3. LEMMA. *Suppose that $S$ is a $P^{\aleph_0}$-embedded subset of a topological space $X$. If $f\in C(S)$, if $Z=Z_S(f)\neq\varnothing$, and if $f\geqq\mathbf{0}$, then there exists $g\in C(X)$ such that $g|S=f$.*

**Proof.** Suppose that $f\in C(S)$, that $Z=Z_S(f)\neq\varnothing$, and that $f\geqq\mathbf{0}$. By 2.2, the pseudometric $\psi_f$ on $S$ associated with $f$ is a continuous $\aleph_0$-separable pseudometric on $S$, and therefore, by hypothesis, there exists a continuous pseudometric $d$ on $X$ such that $d|S\times S=\psi_f$. We then define a map $g$ from $X$ into $\boldsymbol{R}$ by setting

$$g(x) = d(x,Z) = \inf\{d(x,z) : z\in Z\},$$

for all $x\in X$. It is evident that $g\in C(X)$. Moreover, if $x\in S$, then, since $f\geqq\mathbf{0}$, we have

$$g(x) = d(x,Z) = \psi_f(x,Z) = \inf\{|f(x)-f(z)| : z\in Z\} = f(x),$$

and it follows that $g|S=f$.

2.4. THEOREM. *Suppose that $X$ is a topological space and that $S\subset X$. Then the following statements are equivalent:*

(1) *$S$ is $C$-embedded in $X$.*

(2) *$S$ is $P^{\aleph_0}$-embedded in $X$.*

**Proof.** In [12], Shapiro proves that the first statement implies the second. We will now prove the converse. Thus assume that $S$ is $P^{\aleph_0}$-embedded in $X$. If $S=\varnothing$, then there is nothing to prove. Hence assume that $S$ is nonempty, and that $f\in C(X)$.

Let $a \in S$ be arbitrary, set $f(a)=\alpha$, and set $g=(f \vee \boldsymbol{\alpha})-\boldsymbol{\alpha}$ and $h=-((f \wedge \boldsymbol{\alpha})-\boldsymbol{\alpha})$. Then, by 2.3, there exist functions $g_0$ and $h_0$ in $C(X)$ such that $g_0|S=g$ and $h_0|S=h$. Finally, we set $f_0=(g_0-h_0)+\boldsymbol{\alpha}$ to get the desired continuous real-valued extension of $f$ to $X$.

We conclude this section by giving two examples relating to $\aleph_0$-separable pseudometrics.

2.5. Example. It is clear that every continuous pseudometric on a separable space is $\aleph_0$-separable. The converse of this statement is not true. In fact, let $A$ be a set of cardinality $c$, let $\{0, 1\}$ be the two-point discrete space, and let $Y=\{0, 1\}^A$, i.e. $Y$ is the space of all functions from $A$ into $\{0, 1\}$, equipped with the product topology. Finally, let

$$X = \{y \in Y : y(a) \neq 0 \text{ for at most countably many } a \in A\}.$$

By a result due to Corson [5, Theorem 2], $X$ is a dense $C$-embedded subset of $Y$. But then, by [12, Theorem 3.3], every continuous pseudometric on $X$ has a continuous pseudometric extension to $Y$, since $|Y| \leqq 2^c$. Moreover, Comfort [4] noted that $Y$ is separable, but $X$ is not separable. Now suppose that $d$ is a continuous pseudometric on $X$. Then there exists a continuous pseudometric extension $r$ of $d$ to $Y$. But then, since $Y$ is separable, the pseudometric $r$ is $\aleph_0$-separable, i.e. $(Y, r)$ is a separable pseudometric space. Consequently, the subspace $(X, d)$ of $(Y, r)$ is separable and so $d$ is $\aleph_0$-separable. Thus, $X$ is a nonseparable space such that every continuous pseudometric on $X$ is $\aleph_0$-separable.

The following is an example of a normal space $X$ and a closed $P^{\aleph_0}$-embedded subset of $X$ that is not $P^{\aleph_1}$-embedded in $X$.

2.6. Example. If $m$ is an uncountable cardinal number, then Bing [3, Example G] constructs an example of a normal $T_1$-space $X$ containing a closed discrete subset $Y$ such that $|Y|=m$, and such that there exists no pairwise disjoint family $(G_y)_{y \in Y}$ of open subsets of $X$ such that $y \in G_y$ for each $y \in Y$. For this example, take $m=\aleph_1$. Since $X$ is normal and $Y$ is closed in $X$, it follows that $Y$ is $C$-embedded in $X$. Therefore, by 2.4, $Y$ is $P^{\aleph_0}$-embedded in $X$. We assert that $Y$ is not $P^{\aleph_1}$-embedded in $X$. In fact, assume the contrary, and let $\rho$ be the metric on $Y$ defined by $\rho(x, x)=0$ for $x \in Y$, and $\rho(x, y)=1$ for $x, y \in Y$ and $x \neq y$. Since $Y$ is a discrete space, $\rho$ is a continuous pseudometric on $Y$, and since $|Y|=\aleph_1$, $\rho$ is $\aleph_1$-separable. Therefore, there is a continuous pseudometric $r$ on $X$ such that $r|Y \times Y=\rho$. But then the family $(S_r(y, 1/2))_{y \in Y}$ is a pairwise disjoint family of open subsets of $X$ such that $y \in S_r(y, 1/2)$ for each $y \in Y$. This contradiction implies that $\rho$ has no continuous pseudometric extension to $X$, and it follows that $Y$ is not $P^{\aleph_1}$-embedded in $X$.

3. **The main theorems.** We begin this section with a few preliminary results.

3.1. Proposition. *Suppose that $(X, \mathcal{U})$ is a uniform space and that $S$ is a uniform subspace of $X$. If $\mathfrak{V}=(V_\alpha)_{\alpha \in I}$ is a uniform cover of $S$, then there exists an open uniform cover $\mathfrak{W}=(W_\alpha)_{\alpha \in I}$ of $X$ such that $W_\alpha \cap S \subset V_\alpha$ for each $\alpha \in I$.*

**Proof.** The result is trivial if $S=\varnothing$. Therefore suppose that $S$ is nonempty and let $\mathfrak{V}=(V_\alpha)_{\alpha\in I}$ be a uniform cover of $S$. Then there exists an element $V$ in the relative uniformity $\mathscr{U}|S\times S$ on $S$ such that $(V(x))_{x\in S}<\mathfrak{V}$. Let $U\in\mathscr{U}$ such that $U\cap(S\times S)=V$, and choose an open symmetric $W\in\mathscr{U}$ such that $W\circ W\subset U$. Set $\mathfrak{R}=(W(x))_{x\in X}$. Then $\mathfrak{R}$ is an open uniform cover of $X$. We claim that $\mathfrak{R}|S<\mathfrak{V}$. Suppose that $x\in X$ and that $W(x)\cap S\neq\varnothing$. Choose $y\in W(x)\cap S$, and let $z\in W(x)\cap S$. Then we have $(x,y)\in W$ and $(x,z)\in W$. But $W$ is symmetric, so we have $(y,x)\in W$, hence $(y,z)\in W\circ W\subset U$. Also, $(y,z)\in S\times S$, so $(y,z)\in V$, whence $z\in V(y)$, and it follows that $W(x)\cap S\subset V(y)$. Therefore we have

$$\mathfrak{R}|S < (V(x))_{x\in S} < \mathfrak{V}.$$

Now let $\pi$ be a map from $X$ into $I$ such that, for each $x\in X$, we have $W(x)\cap S\subset V_{\pi(x)}$. Then, for each $\alpha\in I$, we set

$$W_\alpha = \bigcup_{x\in\pi^{-1}(\alpha)} W(x),$$

and we set $\mathfrak{W}=(W_\alpha)_{\alpha\in I}$. It is then clear that $\mathfrak{W}$ is an open cover of $X$, and that $W_\alpha\cap S\subset V_\alpha$ for each $\alpha\in I$. Moreover, since $\mathfrak{R}$ is a uniform cover of $X$ and since $\mathfrak{R}<\mathfrak{W}$, it follows that $\mathfrak{W}$ is also a uniform cover of $X$. This completes the proof.

3.2. THEOREM. *Suppose that $(X,\mathscr{U})$ is a uniform space, that $(U_n)_{n\in N}$ is a sequence of symmetric elements of $\mathscr{U}$, and set $U_0=X\times X$. If $U_{n+1}^3\subset U_n$ for each $n\in N$, then there exists a uniformly continuous pseudometric $d$ on $X$ such that $d\leqq 1$, and such that, for each nonnegative integer $n$, we have*

$$(*)\qquad U_{n+1}\subset\{(x,y)\in X\times X : d(x,y)<2^{-n}\}\subset U_n.$$

This is a particular case of the Metrization Lemma of [11, 6.12]. The uniform continuity of $d$ follows at once from $(*)$ and [11, 6.11].

If $(\mathfrak{V}_n)_{n\in N}$ is a normal sequence of covers of a topological space $X$, and if $d$ is a pseudometric on $X$, then we say that *$d$ is associated with* $(\mathfrak{V}_n)_{n\in N}$ in case the following conditions are satisfied:

(1) $d$ is bounded by 1.

(2) If $k\in N$ and if $d(x,y)<2^{-(k+1)}$, then $x\in \operatorname{st}(y,\mathfrak{V}_k)$.

(3) If $k\in N$ and if $x\in\operatorname{st}(y,\mathfrak{V}_k)$, then $d(x,y)<2^{-(k-3)}$. (Here $\operatorname{st}(y,\mathfrak{V}_k)$ denotes the union of all the members of $\mathfrak{V}_k$ that contain $y$.)

3.3. THEOREM. *If $(\mathfrak{V}_n)_{n\in N}$ is a normal sequence of uniform covers of a uniform space $(X,\mathscr{U})$, then there exists a uniformly continuous pseudometric on $X$ that is associated with $(\mathfrak{V}_n)_{n\in N}$.*

**Proof.** For each $n\in N$, let $\mathfrak{V}_n=(V_n(\alpha))_{\alpha\in I_n}$ and set $V_n=\bigcup\{V_n(\alpha)\times V_n(\alpha) : \alpha\in I_n\}$. It is easily seen that $(V_{2n})_{n\in N}$ satisfies the hypothesis of 3.2, i.e. for each $n\in N$, $V_{n+2}^3\subset V_n$. Then we apply 3.2 to obtain a uniformly continuous pseudometric $d$

on $X$ (bounded by **1**) that satisfies (*) of 3.2. Finally, a routine computation shows that the pseudometric $d$ satisfying (*) of 3.2 is associated with $(\mathfrak{V}_n)_{n\in N}$.

We now sketch the proof of a generalization of a result due to Shapiro [12].

3.4. THEOREM. *Suppose that* $(X, \mathcal{U})$ *is a uniform space, that* $S$ *is a uniform subspace of* $X$, *and that* $\gamma$ *is an infinite cardinal number. Then every uniformly continuous* $\gamma$*-separable pseudometric on* $S$ *has a continuous* $\gamma$*-separable pseudometric extension to* $X$.

**Proof.** The result is immediate if $S=\varnothing$. Thus we may assume that $S$ is nonempty. Let $d$ be a uniformly continuous $\gamma$-separable pseudometric on $S$, and let $\mathcal{U}_d$ denote the uniformity on $S$ generated by $d$. By [11, 6.11], it follows that $\mathcal{U}_d \subset \mathcal{U}|S\times S$. Let $m \in N$ be arbitrary, and set

$$\mathfrak{S}_m = (S_d(x, 2^{-(m+3)}))_{x\in S}.$$

Since $d$ is $\gamma$-separable, there exists a dense subset $A$ of $(S, \mathcal{T}_d)$ such that $|A|\leqq\gamma$. Now set

$$\mathfrak{W}^m = (S_d(a, 2^{-(m+3)}))_{a\in A}, \quad \text{and set} \quad \mathfrak{R}_m = (S_d(x, 2^{-(m+4)}))_{x\in S}.$$

Then $\mathfrak{R}_m$ is a uniform cover of $(S, \mathcal{U}_d)$ and hence is a uniform cover of the subspace $S$ of $(X, \mathcal{U})$. If $x \in S$ and $y \in S_d(x, 2^{-(m+4)})$, then there exists $a \in A \cap S_d(x, 2^{-(m+4)})$, and hence $d(a, y)\leqq d(a, x)+d(x, y)<2^{-(m+3)}$. Therefore $y \in S_d(a, 2^{-(m+3)})$, and so $S_d(x, 2^{-(m+4)})\subset S_d(a, 2^{-(m+3)})$. It follows that $\mathfrak{R}_m<\mathfrak{W}^m$, and hence that $\mathfrak{W}^m$ is a uniform cover of the uniform subspace $S$ of $(X, \mathcal{U})$ such that $\mathfrak{W}^m$ has power at most $\gamma$ and $\mathfrak{W}^m<\mathfrak{S}_m$. We now apply 3.1 to obtain an open uniform cover $\mathfrak{V}^m$ of the uniform space $(X, \mathcal{U})$ such that $\mathfrak{V}^m$ is of power at most $\gamma$ and such that $\mathfrak{V}^m|S<\mathfrak{W}^m$. But then $\mathfrak{V}^m$ is a normal open cover of $X$ of power at most $\gamma$, so we may apply [12, Lemma 2.6] to obtain a normal sequence $(\mathfrak{V}_i^m)_{i\in N}$ of open covers of $X$ such that $\mathfrak{V}_1^m<\mathfrak{V}^m$ and such that $\mathfrak{V}_i^m$ is of power at most $\gamma$ for each $i \in N$. Having done this for each $m \in N$, the remainder of the argument follows, verbatim, that given by Shapiro [12, pp. 894–896].

The following corollary of 3.4 was first noticed by Shapiro [13].

3.5. COROLLARY. *If* $X$ *is a uniform space, and if* $S$ *is a uniform subspace of* $X$, *then every uniformly continuous pseudometric on* $S$ *has a continuous pseudometric extension to* $X$.

**Proof.** Set $\gamma=|S|+\aleph_0$. Then every pseudometric on $S$ is $\gamma$-separable, and so the result follows immediately from 3.4.

In the next theorem we restrict our attention to bounded pseudometrics.

3.6. THEOREM. *Suppose that* $(X, \mathcal{U})$ *is a uniform space, that* $S$ *is a uniform subspace of* $X$, *and that* $\gamma$ *is an infinite cardinal number. Moreover, suppose that* $X$ *satisfies the condition*:

(*) *If* $\mathfrak{V}$ *is a uniform cover of* $X$ *of power at most* $\gamma$, *then there exists a uniform cover* $\mathfrak{W}$ *of* $X$ *of power at most* $\gamma$ *such that* $\mathfrak{W}<^*\mathfrak{V}$.

*Then every bounded uniformly continuous $\gamma$-separable pseudometric on $S$ has a bounded uniformly continuous $\gamma$-separable pseudometric extension to $X$.*

**Proof.** The result is trivial in case $S=\varnothing$. Thus we may assume that $S$ is non-empty. Let $d$ be a bounded uniformly continuous $\gamma$-separable pseudometric on $S$, choose $k\in N$ such that $d\leqq 2^k$, and let $K=\{n\in \boldsymbol{Z} : n\geqq -k\}$, where $\boldsymbol{Z}$ denotes the set of all integers. For each $m\in K$, let

$$\mathfrak{S}_m = (S_d(x, 2^{-(m+3)}))_{x\in S}.$$

We now duplicate the argument used in 3.4 to find, for each $m\in K$, a uniform cover $\mathfrak{W}^m$ of the subspace $S$ such that $\mathfrak{W}^m$ has power at most $\gamma$, and such that $\mathfrak{W}^m<\mathfrak{S}_m$. Again, for each $m\in K$, we apply 3.1 to obtain a uniform cover $\mathfrak{V}^m$ of the uniform space $X$ such that $\mathfrak{V}^m$ has power at most $\gamma$, and such that $\mathfrak{V}^m|S<\mathfrak{W}^m$. Then, for each $m\in K$, we can find, by virtue of the hypothesis $(*)$, a sequence $(\mathfrak{V}^m_i)_{i\in N}$ of uniform covers of $X$, each having power at most $\gamma$, such that $\mathfrak{V}^m_1<\mathfrak{V}^m$ and $\mathfrak{V}^m_{i+1}<^*\mathfrak{V}^m_i$ for each $i\in N$.

Now, for each $m\in K$ and $i\in N$, we set

$$\mathfrak{U}^m = \bigwedge_{j=-k}^{m} \mathfrak{V}^j,$$

and

$$\mathfrak{U}^m_i = \bigwedge_{j=-k}^{m} \mathfrak{V}^j_i.$$

Then, for each $m\in K$ and $i\in N$, the following are true:

(i) $\mathfrak{U}^m$ and $\mathfrak{U}^m_i$ are uniform covers of $X$ of power at most $\gamma$,

(ii) $\mathfrak{U}^m_{i+1}<^*\mathfrak{U}^m_i$ and $\mathfrak{U}^m_1<\mathfrak{U}^m$,

(iii) $\mathfrak{U}^{m+1}_i<\mathfrak{U}^m_i$ and $\mathfrak{U}^{m+1}<\mathfrak{U}^m$, and

(iv) $\mathfrak{U}^m|S<\mathfrak{S}_m$.

Again suppose that $m\in K$. Then, by (i) and (ii), $(\mathfrak{U}^m_i)_{i\in N}$ is a normal sequence of uniform covers of $X$, and so there exists, by 3.3, a uniformly continuous pseudometric $r_m$ on $X$ that is associated with $(\mathfrak{U}^m_i)_{i\in N}$, and $r_m$ is $\gamma$-separable. By (ii) and (iv), we also have

$(**)$ If $x, y\in S$ and $r_m(x, y) < 2^{-3}$, then $d(x, y) < 2^{-(m+2)}$.

Now define a map $r\colon X\times X\to \boldsymbol{R}^+$ by the formula $r(x, y)=\sum_{m\in K} 2^{-(m-3)}r_m(x, y)$. It is easy to verify that $r$ is a bounded uniformly continuous $\gamma$-separable pseudometric on $X$. Moreover, from $(**)$ it follows that

$(***)$ $$r|S\times S \geqq d.$$

Next, for each $x, y\in X$, we set

$$f(x, y) = \inf\{r(x, a)+d(a, b)+r(b, y) : a, b\in S\},$$

and

$$d_0(x, y) = \min\{r(x, y), f(x, y)\}.$$

Evidently, $d_0$ is a nonnegative symmetric function that is zero on the diagonal of $X\times X$. The triangle inequality is easily proved by considering the various cases that arise. Therefore $d_0$ is a pseudometric on $X$. Since $d_0 \leqq r$, it follows that $d_0$ is bounded, uniformly continuous, and $\gamma$-separable. Finally, by (***), $d_0$ is an extension of $d$. This completes the proof.

An immediate corollary of Theorem 3.6 is the following result due to J. R. Isbell.

3.7. COROLLARY (ISBELL [8]). *If $X$ is a uniform space and if $S$ is a uniform subspace of $X$, then every bounded uniformly continuous pseudometric on $S$ has a bounded uniformly continuous pseudometric extension to $X$.*

**Proof.** Suppose that $X$ is a uniform space and that $S$ is a uniform subspace of $X$. Set $\gamma = 2^{|X|} + \aleph_0$. Then every uniform cover of power at most $\gamma$ has a uniform star-refinement, and if we delete any repeated members from the latter cover, then it will still be a uniform star-refinement of the first cover, but with power at most $\gamma$. Therefore the result now follows immediately from 3.6, since every pseudometric on $X$ is $\gamma$-separable.

S. Ginsburg and J. R. Isbell proved in [7] that every countable uniform cover of a uniform space $X$ has a countable uniform star-refinement. Therefore, for the case $\gamma = \aleph_0$, the hypothesis (*) of 3.6 is redundant, and we obtain another immediate corollary of Theorem 3.6.

3.8. COROLLARY. *If $X$ is a uniform space and if $S$ is a uniform subspace of $X$, then every bounded uniformly continuous $\aleph_0$-separable pseudometric on $S$ has a bounded uniformly continuous $\aleph_0$-separable pseudometric extension to $X$.*

It is remarked in [9, p. 52] that it is still an open question whether or not hypothesis (*) of Theorem 3.6 is vacuously satisfied in case $\gamma > \aleph_0$. However, we now show that the argument given by Ginsburg and Isbell in [7] can be generalized to provide an answer to this question for any infinite cardinal number $\gamma$ if the generalized continuum hypothesis is assumed.

3.9. THEOREM. *Assume the generalized continuum hypothesis. Suppose that $X$ is a set, that $\mathfrak{U}$ and $\mathfrak{V}$ are covers of $X$ such that $\mathfrak{V}^{**} < \mathfrak{U}$, and that $\gamma$ is an infinite cardinal number. If $\mathfrak{U}$ is of power at most $\gamma$, then there exists a cover $\mathfrak{W}$ of $X$ of power at most $\gamma$ such that $\mathfrak{V} < \mathfrak{W} <^* \mathfrak{U}$.*

**Proof.** We may clearly assume that the power of $\mathfrak{U}$ is infinite. Let $\aleph_\alpha$ be the power of $\mathfrak{U}$, and let $\omega_\alpha$ be the initial ordinal number of power $\aleph_\alpha$. Then we may write $\mathfrak{U} = (U_i)_{i\in I}$, where $I$ is the set of all ordinal numbers less than $\omega_\alpha$. If $i \in I$, let $W(i)$ denote the set of all $s \in I$ such that $s < i$. Let $i \in I$ and suppose that $|W(i)| = \aleph_{\alpha'}$. Note that $\aleph_{\alpha'} < \aleph_\alpha$. Then $|\mathscr{P}(W(i))| = 2^{\aleph_{\alpha'}}$, where $\mathscr{P}(W(i))$ is the set of all subsets of $W(i)$. By the generalized continuum hypothesis, $2^{\aleph_{\alpha'}}=\aleph_{\alpha'+1}$, so that $|\mathscr{P}(W(i))|\leqq\aleph_\alpha$. Set

$$\Phi = \bigcup_{i\in I}\mathscr{P}(W(i)),$$

and note that $|\Phi|\leqq|I|\aleph_\alpha=\aleph_\alpha^2=\aleph_\alpha$.

Now suppose that $\mathfrak{B}=(V_\beta)_{\beta\in J}$. For each $\beta\in J$, let $m(\beta)$ be the least $i\in I$ such that $\operatorname{st}(V_\beta,\mathfrak{B})\subset U_i$, let $n(\beta)$ be the least $i\in I$ such that $\operatorname{st}(\operatorname{st}(V_\beta,\mathfrak{B}),\mathfrak{B}^*)\subset U_i$, and let

$$q(\beta) = \{i\in I : V_\beta\subset U_i \text{ and } i\leqq n(\beta)\}.$$

Thus we have defined maps $m\colon J\to I$, $n\colon J\to I$, and $q\colon J\to\Phi$. Note that, for each $\beta\in J$, $m(\beta)\leqq n(\beta)$, $m(\beta)\in q(\beta)$, and $n(\beta)\in q(\beta)$. Set

$$M = \{(j,k,\phi)\in I\times I\times\Phi : j\in\phi\},$$

and, for each $(j,k,\phi)\in M$, set

$$W_{jk\phi} = \bigcup\{V_\beta : m(\beta)=j,\ n(\beta)=k, \text{ and } q(\beta)=\phi\}.$$

Finally, we let

$$\mathfrak{W} = (W_{jk\phi})_{(j,k,\phi)\in M}.$$

It is clear that $\mathfrak{W}$ is a cover of $X$, that $\mathfrak{B}<\mathfrak{W}$, and, since $|M|\leqq|I\times I\times\Phi|=\aleph_\alpha^3=\aleph_\alpha\leqq\gamma$, that $\mathfrak{W}$ is of power at most $\gamma$. We now show that $\mathfrak{W}<^*\mathfrak{U}$. Suppose that $(j,k,\phi)\in M$, $(r,s,\sigma)\in M$, and that $W_{jk\phi}\cap W_{rs\sigma}\neq\varnothing$. Since $j\in\phi$, it follows that $W_{jk\phi}\subset U_j$. Moreover, there exist $\beta,\beta'\in J$ such that $m(\beta)=j$, $n(\beta)=k$, $q(\beta)=\phi$, $m(\beta')=r$, $n(\beta')=s$, $q(\beta')=\sigma$, and such that

$$V_\beta\cap V_{\beta'}\neq\varnothing.$$

Then $V_\beta\subset\operatorname{st}(V_{\beta'},\mathfrak{B})$, so $\operatorname{st}(V_\beta,\mathfrak{B})\subset\operatorname{st}(\operatorname{st}(V_{\beta'},\mathfrak{B}),\mathfrak{B}^*)\subset U_s$, whence $j\leqq s=n(\beta')$. On the other hand, $V_{\beta'}\subset\operatorname{st}(V_\beta,\mathfrak{B})\subset U_j$, whence $j\in\sigma$. Therefore, $W_{rs\sigma}\subset U_j$, and it follows that $\operatorname{st}(W_{jk\phi},\mathfrak{W})\subset U_j$. Consequently, $\mathfrak{W}<^*\mathfrak{U}$.

3.10. Corollary. *Assume the generalized continuum hypothesis. If $X$ is a uniform space and if $\gamma$ is an infinite cardinal number, then every uniform cover of $X$ of power at most $\gamma$ has a uniform star-refinement of power at most $\gamma$.*

**Proof.** Suppose that $X$ is a uniform space, that $\gamma$ is an infinite cardinal number, and that $\mathfrak{U}$ is a uniform cover of $X$ of power at most $\gamma$. Choose a uniform cover $\mathfrak{B}$ of $X$ such that $\mathfrak{B}^{**}<\mathfrak{U}$. Then, by Theorem 3.9, there exists a cover $\mathfrak{W}$ of $X$ of power at most $\gamma$ such that $\mathfrak{B}<\mathfrak{W}<^*\mathfrak{U}$. Since $\mathfrak{B}<\mathfrak{W}$, it follows that $\mathfrak{W}$ is a uniform cover of $X$, and the proof is complete.

3.11. Theorem. *Assume the generalized continuum hypothesis. If $X$ is a uniform space, if $S$ is a uniform subspace of $X$, and if $\gamma$ is an infinite cardinal number, then every bounded uniformly continuous $\gamma$-separable pseudometric on $S$ has a bounded uniformly continuous $\gamma$-separable pseudometric extension to $X$.*

**Proof.** This theorem follows immediately from Theorems 3.6 and 3.10.

Finally, we derive one other corollary of 3.6.

3.12. THEOREM. *Suppose that $X$ is a uniform space, that $S$ is a uniform subspace of $X$, that $\gamma$ is an infinite cardinal number, and that there exists no uniformly discrete subset of $X$ of power greater than $\gamma$. Then every bounded uniformly continuous $\gamma$-separable pseudometric on $S$ has a bounded uniformly continuous $\gamma$-separable pseudometric extension to $X$.*

**Proof.** We must show that hypothesis (*) of Theorem 3.6 is satisfied. Suppose that $\mathfrak{V}$ is a uniform cover of $X$ of power at most $\gamma$, and choose a uniform cover $\mathfrak{R}$ of $X$ such that $\mathfrak{R} <^* \mathfrak{V}$. We now apply [7, 2.3] to obtain a uniform cover $\mathfrak{W}$ of $X$ of power at most $\gamma$ such that $\mathfrak{W} < \mathfrak{R}$. But then, $\mathfrak{W} <^* \mathfrak{V}$. The result now follows immediately from 3.6.

We are now ready to show that a theorem due to Katětov [10] concerning extensions of uniformly continuous real-valued functions may be easily derived from either 3.7 or 3.8. Besides the original proof given by Katětov, we are aware of two other proofs of this result in the literature. One of these was constructed by S. Ginsburg, M. Henriksen, and J. R. Isbell and appears in [7]; the other is given by M. Atsuji in [2]. The crucial step in our proof is contained in the following lemma.

3.13. LEMMA. *Suppose that $X$ is a uniform space and that $S$ is a uniform subspace of $X$. If $f$ is a nonnegative bounded uniformly continuous real-valued function on $S$ whose zero-set is nonempty, then there exists a bounded uniformly continuous real-valued function $g$ on $X$ such that $g|S=f$.*

**Proof.** Suppose that $f$ is a nonnegative bounded uniformly continuous real-valued function on $S$ such that $Z=Z_S(f)$ is nonempty. Then the pseudometric $\psi_f$ on $S$ that is associated with $f$ is a bounded uniformly continuous $\aleph_0$-separable pseudometric. Therefore, by 3.7 or 3.8, there exists a bounded uniformly continuous pseudometric extension $d$ of $\psi_f$ to $X$. Define a real-valued function $g$ on $X$ by $g(x)=d(x,Z)=\inf\{d(x,z) : z \in Z\}$. Then $g$ is a bounded uniformly continuous real-valued function on $X$. If $x \in S$, then, since $f$ is nonnegative,

$$g(x) = d(x,Z) = \psi_f(x,Z) = \inf\{|f(x)-f(z)| : z \in Z\} = f(x),$$

and so $g$ is the desired extension of $f$.

3.14. THEOREM (KATĚTOV [10]). *If $X$ is a uniform space and if $S$ is a uniform subspace of $X$, then every bounded uniformly continuous real-valued function on $S$ has a bounded uniformly continuous real-valued extension to $X$.*

**Proof.** The result is trivial if $S=\varnothing$. Thus we may assume that $S$ is nonempty. Suppose that $f$ is a bounded uniformly continuous real-valued function on $S$. Choose $a \in S$, set $f(a)=\alpha$, and set $g=(f \vee \boldsymbol{\alpha})-\boldsymbol{\alpha}$ and $h=-((f \wedge \boldsymbol{\alpha})-\boldsymbol{\alpha})$. Then $g$ and $h$ are nonnegative bounded uniformly continuous real-valued functions on $S$ such that $a \in Z_S(g) \cap Z_S(h)$. Therefore, by 3.13, $g$ and $h$ have bounded uniformly continuous real-valued extensions $g_0$ and $h_0$, respectively, to $X$. But then $f_0=(g_0-h_0)+\boldsymbol{\alpha}$ is the desired bounded uniformly continuous extension of $f$ to $X$.

For completeness, we remark that, with computations similar to the ones in the proofs of 3.13 and 3.14, 3.5 leads one to the following result (Theorem 3.16) of Shapiro [13].

3.15. LEMMA. *Suppose that X is a uniform space and that S is a uniform subspace of X. If f is a nonnegative uniformly continuous real-valued function on S whose zero-set is nonempty, then there exists a continuous real-valued function g on X such that* $g|S=f$.

3.16. THEOREM (SHAPIRO [13]). *If X is a uniform space and if S is a uniform subspace of X, then every uniformly continuous real-valued function on S has a continuous real-valued extension to X.*

REMARK. Consider the uniform subspace $\boldsymbol{Z}$ of integers in the uniform space $\boldsymbol{R}$ of real numbers. The function $f$ defined on $\boldsymbol{Z}$ by $f(n)=n^2$ is an unbounded uniformly continuous function on $\boldsymbol{Z}$ that has no uniformly continuous real-valued extension to $\boldsymbol{R}$. However, the function $g$ on $\boldsymbol{R}$ defined by $g(x)=x^2$ provides an obviously continuous real-valued extension of $f$ to $\boldsymbol{R}$. Similarly, $\psi_f$ is an unbounded uniformly continuous $\aleph_0$-separable pseudometric on $\boldsymbol{Z}$ that has no uniformly continuous pseudometric extension to $\boldsymbol{R}$, whereas $\psi_g$ is a continuous $\aleph_0$-separable pseudometric extension of $\psi_f$ to $\boldsymbol{R}$. Thus boundedness is needed if one desires uniformly continuous extensions, and contrapositively, one can expect only continuous extensions in the unbounded case.

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