AN INFINITE SUBALGEBRA OF $\mathrm{EXT}_A(Z_2, Z_2)$

BY

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1. **Introduction.** The day is probably not far off when we will have a complete and effective description of $\mathrm{Ext}_A(Z_2, Z_2)$, where A denotes the Steenrod algebra (mod 2). In the meantime, theorems on the structure of certain submodules of this algebra may be of interest, both for their intrinsic value and as clues to the final solution. The purpose of this note is to present a proof of the existence of a submodule of considerable size and of very regular behavior.

We always use A to denote the mod 2 Steenrod algebra and we write Ext for its cohomology, $\mathrm{Ext} = \mathrm{Ext}_A(Z_2, Z_2)$. Our notation for elements of Ext is based on that of May [3] as extended by one of us [4]. We always imagine Ext as displayed in the plane with the generators of $\mathrm{Ext}^{s,t}$ written at the lattice point whose coordinates are $(t-s, s)$, and the use of such terms as "above x" or "to the right of y" should be understood in this sense. When we say that an element is located "at (m, n)" we mean $t-s=m$, $s=n$. Thus if an element is located at (m, n), its image under the Adams periodicity operator P^1 is located at $(m+8, n+4)$.

Recall that there is an element g located at (20, 4) which generates a polynomial subalgebra of Ext [3], [4]. We will prove the following results.

THEOREM 1. *There exists a submodule* Λ *of* Ext, *containing sixteen elements, and such that the elements*

$$\{P^i g^j \lambda : i \geqq 0, j \geqq 0, \lambda \in \Lambda\}$$

form an independent set in Ext (*as a* Z_2 *vector space*). *The product of any such element with* h_0 *is zero, and none of the elements are divisible by* h_0.

We take for Λ the following sixteen elements:

$d_0 l$, $d_0 m$, $e_0 m$, gm located at (46, 11), (49, 11), (52, 11) and (55, 11) respectively;

$P^1 g^2$, $d_0 e_0 g$, $d_0 g^2$, $e_0 g^2$ located at (48–51–54–57, 12);

$P^1 v$, $d_0 u$, $e_0 u$, gu located at (50–53–56–59, 13);

$P^1 d_0 r$, $P^1 e_0 r$, $P^1 gr$, $d_0 e_0 r$ located at (52–55–58–61, 14).

Thus the sixteen elements of Λ are arrayed evenly upon a parallelogram with vertices at (46, 11), (55, 11), (61, 14), and (52, 14). A display of a portion of Ext containing these elements is given in Appendix 1, Table 1. Table 2 in Appendix 1 displays a portion of the wedge.

Received by the editors March 8, 1967.

(1) This work was supported by the U.S. Army Research Office (Durham). The first-named author is an A. P. Sloan Fellow. The second-named author is now at the University of Chicago.

Some such elements have many aliases, because of relations in Ext, but they are always easily recognizable by their regularity.

THEOREM 2. *There is also a family* T *of six elements, with the property that* $\{g^j\tau : j \geqq 0, \tau \in T\}$ *is likewise an independent set, which does not meet the one generated by* Λ.

These are the elements g^2 at (40, 8), v at (42, 9), w at (45, 9), $d_0 r$ at (44, 10), $e_0 r$ at (47, 10), and gr at (50, 10). (Applying P^1 to these elements produces six of the members of Λ.)

Combining all the $P^i g^j \Lambda$ with all the $g^j T$, we obtain an infinite wedge-shaped diagram, filling out the angle with vertex at g^2 (i.e. at (40, 8)), bounded above by the line $s = \frac{1}{2}(t-s) - 12$, which is "parallel" to the Adams edge, and bounded below by the line $t-s=5s$. Thus this diagram spreads over essentially three-fifths of the diagram of Ext. Above the wedge, we are in the range of Adams periodicity of period 32 (at least stably). Inside the wedge, the picture is not quite so clear; aside from the elements we are discussing, certain other fairly regular patterns appear, and there is some "static" near the lower boundary of the wedge. Below the wedge, Ext still appears quite unpredictable.

The upper boundary of the wedge contains the elements g^2, v, $d_0 r$, $d_0 l$, $P^1 g^2$, $P^1 v$, etc. whereas the lower boundary contains g^2, w, gr, gm, g^3, gw, $g^2 r$, etc. In particular the lower boundary contains all the powers of g.

We will also show the following:

THEOREM 3. *The wedge elements form a subalgebra of* Ext.

The fact that none of these elements are divisible by h_0 follows immediately from the first part of Theorem 1 and the fact that $h_0 g^2 = 0$. (Otherwise, supposing $P^i g^j \lambda = h_0 \mu$, we obtain the contradiction $0 = (h_0 g^2)(\mu) = P^i g^{j+2} \lambda \neq 0$.)

The same argument shows that none of these elements are divisible by h_i for any $i \leqq 5$, since $h_i g^2 = 0$ for $i \leqq 5$ by direct calculation. Presumably the restriction on i is not essential, since it seems likely that, for any i, $h_i g^j$ vanishes when j is sufficiently large.

We remark, for what it is worth, that every one of these elements lies in a bigrading $(t-s, s)$ for which t is divisible by 3.

We will refer to the following elements of Ext which appear in the known range: d_0 at (14, 4), e_0 at (17, 4), g at (20, 4); r at (30, 6), l at (32, 7), m at (35, 7); u at (39, 9), v at (42, 9), w at (45, 9), and z at (41, 10). We will also use the following relations between these elements: $d_0^2 = P^1 g$; $e_0^2 = d_0 g$; $gl = e_0 m$; $d_0 e_0 r = gz$; $h_1 e_0 g = h_0 h_2 m$; $h_1^2 u = h_0 z$; $d_0 v = e_0 u$. Note that the first two of these relations imply that $d_0 e_0^2 g = P^1 g^3$.

In the proof it should be remembered that Ext has been computed explicitly (with some reservations about the multiplicative structure) for $t-s \leqq 70$ [4]. If an argument about $P^i g^j \lambda$ appears to break down for $i=j=0$, say, this is of no consequence, since the statement in question has been verified directly for such initial cases.

2. **Some preliminaries.** In stable homotopy theory we may think of the Hopf maps as maps from S^{n-1} to S^0, where $n=2^j$, $j=0, 1, 2, 3$. Using this notation, any one of the Hopf maps gives rise to a cofibration

$$S^0 \xrightarrow{i} S^0 \cup e^n \xrightarrow{p} S^n$$

and to long exact sequences

$$\cdots \longrightarrow \mathrm{Ext}_R^{s,t}(S^0) \xrightarrow{i_\#} \mathrm{Ext}_R^{s,t}(S^0 \cup e^n) \xrightarrow{p_\#} \mathrm{Ext}_R^{s,t}(S^n) \longrightarrow \cdots$$

where we may take $R=A$ (and Ext (X) means Ext $(H^*(X), Z_2)$) or we may equally well take $R=E^0A$ (and Ext (X) then means Ext $(E^0H^*(X), Z_2)$). In either case it has been shown by Adams [1] that the connecting homomorphism is given by multiplication by h_j (in the obvious sense). Each Ext (X) is an Ext (S^0)-module, and the maps are Ext (S^0)-module morphisms. These complexes and exact sequences play a central role in the proof of the theorem. In addition, they are used in the proofs of Lemma 1(b) and Lemma 2 below.

LEMMA 1. *In* $\mathrm{Ext}_A(S^0)$ *we have the following Massey products*:

(a) $e_0g=\langle h_0, h_1, m\rangle$;

(b) $m=\langle h_2, h_1, r\rangle$;

(c) $z=\langle h_1, h_0, u\rangle$.

Proof. Adams has shown that $\langle h_1, h_0, h_1\rangle=h_0h_2$ and that $\langle h_0, h_1, h_0\rangle=h_1^2$. We can then obtain (a) and (c) from the following calculations (noting zero indeterminacy):

$$h_1e_0g = h_0h_2m = \langle h_1, h_0, h_1\rangle m = h_1\langle h_0, h_1, m\rangle;$$

$$h_0z = h_1^2u = \langle h_0, h_1, h_0\rangle u = h_0\langle h_1, h_0, u\rangle.$$

We need another argument for (b). One proof goes as follows: In $\mathrm{Ext}_A(S^0 \cup e^2)$ (where the attaching map is the stable Hopf map η) we can show that $i_\#(m)=h_2R$ where $p_\#(R)=r$ (and the multiplication is the module action of Ext (S^0)). Since $R=\langle i_\#(1), h_1, r\rangle$, we deduce that $i_\#(m)=i_\#(\langle h_2, h_1, r\rangle)$ and the result follows. (See Appendix 2.)

LEMMA 2. *In* $\mathrm{Ext}_A(S^0 \cup e^8)$ (*where the attaching map is the stable Hopf map* σ) *there is a class* $\boldsymbol{P}$ *such that* $p_\#(\boldsymbol{P})=h_0^4$ *and such that* $\boldsymbol{P}\cdot i_\#(\alpha)=i_\#(P^1\alpha)$ *for any* α *in* $\mathrm{Ext}_A(S^0)$. *This class is a surviving cycle in the Adams spectral sequence for* $S^0 \cup e^8$.

See [2].

3. **Proof of Theorems 1 and 2.** We will prove that the $\{P^ig^j\lambda\}$ are nonzero by working a row at a time, or rather an infinite group of rows at a time. We begin with the known fact that all the elements in question which occur in a dimension s congruent to 0 mod 4 have already been proved nonzero in Ext by a technique originally due to May [4, Chapter 5](2). If $s=4k$, then the upper boundary of the

(2) An elegant new proof has recently been published by A. Zachariou [7].

wedge contains the element $P^{k-2}g^2$ located at $(8k+24, 4k)$. Reading along this row from left to right, we find $P^{k-2}g^2$, $P^{k-3}d_0e_0g$, $P^{k-3}d_0g^2$, and $P^{k-3}e_0g^2$ from $P^{k-3}\Lambda$; then the four elements from this row of $P^{k-4}g\Lambda$; and so forth, until we get $g^{k-3}\Lambda$ and then finally g^k (from $g^{k-2}\mathrm{T}$). One can generate the succession of elements in this row by replacing d_0 by e_0 or e_0 by g at each step; or, if no d_0 or e_0 is visible, using the relation $P^1g=d_0^2$ to replace P^1g by d_0e_0.

We next consider the rows underneath, i.e. the rows for which $s\equiv 3 \pmod 4$. If $s=4k-1$, this row begins with $P^{k-3}d_0l$, $P^{k-3}d_0m$, $P^{k-3}e_0m$, and $P^{k-3}gm$, and proceeds across to $g^{k-2}m$. (We write $P^id_0e_0m$ rather than P^id_0gl when these terms appear at every fourth place in such a row.) We consider the (stable) complex $S^0 \cup e^1$ where the attaching map is of degree 2. Recall that Ext_A for this space is an $\mathrm{Ext}_A(S^0)$-module. Our method is to show that the elements in the row $s=4k$ in $\mathrm{Ext}_A(S^0 \cup e^1)$ contain as "factors" (in the sense of module action) the elements in the row $s=4k-1$ of $\mathrm{Ext}_A(S^0)$; thus these latter elements must be nonzero, since the former are nonzero.

Indeed, $\mathrm{Ext}_A(S^0 \cup e^1)$ contains an element H_1 such that $p_{\#}(H_1)=h_1$; this follows from the long exact sequence in Ext_A for the cofibration. Now $i_{\#}(e_0g)=m\cdot H_1$, where $m \in \mathrm{Ext}_A(S^0)$ acts according to the module action. This follows from Lemma 1(a), since $H_1=\langle i_{\#}(1), h_0, h_1\rangle$. Now P^ig^j is nonzero in $\mathrm{Ext}_A(S^0)$, and projects to a nonzero element under $i_{\#}$ since we have proved that no such element is divisible by h_0. But then

$$\begin{aligned} 0 \neq i_{\#}(P^{i+1}g^{j+3}) &= i_{\#}(P^ig^j\cdot d_0e_0\cdot e_0g) = (P^ig^jd_0e_0)i_{\#}(e_0g) \\ &= (P^ig^jd_0e_0)(mH_1) = (P^ig^jd_0e_0m)(H_1), \end{aligned}$$

where the last step is justified by the fact that $i_{\#}$ is a module morphism. This proves that $P^ig^jd_0e_0m$ is nonzero. It follows immediately that $P^ig^jd_0m$, $P^ig^je_0m$, and P^ig^jm are nonzero, but all the elements we are talking about in the row $s=4k-1$ have this form, except the lead-off element $P^{k-3}d_0l$. However, this one is also nonzero, since when we multiply it by g we obtain $P^{k-3}d_0e_0m$. This shows that all the wedge elements are nonzero in a row $s\equiv 3 \pmod 4$.

We next proceed to the rows $s\equiv 2 \pmod 4$ and use a similar device. We consider $S^0 \cup e^2$ where the attaching map is the stable Hopf map η. In Ext for this space there is a class $H_2=\langle i_{\#}(1), h_1, h_2\rangle$ such that $p_{\#}(H_2)=h_2$, and such that $r\cdot H_2=i_{\#}(m)$ (by Lemma 1(b)). We have proved that $P^ig^jd_0e_0m\neq 0$, and using the same ideas as in the previous case, the calculation

$$0 \neq i_{\#}(P^ig^jd_0e_0m) = (P^ig^jd_0e_0r)\cdot H_2$$

shows that $P^ig^jd_0e_0r$ is nonzero in $\mathrm{Ext}_A(S^0)$. It follows immediately that all the elements in the wedge are nonzero along the rows $s\equiv 2 \pmod 4$.

In the remaining rows we have $s\equiv 1 \pmod 4$. Such a row leads off with $P^{k-3}v$, $P^{k-4}d_0u$, $P^{k-4}e_0u$, and $P^{k-4}gu$, and terminates with $g^{k-3}v$ and $g^{k-3}w$ (from $g^{k-3}\mathrm{T}$). (We write $P^id_0e_0g^ju$ rather than $P^{i+1}g^{j+1}v$ when these elements appear along the row, after the initial place.) We appeal to the complex $S^0 \cup e^2$ again, and use the class $H_0 = \langle i_\#(1), h_1, h_0\rangle$ such that $p_\#(H_0) = h_0$. By Lemma 1(c), $u \cdot H_0 = i_\#(z)$. Using the relation $d_0 e_0 r = gz$, we have

$$0 \neq i_\#(P^i g^{j-1} d_0 e_0 r) = (P^i g^j) i_\#(z) = (P^i g^j u) \cdot H_0$$

and thus $P^i g^j u \neq 0$ (if $j \geqq 1$; but the result follows for $j=0$ by multiplying by g). Since $P^{i+1} g^{j+2} u = P^i g^j d_0 e_0^2 u$, all the elements in these rows are nonzero, except that we do not obtain the last two, $g^j v$ and $g^j w$, in this way. But multiplying these by d_0 gives nonzero products, and so they also are nonzero.

This completes the argument that all the wedge elements are nonzero in Ext.

To see that h_0 times any wedge element is zero, we use Lemma 2. In Ext $(S^0 \cup e^8)$, the operator P^1 corresponds to an actual product, so $h_0 \alpha = 0$ implies $P^1 h_0 \alpha = 0$. (Here we use the fact that none of the wedge elements are divisible by h_3.) Also, $h_0 g^2 = 0$, so the assertion holds whenever the wedge element has g^2 as a factor. The few wedge elements which are not covered by these two cases can be checked individually.

4. **Proof of Theorem 3.** We should perhaps begin by remarking that this theorem is not obvious. Along the lower boundary of the wedge (which forms a subalgebra by itself), we are claiming, for example, that $(gr)^2 = g^5$. This would follow immediately from the relation $r^2 = g^3$; but this latter is *not* a correct relation. Also, most of our present knowledge of the multiplicative structure of Ext is based on the May spectral sequence; but Theorem 3 is completely false if interpreted in the filtered version of Ext which one obtains as E_∞ of the May spectral sequence. For example, $r^2 = 0$ there. In fact, of the ten key relations which we shall list below, eight are false in May's E_∞; only (2) and (3) are correct there, while in the others the left-hand side is zero.

With these comments in mind, we observe that Theorem 3 essentially reduces to proving the following relations:

(1) $r^2 = g^3 + \alpha$ where $d_0 \alpha = e_0 \alpha = g\alpha = 0$;

(2) $rm = gw$; (3) $m^2 = g^2 r$; (4) $rv = e_0 gm$;

(5) $rw = g^2 m$; (6) $mv = e_0 g^3$; (7) $mw = g^4$;

(8) $v^2 = d_0 g^2 r$; (9) $vw = e_0 g^2 r$; (10) $w^2 = g^3 r$.

For the proof of (1) we refer the reader to [5]. The relations (2) and (3) hold in May's E_∞ and project to bona fide relations in Ext. For the remaining relations, we will argue with Massey products.

May has given the Massey products $v = \langle h_1, h_2, e_0 g\rangle$ and $w = \langle h_1, h_2, g^2\rangle$. We need three others:

LEMMA. (a) $g^2 = \langle h_2, h_1, m\rangle$; (b) $e_0 r = \langle h_2, h_1, v\rangle$; (c) $gr = \langle h_2, h_1, w\rangle$.

In each case we observe that the triple product makes sense, and show that its product with a certain element is nonzero; there is no indeterminacy, and the result follows. In the first case, we have

$$g^3\langle h_2, h_1, m\rangle = g\langle g^2, h_2, h_1\rangle m = gwm$$

and by (1), (2), and (3) we have $gwm = rm^2 = g^2r^2 = g^5$, from which part (a) follows. In the next case, if we multiply e_0r by e_0g we get the wedge element d_0g^2r, while

$$e_0g\langle h_2, h_1, v\rangle = \langle e_0g, h_2, h_1\rangle v = v^2$$

which not only gives part (b) of the lemma but also part (8) of the theorem. Similarly we can get (c) of the lemma and (9) of the theorem.

We prove (4) by the calculation

$$rv = r\langle h_1, h_2, e_0g\rangle = \langle r, h_1, h_2\rangle e_0g = e_0gm$$

and a similar argument gives (5). We get (6) from

$$mv = m\langle h_1, h_2, e_0g\rangle = \langle m, h_1, h_2\rangle e_0g = e_0g^3$$

(using (a) of the lemma) and (7) similarly. Finally (10) is obtained by formulas similar to those used for (8) and (9). This completes the proof of Theorem 3.

We note in particular that *every wedge element generates a polynomial subalgebra of* Ext.

5. **Some open questions.**

1. All the wedge elements appear in the Iwai-Shimada calculation of $H^*(A_2)$ [6] where A_2 is the subalgebra of A generated by Sq^1, Sq^2, and Sq^4. It is interesting to ask what phenomenon analogous to the wedge should appear (in addition to the wedge itself) in $H^*(A_3)$.

2. It is an empirical fact, for $t-s \leqq 70$, that the lower boundary of the wedge delineates the zone of periodicity. Above this lower boundary, everything is periodic with period 8 (in $t-s$) except elements which form a nonzero product with h_0^4; these have period 16. This line lies much deeper than any periodicity theorem which has been proved to date.

3. It is not hard to calculate δ_4 in the Adams spectral sequence for the wedge elements and show that every wedge element either bounds or is bounded by another wedge element, except of course near the upper boundary (where some δ_2's and δ_3's reach out to hit elements lying above the wedge) or near the lower boundary (where some wedge elements are permanent cycles, but we do not know what, if anything, they might bound). However, some of these calculations are fictitious, since we do not have a proof that the elements in question survive to E_4 in the Adams spectral sequence. If one could settle this, one could prove that none of the wedge elements survive (except possibly some along the lower boundary).

We indicate the calculations of δ_4. It is well known that d_0 and g are permanent cycles. Also $P^n d_0$ is a permanent cycle, by virtue of its location; it follows that

$P^n g = P^{n-1} d_0^2$ is also a permanent cycle. It has been proved [2] that $\delta_4(P^n e_0 g) = P^{n+2} g$ (for $n \geqq 0$). From this it follows immediately that

$$\delta_4(P^i e_0 g^j) = P^{i+2} g^j \qquad (i \geqq 0, j \geqq 1)$$

and that

$$\delta_4(P^i d_0 e_0 g^j) = P^{i+2} d_0 g^j \qquad (i \geqq 0, j \geqq 1).$$

This takes care of almost all the wedge elements for which $s = 4k$. Along each such row the wedge elements alternately kill or are killed by a δ_4, until we approach the lower boundary of the wedge (at the right-hand end of the row), where the last few permanent cycles may survive, since the δ_4's which might kill them (according to the above scheme) now originate below the boundary of the wedge. Indeed, g^2 itself represents a homotopy element [2].

In the rows for which $s = 4k-1$, we use the fact that $e_0 m$ is a permanent cycle (there is nothing for it to hit, except elements known to survive; see [2]). This gives

$$\delta_4(d_0 g^2 m) = \delta_4(e_0 m \cdot e_0 g) = e_0 m \cdot P^2 g = d_0 g \cdot P^1 d_0 l$$

from which it follows that $\delta_4(gm) = P^1 d_0 l$. From this we obtain $\delta_4(P^i d_0 g^{j+1} m) = P^{i+2} e_0 g^j m$ ($i \geqq 0, j \geqq 0$) and also

$$\delta_4(P^i g^{j+1} m) = P^{i+1} d_0 e_0 g^{j-1} m \, (= P^{i+1} d_0 g^j l) \qquad (i \geqq 0, j \geqq 0).$$

Again this cleans out the rows in question in an alternating pattern, except near the lower boundary of the wedge, and also near the upper boundary, where we have $\delta_3(P^n d_0 m) = P^{n+1} h_1 u$ [2].

We argue similarly in the rows for which $s = 4k-2$. We find $P^1 e_0 r$ and $d_0 e_0 r$ ($= gz$) to be permanent cycles, and, starting again from $\delta_4(e_0 g) = P^2 g$, we can show that $\delta_4(P^i g^j r) = P^{i+1} d_0 e_0 g^{j-2} r$ and that $\delta_4(P^i d_0 g^j r) = P^{i+2} e_0 g^{j-1} r$. This cleans out these rows, except near the boundaries of the wedge. Near the upper boundary one has $\delta_3(P^n d_0 r) = P^{n+1} h_1 d_0 g$ [2].

The remaining rows also follow the pattern; from the differential $\delta_4(e_0 u) = P^2 u$ [2] we can obtain $\delta_4(P^i g^j v) = P^{i+1} d_0 g^{j-1} u$ and $\delta_4(P^i e_0 g^j u) = P^{i+2} g^j u$. At the upper boundary, $\delta_2(P^n v) = P^n h_0 z$.

Appendix 1. We present two tables here for reference.

In Table 1 we give a graphical presentation of a portion of Ext. Recall that the vertical coordinate represents s and the horizontal coordinate $t-s$. The range displayed is $30 \leqq t-s \leqq 61$. The first wedge element, g^2, is found at (40, 8), and the wedge opens out slowly so that in column 61 there are three wedge elements present (where $s = 14, 17, 20$). Of course Ext contains many other elements in this range. Some are indecomposable, like r at (30, 6) or n at (31, 5). Others are decomposable and have been entered in the table as products, like h_4^2 at (30, 2) or like the wedge elements themselves. Many elements contain h_0 or h_1 as a factor, and we find it

Appendix I

Table I

$\mathrm{Ext}_A(Z_2, Z_2)$ for $t-s \le 61$

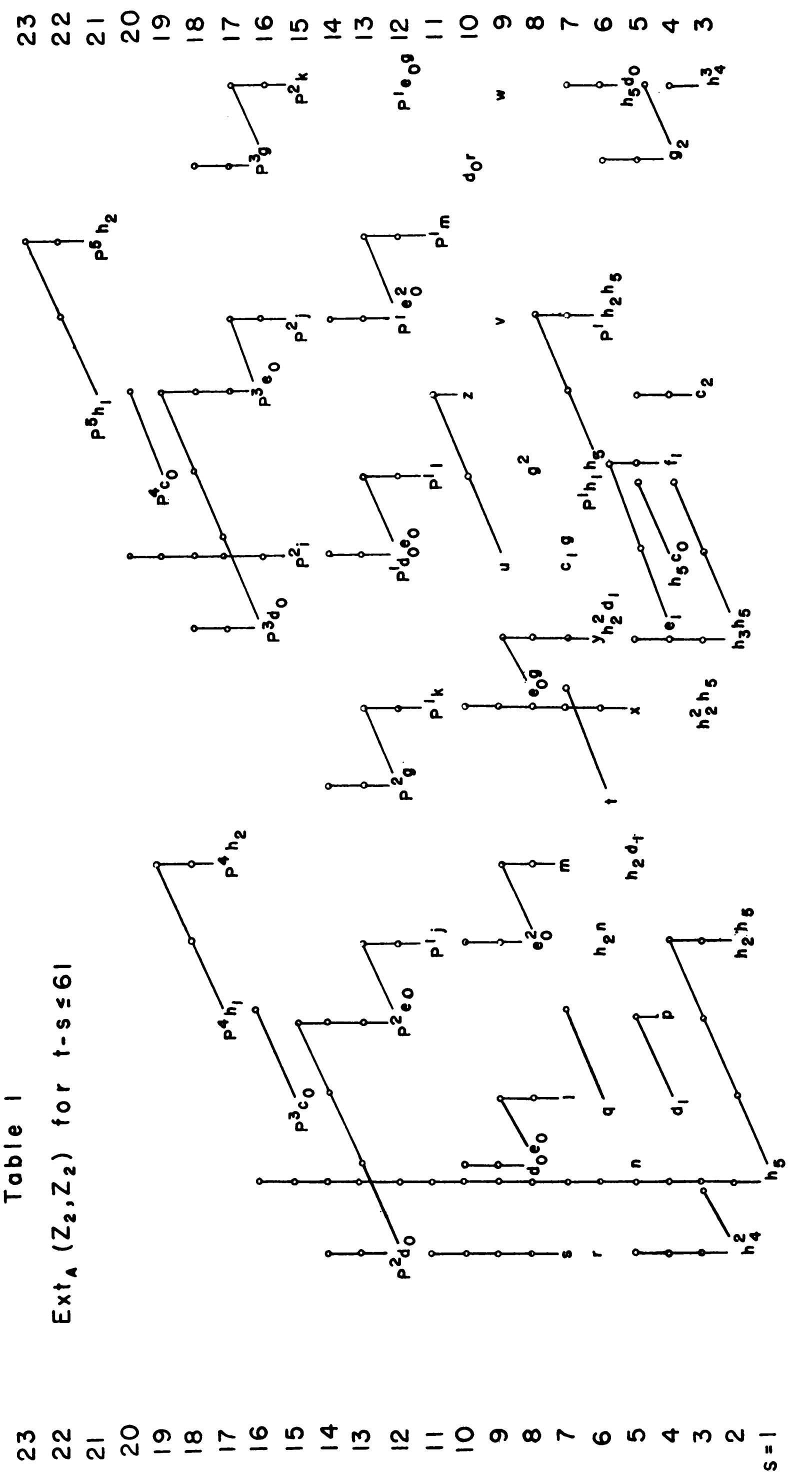

Table I (continued)

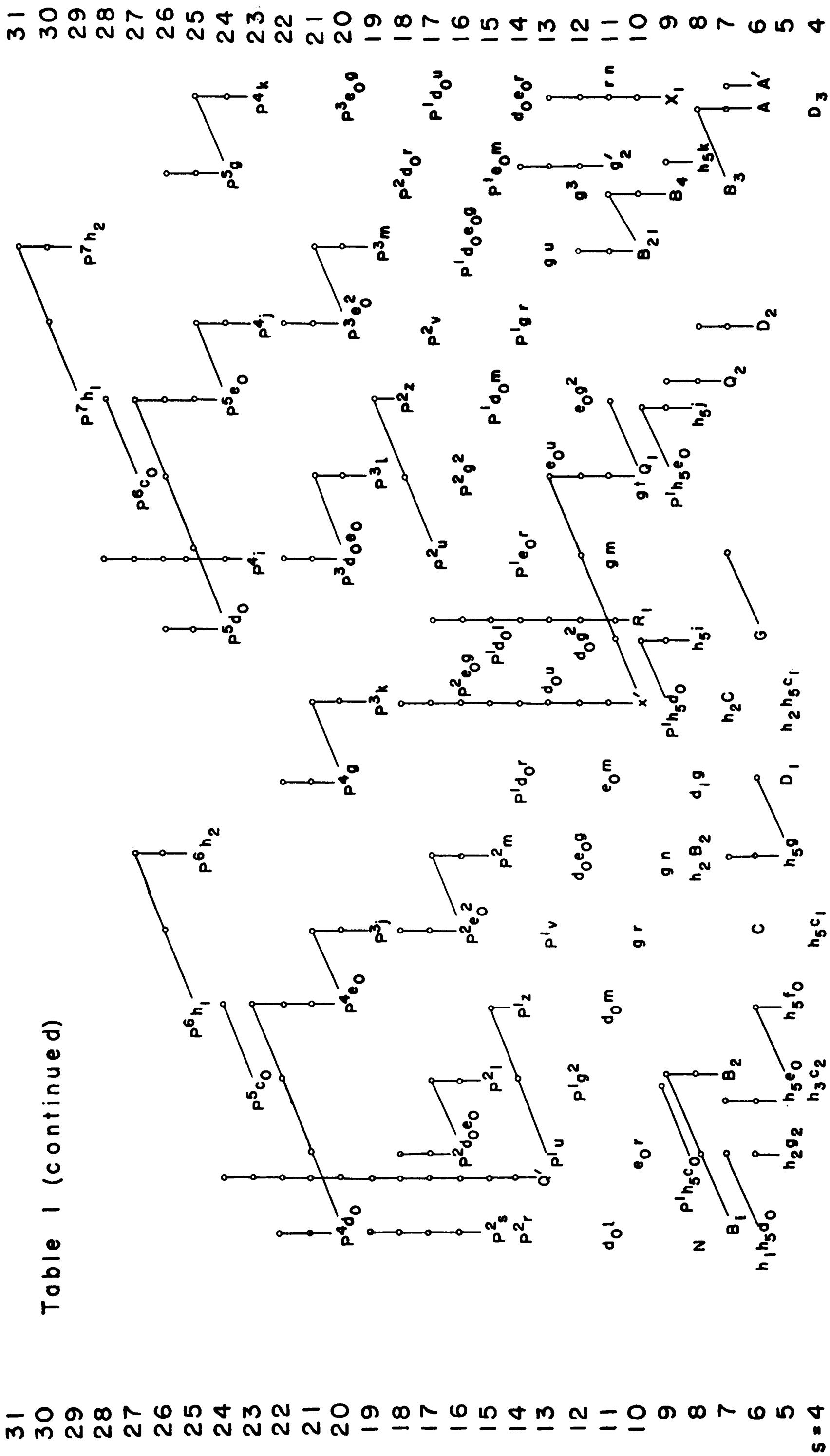

Table 2
The beginning of the wedge
In this table only the wedge elements in a given bigrading appear

24																															
																															P^3dl
22																													P^3dr		
																											P^3v			P^2du	
20																									P^3g^2			P^2deg			P^2dg^2
																						P^2dl			P^2dm			P^2em			
18																					P^2dr			P^2er			P^2gr			$Pder$	
																		P^2v			Pdu			Peu			Pgu			deu
16																	P^2g^2			$Pdeg$			Pdg^2			Peg^2			Pg^3		
														Pdl			Pdm			Pem			Pgm			dem			dgm	
14													Pdr			Per			Pgr			der			dgr			egr			g^2r
											Pv			du			eu			gu			gv			gw					
12									Pg^2			deg			dg^2			eg^2			g^3										
							dl			dm			em			gm															
10					dr			er			gr																				
			v			w																									
8	g^2																														
	40		42		44		46		48		50		52		54		56		58		60		62		64		66		68		70

convenient to indicate them simply by a small dot or circle at the appropriate location, joined by a vertical or diagonal line to the other factor. Thus the element at (30, 5) is $h_0^3(h_4^2)$. If two lines converge, this indicates a relation; thus, for example, $h_1d_1=h_0p$ at (33, 5). (If two different dots or circles are shown, there is no relation; for example, $h_1t\neq h_0^2x$ at (37,7).) The Adams periodicity operator is denoted by P or P^1.

For detailed information about the algebra generators, the reader is referred to [4], Chapter 4 and appendices.

In Table 2 we present a display from which all other elements of Ext have been deleted and only the wedge elements presented, for $t-s\leqq 70$. For typographical reasons, subscripts have been dropped from the notation: thus in Table 2 we write *Pdeg* for the element usually written $P^1d_0e_0g$ (located at (59, 16)). From Table 2 the reader should be able to extrapolate the wedge indefinitely.

APPENDIX 2. In Lemma 1 of §2 we indicated that the Massey product $m=\langle h_2, h_1, r\rangle$ follows from the fact that $i_\#(m)=h_2R$ in $\text{Ext}_A(S^0\cup e^2)$, where $R=\langle i_\#(1), h_1, r\rangle$ and $p_\#(R)=r$. We now indicate how this relation is obtained.

In the May spectral sequence for S^0 we have:

(1) $h_2^2h_4(b_2^0)^2=\delta_2(h_0^2b_2^0b_3^1)$;

(2) $h_2^3g=\delta_2(C)$, where $C=(b_2^1)^3+h_1^2b_3^0h_1(1)$;

(3) $\delta_4[h_2(b_3^0)^2]=h_2h_4(b_2^0)^2+h_2^2g$;

(4) $\delta_2[b_3^0(b_2^1)^2]=h_1D$, where $D=(b_2^1)^3+b_2^0b_2^1h_1(1)$.

From (1)–(3) we find that $\delta_4[h_2^2(b_3^0)^2]=0$ in E_4, and $h_2^2(b_3^0)^2=r$ survives to E_∞.

We may write E_2 of the May spectral sequence for $S^0\cup e^2$ by means of the long exact sequence in Ext_{E^0A} arising from the cofibration. Using naturality arguments, we find that (1) pulls back into $S^0\cup e^2$, but that there is an element $(h_2^3g)^*$ such that $p_\#(h_2^3g)^*=h_2^3g$, which does not bound in E_2 for $S^0\cup e^2$, since $h_1C\neq 0$. Thus we can verify that the natural candidate for R, namely $p_\#^{-1}(h_2^2(b_3^0)^2)$, is not a permanent cycle.

On the other hand, $\delta_2(i_\#(b_3^0(b_2^1)^2))=i_\#(h_1D)$ by (4), but $i_\#(h_1D)=0$ by exactness. We can verify that $i_\#(b_3^0(b_2^1)^2)=R$ survives, and that $p_\#(R)=r$. But $h_2R=i_\#(h_2b_3^0(b_2^1)^2)=i_\#(m)$ since by May's definition $m=h_2b_3^0(b_2^1)^2$. This proves $h_2R=m$ and the rest is easy.

Further discussion of this type of calculation will appear elsewhere ([5]).

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