AN INFINITE SUBALGEBRA OF $\text{Ext}_A(Z_2, Z_2)$

BY

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1. Introduction. The day is probably not far off when we will have a complete and effective description of $\text{Ext}_A(Z_2, Z_2)$, where $A$ denotes the Steenrod algebra (mod 2). In the meantime, theorems on the structure of certain submodules of this algebra may be of interest, both for their intrinsic value and as clues to the final solution. The purpose of this note is to present a proof of the existence of a submodule of considerable size and of very regular behavior.

We always use $A$ to denote the mod 2 Steenrod algebra and we write $\text{Ext}$ for its cohomology, $\text{Ext} = \text{Ext}_A(Z_2, Z_2)$. Our notation for elements of $\text{Ext}$ is based on that of May [3] as extended by one of us [4]. We always imagine $\text{Ext}$ as displayed in the plane with the generators of $\text{Ext}^{s,i}$ written at the lattice point whose coordinates are $(i-s, s)$, and the use of such terms as "above $x$" or "to the right of $y$" should be understood in this sense. When we say that an element is located "at $(m, n)$" we mean $i-s=m, s=n$. Thus if an element is located at $(m, n)$, its image under the Adams periodicity operator $P^1$ is located at $(m+8, n+4)$.

Recall that there is an element $g$ located at $(20, 4)$ which generates a polynomial subalgebra of $\text{Ext}$ [3], [4]. We will prove the following results.

**Theorem 1.** There exists a submodule $\Lambda$ of $\text{Ext}$, containing sixteen elements, and such that the elements

$$\{P^i g^j \lambda : i \geq 0, j \geq 0, \lambda \in \Lambda\}$$

form an independent set in $\text{Ext}$ (as a $Z_2$ vector space). The product of any such element with $h_0$ is zero, and none of the elements are divisible by $h_0$.

We take for $\Lambda$ the following sixteen elements:

- $d_0l, d_0m, e_0m, gm$ located at $(46, 11), (49, 11), (52, 11)$ and $(55, 11)$ respectively;
- $P^1g^2, d_0e_0g, d_0g^2, e_0g^2$ located at $(48-52-56-60, 12)$;
- $P^1u, d_0u, e_0u, gu$ located at $(50-54-58-62, 13)$;
- $P^1d_0r, P^1e_0r, P^1gr, d_0e_0r$ located at $(52-56-60-64, 14)$.

Thus the sixteen elements of $\Lambda$ are arrayed evenly upon a parallelogram with vertices at $(46, 11), (55, 11), (64, 14), (52, 14)$. A display of a portion of $\text{Ext}$ containing these elements is given in Appendix 1, Table 1. Table 2 in Appendix 1 displays a portion of the wedge.

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Some such elements have many aliases, because of relations in \( Ext \), but they are always easily recognizable by their regularity.

**Theorem 2.** There is also a family \( T \) of six elements, with the property that \( \{g^r : j \geq 0, r \in T\} \) is likewise an independent set, which does not meet the one generated by \( \Lambda \).

These are the elements \( g^2 \) at \((40, 8)\), \( v \) at \((42, 9)\), \( w \) at \((45, 9)\), \( d_0 r \) at \((44, 10)\), \( e_0 r \) at \((47, 10)\), and \( g r \) at \((50, 10)\). (Applying \( P^1 \) to these elements produces six of the members of \( \Lambda \).)

Combining all the \( P^i g^j \Lambda \) with all the \( g'T \), we obtain an infinite wedge-shaped diagram, filling out the angle with vertex at \( g^2 \) (i.e. at \((40, 8)\)), bounded above by the line \( s = \frac{1}{4}(t-s) - 12 \), which is “parallel” to the Adams edge, and bounded below by the line \( t-s = 5s \). Thus this diagram spreads over essentially three-fifths of the diagram of \( Ext \). Above the wedge, we are in the range of Adams periodicity of period 32 (at least stably). Inside the wedge, the picture is not quite so clear; aside from the elements we are discussing, certain other fairly regular patterns appear, and there is some “static” near the lower boundary of the wedge. Below the wedge, \( Ext \) still appears quite unpredictable.

The upper boundary of the wedge contains the elements \( g^2, v, d_0 r, d_0 l, P^1 g^2, P^1 v, \) etc. whereas the lower boundary contains \( g^2, w, g r, g m, g^3, g w, g^2 r, \) etc. In particular the lower boundary contains all the powers of \( g \).

We will also show the following:

**Theorem 3.** The wedge elements form a subalgebra of \( Ext \).

The fact that none of these elements are divisible by \( h_0 \) follows immediately from the first part of Theorem 1 and the fact that \( h_0 g^2 = 0 \). (Otherwise, supposing \( P^i g^j \Lambda = h_0 h_i \), we obtain the contradiction \( 0 = (h_0 g^2)(\mu) = P^i g^{i+2} \lambda \neq 0 \).)

The same argument shows that none of these elements are divisible by \( h_i \) for any \( i \leq 5 \), since \( h_0 g^2 = 0 \) for \( i \leq 5 \) by direct calculation. Presumably the restriction on \( i \) is not essential, since it seems likely that, for any \( i \), \( h_i g^j \) vanishes when \( j \) is sufficiently large.

We remark, for what it is worth, that every one of these elements lies in a bigrading \((t-s, s)\) for which \( t \) is divisible by 3.

We will refer to the following elements of \( Ext \) which appear in the known range: \( d_0 \) at \((14, 4)\), \( e_0 \) at \((17, 4)\), \( g \) at \((20, 4)\); \( r \) at \((30, 6)\), \( l \) at \((32, 7)\), \( m \) at \((35, 7)\); \( u \) at \((39, 9)\), \( v \) at \((42, 9)\), \( w \) at \((45, 9)\), and \( z \) at \((41, 10)\). We will also use the following relations between these elements: \( d_0^2 = P^1 g; e_0^2 = d_0 g; g l = e_0 m; d_0 e_0 r = g z; h_1 e_0 g = h_0 h_2 m; h_1^2 u = h_0 z; d_0 v = e_0 u. \) Note that the first two of these relations imply that \( d_0 e_0 g^2 = P^1 g^3 \).

In the proof it should be remembered that \( Ext \) has been computed explicitly (with some reservations about the multiplicative structure) for \( t-s \leq 70 \) [4]. If an argument about \( P^i g^j \Lambda \) appears to break down for \( i = j = 0 \), say, this is of no consequence, since the statement in question has been verified directly for such initial cases.
2. **Some preliminaries.** In stable homotopy theory we may think of the Hopf maps as maps from $S^{n-1}$ to $S^0$, where $n=2^j$, $j=0,1,2,3$. Using this notation, any one of the Hopf maps gives rise to a cofibration

$$S^0 \overset{i}{\longrightarrow} S^0 \cup e^n \overset{p}{\longrightarrow} S^n$$

and to long exact sequences

$$\cdots \longrightarrow \text{Ext}_{\mathbb{Z}_2}^1(S^0) \overset{i_\#}{\longrightarrow} \text{Ext}_{\mathbb{Z}_2}^1(S^0 \cup e^n) \overset{p_\#}{\longrightarrow} \text{Ext}_{\mathbb{Z}_2}^1(S^n) \longrightarrow \cdots$$

where we may take $R=\mathbb{Z}$ (and $\text{Ext}(X)$ means $\text{Ext}(H^*(X), \mathbb{Z}_2)$) or we may equally well take $R=E^0\mathbb{Z}$ (and $\text{Ext}(X)$ then means $\text{Ext}(E^0H^*(X), \mathbb{Z}_2)$). In either case it has been shown by Adams [1] that the connecting homomorphism is given by multiplication by $h$ (in the obvious sense). Each $\text{Ext}(X)$ is an $\text{Ext}(S^0)$-module, and the maps are $\text{Ext}(S^0)$-module morphisms. These complexes and exact sequences play a central role in the proof of the theorem. In addition, they are used in the proofs of Lemma 1(b) and Lemma 2 below.

**Lemma 1.** In $\text{Ext}_A(S^0)$ we have the following Massey products:

(a) $e_0g = \langle h_0, h_1, m \rangle$;

(b) $m = \langle h_2, h_1, r \rangle$;

(c) $z = \langle h_1, h_0, u \rangle$.

**Proof.** Adams has shown that $\langle h_1, h_0, h_1 \rangle = h_0h_2$ and that $\langle h_0, h_1, h_0 \rangle = h_0^2$. We can then obtain (a) and (c) from the following calculations (noting zero indeterminacy):

$$h_1e_0g = h_0h_2m = \langle h_1, h_0, h_1 \rangle m = h_1(h_0, h_1, m);$$

$$h_0z = h_0^2u = \langle h_0, h_1, h_0 \rangle u = h_0(h_1, h_0, u).$$

We need another argument for (b). One proof goes as follows: In $\text{Ext}_A(S^0 \cup e^n)$ (where the attaching map is the stable Hopf map $\eta$) we can show that $i_{\#}(m) = h_2R$ where $p_{\#}(R) = r$ (and the multiplication is the module action of $\text{Ext}(S^0)$). Since $R = \langle i_{\#}(1), h_1, r \rangle$, we deduce that $i_{\#}(m) = i_{\#}(\langle h_2, h_1, r \rangle)$ and the result follows. (See Appendix 2.)

**Lemma 2.** In $\text{Ext}_A(S^0 \cup e^n)$ (where the attaching map is the stable Hopf map $\eta$) there is a class $P$ such that $p_{\#}(P) = h_0^3$ and such that $P \cdot i_{\#}(\alpha) = i_{\#}(P \cdot \alpha)$ for any $\alpha$ in $\text{Ext}_A(S^0)$. This class is a surviving cycle in the Adams spectral sequence for $S^0 \cup e^n$.

See [2].

3. **Proof of Theorems 1 and 2.** We will prove that the $\{P'g', \lambda\}$ are nonzero by working a row at a time, or rather an infinite group of rows at a time. We begin with the known fact that all the elements in question which occur in a dimension $s$ congruent to 0 mod 4 have already been proved nonzero in $\text{Ext}$ by a technique originally due to May [4, Chapter 5](2). If $s = 4k$, then the upper boundary of the

(2) An elegant new proof has recently been published by A. Zachariou [7].
wedge contains the element $P^{k-3}g^2$ located at $(8k + 24, 4k)$. Reading along this row from left to right, we find $P^{k-2}g^2$, $P^{k-3}d_0e_0g$, $P^{k-3}d_0g^2$, and $P^{k-3}e_0g^2$ from $P^{k-3}g\Lambda$; then the four elements from this row of $P^{k-4}g\Lambda$; and so forth, until we get $g^{k-3}g\Lambda$ and then finally $g^6$ (from $g^{k-2}T$). One can generate the succession of elements in this row by replacing $d_0$ by $e_0$ or $e_0$ by $g$ at each step; or, if no $d_0$ or $e_0$ is visible, using the relation $P^1g = d_0^2$ to replace $P^1g$ by $d_0e_0$.

We next consider the rows underneath, i.e. the rows for which $s \equiv 3 \pmod{4}$. If $s = 4k - 1$, this row begins with $P^{k-3}d_0l$, $P^{k-3}d_0m$, $P^{k-3}e_0m$, and $P^{k-3}gm$, and proceeds across to $g^{k-2}m$. (We write $P^l d_0e_0m$ rather than $P^l d_0g l$ when these terms appear at every fourth place in such a row.) We consider the (stable) complex $S^0 \cup e^1$ where the attaching map is of degree 2. Recall that Ext for this space is an Ext(A)-module. Our method is to show that the elements in the row $s = 4k$ in Ext(A) ($S^0 \cup e^1$) contain as "factors" (in the sense of module action) the elements in the row $s = 4k - 1$ of Ext(A ($S^0$); thus these latter elements must be nonzero, since the former are nonzero.

Indeed, Ext(A ($S^0 \cup e^1$) contains an element $H_1$ such that $p\#(H_1) = h_1$; this follows from the long exact sequence in Ext for the cofibration. Now $i\#(g_0g) = m \cdot H_1$, where $m \in Ext(A (S^0))$ acts according to the module action. This follows from Lemma 1(a), since $H_1 = (i\#(1), h_0, h_1)$. Now $P^1g^l$ is nonzero in Ext(A ($S^0$), and projects to a nonzero element under $i\#$ since we have proved that no such element is divisible by $h_0$. But then

$$0 \neq i\#(P^{1+1}g^{i+3}) = i\#(P^1g^l, d_0e_0, e_0g) = (P^l g^l d_0e_0) i\#(e_0g)$$

$$= (P^l g^l d_0e_0)(m H_1) = (P^l g^l d_0e_0m)(H_1),$$

where the last step is justified by the fact that $i\#$ is a module morphism. This proves that $P^1g^l d_0e_0m$ is nonzero. It follows immediately that $P^l g^l d_0m$, $P^l g^l e_0m$, and $P^l g^l m$ are nonzero, but all the elements we are talking about in the row $s = 4k - 1$ have this form, except the lead-off element $P^{k-3}d_0l$. However, this one is also nonzero, since when we multiply it by $g$ we obtain $P^{k-3}d_0g l$. This shows that all the wedge elements are nonzero in a row $s \equiv 3 \pmod{4}$.

We next proceed to the rows $s \equiv 2 \pmod{4}$ and use a similar device. We consider $S^0 \cup e^2$ where the attaching map is the stable Hopf map $\eta$. In Ext for this space there is a class $H_2 = (i\#(1), h_1, h_2)$ such that $p\#(H_2) = h_2$, and such that $r \cdot H_2 = i\#(m)$ (by Lemma 1(b)). We have proved that $P^1g^l d_0e_0m \neq 0$, and using the same ideas as in the previous case, the calculation

$$0 \neq i\#(P^1g^l d_0e_0m) = (P^l g^l d_0e_0\sigma r) \cdot H_2$$

shows that $P^1g^l d_0e_0\sigma r$ is nonzero in Ext(A ($S^0$). It follows immediately that all the elements in the wedge are nonzero along the rows $s \equiv 2 \pmod{4}$.

In the remaining rows we have $s \equiv 1 \pmod{4}$. Such a row leads off with $P^{k-3}v$, $P^{k-4}d_0u$, $P^{k-4}e_0u$, and $P^{k-4}u$, and terminates with $g^{k-3}v$ and $g^{k-3}w$ (from $g^{k-3}T$). (We write $P^l d_0e_0g^l u$ rather than $P^{1+1}g^{i+1}v$ when these elements appear
along the row, after the initial place.) We appeal to the complex \( S^0 \cup e^2 \) again, and use the class \( H_0 = \langle i_\#(1), h_1, h_0 \rangle \) such that \( p_\#(H_0) = h_0 \). By Lemma 1(c), \( u \cdot H_0 = i_\#(z) \).

Using the relation \( d_0 e^r = g z \), we have

\[
0 \neq i_\#(P^1 g^{l-1} d_0 e^r) = (P^1 g^l) i_\#(z) = (P^1 g^l u) \cdot H_0
\]

and thus \( P^1 g^l u \neq 0 \) (if \( j \geq 1 \); but the result follows for \( j = 0 \) by multiplying by \( g \)). Since \( P^1 + g^l + 2u = P^1 g^l d_0 e^3 u \), all the elements in these rows are nonzero, except that we do not obtain the last two, \( g^l v \) and \( g^l w \), in this way. But multiplying these by \( d_0 \) gives nonzero products, and so they also are nonzero.

This completes the argument that all the wedge elements are nonzero in \( \text{Ext} \).

To see that \( h_0 \) times any wedge element is zero, we use Lemma 2. In \( \text{Ext} (S^0 \cup e^2) \), the operator \( P^1 \) corresponds to an actual product, so \( h_0 a = 0 \) implies \( P^1 h_0 a = 0 \). (Here we use the fact that none of the wedge elements are divisible by \( h_3 \).) Also, \( h_0 g^2 = 0 \), so the assertion holds whenever the wedge element has \( g^2 \) as a factor. The few wedge elements which are not covered by these two cases can be checked individually.

4. Proof of Theorem 3. We should perhaps begin by remarking that this theorem is not obvious. Along the lower boundary of the wedge (which forms a subalgebra by itself), we are claiming, for example, that \( (gr)^2 = g^5 \). This would follow immediately from the relation \( r^2 = g^3 \); but this latter is not a correct relation. Also, most of our present knowledge of the multiplicative structure of \( \text{Ext} \) is based on the May spectral sequence; but Theorem 3 is completely false if interpreted in the filtered version of \( \text{Ext} \) which one obtains as \( E_\infty \) of the May spectral sequence. For example, \( r^2 = 0 \) there. In fact, of the ten key relations which we shall list below, eight are false in May’s \( E_\infty \); only (2) and (3) are correct there, while in the others the left-hand side is zero.

With these comments in mind, we observe that Theorem 3 essentially reduces to proving the following relations:

1. \( r^2 = g^3 + \alpha \) where \( d_0 \alpha = e_0 \alpha = g \alpha = 0 \);
2. \( rm = gw \);
3. \( m^2 = g^2r \);
4. \( rv = e_0 gm \);
5. \( rw = g^2m \);
6. \( mv = e_0 g^3 \);
7. \( mw = g^4 \);
8. \( v^2 = d_0 g^2r \);
9. \( vw = e_0 g^2r \);
10. \( w^2 = g^3r \).

For the proof of (1) we refer the reader to [5]. The relations (2) and (3) hold in May’s \( E_\infty \) and project to bona fide relations in \( \text{Ext} \). For the remaining relations, we will argue with Massey products.

May has given the Massey products \( v = \langle h_1, h_2, e_0g \rangle \) and \( w = \langle h_1, h_2, g^2 \rangle \). We need three others:

**Lemma.**

(a) \( g^2 = \langle h_2, h_1, m \rangle \);
(b) \( e_0 r = \langle h_2, h_1, v \rangle \);
(c) \( gr = \langle h_2, h_1, w \rangle \).

In each case we observe that the triple product makes sense, and show that its
product with a certain element is nonzero; there is no indeterminacy, and the result follows. In the first case, we have
\[ g^3 \langle h_2, h_1, m \rangle = g \langle g^2, h_2, h_1 \rangle m = g^m \]
and by (1), (2), and (3) we have \( g^m = r^m = g^2 r^2 = g^5 \), from which part (a) follows. In the next case, if we multiply \( e_0 r \) by \( e_0 g \) we get the wedge element \( d_0 g^2 r \), while
\[ e_0 g \langle h_2, h_1, v \rangle = \langle e_0 g, h_2, h_1 \rangle v = v^2 \]
which not only gives part (b) of the lemma but also part (8) of the theorem. Similarly we can get (c) of the lemma and (9) of the theorem.

We prove (4) by the calculation
\[ rv = r \langle h_1, h_2, e_0 g \rangle = \langle r, h_1, h_2 \rangle e_0 g = e_0 g m \]
and a similar argument gives (5). We get (6) from
\[ mv = m \langle h_1, h_2, e_0 g \rangle = \langle m, h_1, h_2 \rangle e_0 g = e_0 g^2 \]
(using (a) of the lemma) and (7) similarly. Finally (10) is obtained by formulas similar to those used for (8) and (9). This completes the proof of Theorem 3.

We note in particular that every wedge element generates a polynomial subalgebra of Ext.

5. Some open questions.

1. All the wedge elements appear in the Iwai-Shimada calculation of \( H^*(A_2) \) [6] where \( A_2 \) is the subalgebra of \( A \) generated by \( SQ^1, SQ^2, \) and \( SQ^4 \). It is interesting to ask what phenomenon analogous to the wedge should appear (in addition to the wedge itself) in \( H^*(A_3) \).

2. It is an empirical fact, for \( t - s \leq 70 \), that the lower boundary of the wedge delineates the zone of periodicity. Above this lower boundary, everything is periodic with period 8 (in \( t - s \)) except elements which form a nonzero product with \( h^5_5 \); these have period 16. This line lies much deeper than any periodicity theorem which has been proved to date.

3. It is not hard to calculate \( \delta_4 \) in the Adams spectral sequence for the wedge elements and show that every wedge element either bounds or is bounded by another wedge element, except of course near the upper boundary (where some \( \delta_2 \)'s and \( \delta_3 \)'s reach out to hit elements lying above the wedge) or near the lower boundary (where some wedge elements are permanent cycles, but we do not know what, if anything, they might bound). However, some of these calculations are fictitious, since we do not have a proof that the elements in question survive to \( E_4 \) in the Adams spectral sequence. If one could settle this, one could prove that none of the wedge elements survive (except possibly some along the lower boundary).

We indicate the calculations of \( \delta_4 \). It is well known that \( d_0 \) and \( g \) are permanent cycles. Also \( P^n d_0 \) is a permanent cycle, by virtue of its location; it follows that
$P^n g = P^{n-1}d_0^2$ is also a permanent cycle. It has been proved [2] that $\delta_4(P^n e_0 g) = P^{n+2}g$ (for $n \geq 0$). From this it follows immediately that

$$\delta_4(P^i e_0 g^j) = P^{i+2}g^j \quad (i \geq 0, j \geq 1)$$

and that

$$\delta_4(P^i d_0 e_0 g^j) = P^{i+2}d_0 g^j \quad (i \geq 0, j \geq 1).$$

This takes care of almost all the wedge elements for which $s=4k$. Along each such row the wedge elements alternately kill or are killed by a $\delta_4$, until we approach the lower boundary of the wedge (at the right-hand end of the row), where the last few permanent cycles may survive, since the $\delta_4$'s which might kill them (according to the above scheme) now originate below the boundary of the wedge. Indeed, $g^2$ itself represents a homotopy element [2].

In the rows for which $s=4k-1$, we use the fact that $e_0 m$ is a permanent cycle (there is nothing for it to hit, except elements known to survive; see [2]). This gives

$$\delta_4(d_0^2 m) = \delta_4(e_0 m \cdot e_0 g) = e_0 m \cdot P^2 g = d_0 g \cdot P^1 d_0 l$$

from which it follows that $\delta_4(gm) = P^1 d_0 l$. From this we obtain $\delta_4(P^i d_0 g^{i+1} m) = P^{i+2} e_0 g^i m (i \geq 0, j \geq 0)$ and also

$$\delta_4(P^i g^{i+1} m) = P^{i+1} d_0 e_0 g^{i-1} m (= P^{i+1} d_0 g^i l) \quad (i \geq 0, j \geq 0).$$

Again this cleans out the rows in question in an alternating pattern, except near the lower boundary of the wedge, and also near the upper boundary, where we have $\delta_3(P^n d_0 m) = P^{n+1} h_1 u [2]$.

We argue similarly in the rows for which $s=4k-2$. We find $P^1 e_0 r$ and $d_0 e_0 r (= g^2)$ to be permanent cycles, and, starting again from $\delta_4(e_0 g) = P^2 g$, we can show that $\delta_4(P^i g^i r) = P^{i+1} d_0 e_0 g^{i-2} r$ and that $\delta_4(P^i d_0 g^i r) = P^{i+2} e_0 g^{i-2} r$. This cleans out these rows, except near the boundaries of the wedge. Near the upper boundary one has $\delta_3(P^n d_0 r) = P^{n+1} h_1 d_0 g [2]$. The remaining rows also follow the pattern; from the differential $\delta_4(e_0 u) = P^2 u [2]$ we can obtain $\delta_4(P^i g^i v) = P^{i+1} d_0 g^{i-1} u$ and $\delta_4(P^i e_0 g^i u) = P^{i+2} g^i u$. At the upper boundary, $\delta_3(P^n v) = P^n h_0 z$.

APPENDIX 1. We present two tables here for reference.

In Table 1 we give a graphical presentation of a portion of Ext. Recall that the vertical coordinate represents $s$ and the horizontal coordinate $t-s$. The range displayed is $30 \leq t-s \leq 61$. The first wedge element, $g^2$, is found at $(40, 8)$, and the wedge opens out slowly so that in column 61 there are three wedge elements present (where $s = 14, 17, 20$). Of course Ext contains many other elements in this range. Some are indecomposable, like $r$ at $(30, 6)$ or $n$ at $(31, 5)$. Others are decomposable and have been entered in the table as products, like $h_0^2$ at $(30, 2)$ or like the wedge elements themselves. Many elements contain $h_0$ or $h_1$ as a factor, and we find it
Appendix 1
Table 1

$\text{Ext}_A(Z_2, Z_2)$ for $t-s \leq 61$

\[
\begin{array}{cccccccccccccccccccccccccccccc}
\end{array}
\]

$(t-s) = 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45$
AN INFINITE SUBALGEBRA OF EXT$_{\lambda}(\mathbb{Z}_2, \mathbb{Z}_2)$

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Table 1 (Continued)

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(1-s) = 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61
Table 2
The beginning of the wedge
In this table only the wedge elements in a given bigrating appear
AN INFINITE SUBALGEBRA OF EXT, (Z₂, Z₂)

convenient to indicate them simply by a small dot or circle at the appropriate location, joined by a vertical or diagonal line to the other factor. Thus the element at (30, 5) is \( h_2^2(h_1^3) \). If two lines converge, this indicates a relation; thus, for example, \( h_1d_1 = h_0p \) at (33, 5). (If two different dots or circles are shown, there is no relation; for example, \( h_2t \neq h_0^2x \) at (37, 7).) The Adams periodicity operator is denoted by \( P \) or \( P^1 \).

For detailed information about the algebra generators, the reader is referred to [4], Chapter 4 and appendices.

In Table 2 we present a display from which all other elements of Ext have been deleted and only the wedge elements presented, for \( t-s \leq 70 \). For typographical reasons, subscripts have been dropped from the notation: thus in Table 2 we write \( P_{deg} \) for the element usually written \( P^1d_0e_0g \) (located at (59, 16)). From Table 2 the reader should be able to extrapolate the wedge indefinitely.

APPENDIX 2. In Lemma 1 of §2 we indicated that the Massey product \( m = \langle h_2, h_1, r \rangle \) follows from the fact that \( i\#(m) = h_2R \) in \( Ext_A(S^0 \cup e^2) \), where \( R = \langle i\#(1), h_1, r \rangle \) and \( p\#(R) = r \). We now indicate how this relation is obtained.

In the May spectral sequence for \( S^0 \) we have:

1. \( h_2^2h_4(b_0^3)^2 = \delta_2(h_2^2h_0^3b_1^3) \);
2. \( h_3^2g = \delta_2(C) \), where \( C = (b_3^2)^3 + h_2^2h_0^3h_1(1) \);
3. \( \delta_4[h_4(b_0^3)^2] = h_4h_4(b_0^3)^2 + h_3^2g \);
4. \( \delta_2(b_3^2(b_1^3)^2) = h_1D \), where \( D = (b_1^3)^3 + b_0^5b_1^2h_1(1) \).

From (1)–(3) we find that \( \delta_4[h_2^2(b_0^3)^2] = 0 \) in \( E_4 \), and \( h_3^2(b_0^3)^2 = r \) survives to \( E_{\infty} \).

We may write \( E_2 \) of the May spectral sequence for \( S^0 \cup e^2 \) by means of the long exact sequence in \( Ext_{S^0} \) arising from the cofibration. Using naturality arguments, we find that (1) pulls back into \( S^0 \cup e^2 \), but that there is an element \((h_2^2g)^*\) such that \( p\#(h_2^2g)^* = h_2^2g \), which does not bound in \( E_2 \) for \( S^0 \cup e^2 \), since \( h_1C \neq 0 \). Thus we can verify that the natural candidate for \( R \), namely \( p\#^{-1}(h_2^2(b_0^3)^2) \), is not a permanent cycle.

On the other hand, \( \delta_2(i\#(b_0^3(b_1^3)^2)) = i\#(h_1D) \) by (4), but \( i\#(h_1D) = 0 \) by exactness. We can verify that \( i\#(b_0^3(b_1^3)^2) = R \) survives, and that \( p\#(R) = r \). But \( h_2R = i\#(h_2b_0^3(b_1^3)^2) = i\#(m) \) since by May’s definition \( m = h_2b_0^3(b_1^3)^2 \). This proves \( h_2R = m \) and the rest is easy.

Further discussion of this type of calculation will appear elsewhere ([5]).

Bibliography


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