BISIMPLE INVERSE SEMIGROUPS

BY

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In [1] Clifford showed that the structure of any bisimple inverse semigroup with identity is uniquely determined by that of its right unit subsemigroup. The object of this paper is to show that the structure of any bisimple inverse semigroup with or without identity is determined by that of any of its $\mathscr{R}$-classes.

Let us define a *right partial semigroup* S to be a set S together with a partial binary operation satisfying the following condition:

(A) if, for elements a, b, c of S, $a(bc)$ is defined then so also is $(ab)c$ defined and then $a(bc)=(ab)c$.

We say that a right partial semigroup S is *isomorphic* with a right partial semigroup T if there exists a bijection ϕ of S onto T such that ab is defined if and only if $a\phi\, b\phi$ is defined and such that if ab is defined then $(ab)\phi=a\phi\, b\phi$.

We define an RP-system (R, P) to be a right partial semigroup R together with a subsemigroup P of R such that:

P(1) ab is defined if and only if $a \in P$, for all a, b in R;

P(2) R has a left identity contained in P;

P(3) $ac=bc$ implies that $a=b$ for all $a, b \in P$, $c \in R$;

P(4) for all $a, b \in R$, $Pa \cap Pb=Pc$ for some $c \in R$.

It then follows from P(1) and P(3) that any left identity of R contained in P is, in fact, a two-sided identity for P and so is unique.

Now consider any $\mathscr{R}$-class R of a bisimple inverse semigroup S. If we define the partial binary operation $\circ$ in R by the rule that $a \circ b=ab$ if and only if $ab \in R$, then with respect to this operation R is a right partial semigroup with a subsemigroup P such that (R, P) is an RP-system (Theorem 1.4).

Note 1. RP-systems can be obtained from systems having products more generally defined than stipulated in P(1). We then just ignore products which do not satisfy the condition P(1) (cf. Examples 1, 2 in §6).

Note 2. In particular, R could be a lattice ordered group and P the positive cone of R.

We show that for any RP-system (R, P) there exists a bisimple inverse semigroup, which we denote by $R^{-1} \circ R$, some $\mathscr{R}$-class of which is isomorphic with R. Conversely, for any $\mathscr{R}$-class R of a bisimple inverse semigroup S there is a subsemigroup P of R such that (R, P) is an RP-system and $R^{-1} \circ R$ is isomorphic with S.

Received by the editors March 22, 1966

[1] The author acknowledges support from the National Science Foundation Grant GP 1791 to Tulane University.

It must be admitted that this only transfers the problem of classifying bisimple inverse semigroups to the equally obscure task of classifying RP-systems. However, we do at least have a plentiful supply of RP-systems as subsystems of lattice ordered groups and unique factorization domains (cf. §6).

In §4 we find necessary and sufficient conditions for two RP-systems to yield isomorphic bisimple inverse semigroups under our construction.

We conclude by showing that the maximum group homomorphic image of the bisimple inverse semigroup obtained from an RP-system (R, P) is the group of left quotients of the maximum cancellative homomorphic image of P. This interprets for RP-systems a result of Warne's ([6, Theorem 2.1]).

We adopt the notation and terminology of [2]. In particular, two elements of a semigroup S are said to be $\mathscr{L}$-($\mathscr{R}$-) equivalent if they generate the same principal left (right) ideal of S. We write $\mathscr{H}=\mathscr{L}\cap\mathscr{R}$ and $\mathscr{D}=\mathscr{L}\circ\mathscr{R}=\mathscr{R}\circ\mathscr{L}$. Then $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$, and $\mathscr{D}$ are equivalence relations on S such that $\mathscr{H}\subseteq\mathscr{L}\subseteq\mathscr{D}$ and $\mathscr{H}\subseteq\mathscr{R}\subseteq\mathscr{D}$. We denote by $L_a(R_a, H_a)$ the $\mathscr{L}$-($\mathscr{R}$-, $\mathscr{H}$-) class of S containing the element a. S is said to be *bisimple* if it contains only one $\mathscr{D}$-class.

The elementary properties of inverse semigroups will be found in [2, §1.9]. In particular, we have [2, Lemma 1.19].

LEMMA 1.1. *For any idempotents e, f of an inverse semigroup S*

$$Se \cap Sf = Sef.$$

Thus for any elements a, b of S

$$Sa \cap Sb = Sa^{-1}a \cap Sb^{-1}b = Sa^{-1}ab^{-1}b.$$

An *inverse subsemigroup* of an inverse semigroup S is a subsemigroup T of S such that the inverse of every element in T also belongs to T.

For any inverse semigroup S we denote by E_S the semilattice of idempotents of S.

Let S be a semigroup with an identity element 1. If u and v are elements of S such that $uv=1$, then we call u a *right unit* and v a *left unit* of S. An element which is both a left and right unit is called a *unit* and the set of all units is a subgroup of S called the *unit group* of S. The set of all right units is a subsemigroup of S and is called the *right unit subsemigroup* of S. If for a right unit u of S there exists a right unit v of S such that $uv=1$ then u is a unit of S. Moreover, the unit group of S is just the unit group of P ([2, p. 21]). Then we have almost immediately from [7, Lemma 1.2] the following lemma.

LEMMA 1.2. *Let e be any idempotent of an inverse semigroup S. Then eSe is an inverse subsemigroup of S with identity e which is bisimple if S is bisimple. Let P_e be the right unit subsemigroup of eSe. Then $P_e=R_e\cap eSe=\{a\in R_e : ae=a\}$. Moreover, the unit group of P_e is just H_e.*

For any idempotent e of an inverse semigroup S we shall denote by P_e the right unit subsemigroup of eSe.

LEMMA 1.3. *Let e be any idempotent of an inverse semigroup S. Then* $P_e a = Sa \cap R_e$, *for all elements a of* R_e. *Consequently, if S is bisimple, then for all elements a, b of* R_e, *there exists an element c in* R_e *such that* $P_e a \cap P_e b = P_e c$.

Proof. For all $p \in R_e$, $a \in R_e$,

$$(pa)(pa)^{-1} = paa^{-1}p^{-1} = pep^{-1} = pp^{-1} = e.$$

Therefore $P_e a \subseteq Sa \cap R_e$. Now let x be an element of $Sa \cap R_e$ and a be an element of R_e. Then $x = sa$, for some element s of S, $xx^{-1} = e$ and $aa^{-1} = e$. Hence,

$$e = xx^{-1} = saa^{-1}s^{-1} = ses^{-1} = (se)(se)^{-1}$$

and so $se \in R_e \cap eSe = P_e$. Thus $x = sa = s(ea) = (se)a \in P_e a$.

Now let a and b be any elements of R_e. Then, by Lemma 1.1,

$$P_e a \cap P_e b = Sa \cap R_e \cap Sb \cap R_e = Sa^{-1}a \cap Sb^{-1}b \cap R_e = Sa^{-1}ab^{-1}b \cap R_e.$$

Let $c \in L_{a^{-1}ab^{-1}b} \cap R_e$. Such an element c exists as S is bisimple. Then $Sa^{-1}ab^{-1}b = Sc$ and so

$$P_e a \cap P_e b = Sc \cap R_e = P_e c.$$

THEOREM 1.4. *Let e be any idempotent of a bisimple inverse semigroup S. Define the partial binary operation* $\circ$ *on* R_e *as follows: for any elements a, b of* R_e, $a \circ b$ *is defined if and only if ab is an element of* R_e *and then* $a \circ b = ab$. *Then, with respect to this partial binary operation* (R_e, P_e) *is an RP-system.*

Proof. Suppose that $ab \in R_e$ for some elements a, b of R_e. Then $e = (ab)(ab)^{-1} = abb^{-1}a^{-1} = aea^{-1}$ and consequently, $ae = aa^{-1}ae = aea^{-1}a = ea = a$. Hence $a \in P_e$. Conversely, for any element a of P_e and any element b of R_e, $(ab)(ab)^{-1} = abb^{-1}a^{-1} = aea^{-1} = aa^{-1} = e$. Thus $a \circ b$ is defined if and only if $a \in P_e$, and so P(1) holds for the pair (R_e, P_e).

We now prove that R_e satisfies condition (A). Suppose that $a \circ (b \circ c)$ is defined. Then a and b are necessarily elements of P_e. Hence, as P_e is a subsemigroup of eSe and so of S, $ab \in P_e \subseteq R_e$. Thus $a \circ b$ is defined and contained in P_e. Consequently $(a \circ b) \circ c$ is also defined. We then have

$$a \circ (b \circ c) = a \circ (bc) = a(bc)$$

while

$$(a \circ b) \circ c = (ab) \circ c = (ab)c$$

and these are equal as S is a semigroup.

Now, for any element r of R_e, $rr^{-1} = e \in P_e$ and so

$$e \circ r = er = (rr^{-1})r = r.$$

Thus P(2) holds.

To prove P(3) suppose that $a \circ c = b \circ c$ for some elements a, b of P_e and some element c of R_e. Then $ac = bc$ and so $acc^{-1} = bcc^{-1}$; that is, $ae = be$. Hence $a = b$, since a and b are elements of P_e for which e is a two-sided identity.

That R_e and P_e satisfy condition P(4) follows immediately from Lemma 1.3.

Henceforth, when considering (R_e, P_e) as an RP-system we shall not distinguish between the operation $\circ$ in R_e and the multiplication in S.

2. Throughout this section, let (R, P) be an RP-system.

LEMMA 2.1. *The relation $\mathscr{L}'$ defined on R by $(a, b) \in \mathscr{L}'$ if and only if $Pa = Pb$ is an equivalence relation on R and $(a, b) \in \mathscr{L}'$ if and only if $a = ub$ for some unit u of P.*

Proof. Clearly $\mathscr{L}'$ is an equivalence. Now, if $Pa = Pb$ then $a = ub$ and $b = va$ for some elements u, v of P. Hence $a = uva$ and so, by P(3), $uv = 1$, the identity of P. Thus u and v are units of P. Conversely, $a = ub$, for some unit u of P, implies that $Pa = Pub = Pb$.

Note. If R is an $\mathscr{R}$-class of a bisimple inverse semigroup then it follows from Lemma 1.3 that $\mathscr{L}'$ is just the restriction of $\mathscr{L}$ to R.

We denote the $\mathscr{L}'$-class containing the element a by L'_a. Partially order the set $P(\mathscr{L}')$ of $\mathscr{L}'$-classes by writing $L'_a < L'_b$ if and only if $Pa \subset Pb$. Then $P(\mathscr{L}')$ is a semilattice on account of P(4). Select and keep fixed a representative from each of the $\mathscr{L}'$-classes. If $Pa \cap Pb = Pc$ then let $a \vee b$ denote the representative from the $\mathscr{L}'$-class L'_c containing the element c.

Define $a * b$, for all elements a, b of R, by $(a * b)b = a \vee b$. Then $a * b$ is an element of P, for all elements a, b of R, and is, moreover, on account of P(3), uniquely determined (cf. [1]). Also, since $a \vee b = b \vee a$, we have $(a * b)b = (b * a)a$.

THEOREM 2.2. *Let (R, P) be an RP-system and let an operation $*$ be defined on R as above. Let $R^{-1} \circ R$ denote $R \times R$ under the multiplication*

$$(a, b)(c, d) = ((c * b)a, (b * c)d)$$

where we identify the pairs (a, b), (x, y) if and only if $a = ux$, $b = uy$ for some unit u of P. Then $R^{-1} \circ R$ is a bisimple inverse semigroup such that $E_{R^{-1} \circ R}$, the semilattice of idempotents of $R^{-1} \circ R$, is isomorphic with $P(\mathscr{L}')$ and, for some $\mathscr{R}$-class R' of $R^{-1} \circ R$, R' is isomorphic with R. $R^{-1} \circ R$ has an identity if and only if $Pa = R$ for some element a of R, and then (a, a) is the identity element of $R^{-1} \circ R$. Conversely, if S is a bisimple inverse semigroup, then for any idempotent e of S, (R_e, P_e) is an RP-system and S is isomorphic with $R_e^{-1} \circ R_e$.

Proof. We prove the theorem by means of a sequence of lemmas.

For any set X, the one-to-one partial transformations of X form an inverse semigroup [2, p. 29] which we call the *symmetric inverse semigroup* on X and denote by $\mathscr{I}_X$. Then, from the proof of [5, Lemma 2], we have the lemma:

Lemma 2.3. *Let $\mathscr{I}_X$ be the symmetric inverse semigroup on some set X. Then, for any elements α, β of $\mathscr{I}_X$, $(\alpha, \beta) \in \mathscr{R}$ if and only if the domain of α = the domain of β.*

For any mapping α, we shall denote the domain of α by $U(\alpha)$.

For each element r of R we define a mapping ρ_r as follows:

$$U(\rho_r) = P \quad \text{and} \quad p\rho_r = pr \text{ for all } p \in P.$$

Then by P(3), ρ_r is a one-to-one mapping of P into R, that is, a one-to-one partial transformation of R with domain P. Now, for any elements r, s of R, $\rho_r = \rho_s$ implies, in particular, that $1\rho_r = 1\rho_s$ and so that $r = s$. Thus the mapping $\rho: r \to \rho_r$ is a one-to-one mapping of R into $\mathscr{I}_R$ such that, by Lemma 2.3, $R\rho$ is contained in a single $\mathscr{R}$-class of $\mathscr{I}_R$.

We point out that if $a \in P$, then clearly $\rho_a\rho_r = \rho_{ar}$, for all elements r of R since ρ_a maps P onto $Pa \subseteq P$. Also, if a is a unit of P, then $\rho_a^{-1} = \rho_{a^{-1}}$.

Let $S = (R\rho)^{-1}(R\rho)$. Then the following lemma establishes that S is a subsemigroup of $\mathscr{I}_R$.

Lemma 2.4. *For any elements a, b, c, d of R, $\rho_b\rho_c^{-1} = \rho_{c*b}^{-1}\rho_{b*c}$ and consequently $(\rho_a^{-1}\rho_b)(\rho_c^{-1}\rho_d) = \rho_{(c*b)a}^{-1}\rho_{(b*c)d}$ and S is a subsemigroup of $\mathscr{I}_R$.*

Proof. Let a, b, c, d, be any elements of R. We have $U(\rho_b) = P$ and so, for any element x of P, $x \in U(\rho_b\rho_c^{-1})$ if and only if $x\rho_b \in U(\rho_c^{-1}) = Pc$; that is, if and only if $xb \in Pb \cap Pc = Pb \vee c = P((c * b)b)$ or, equivalently, if and only if $xb = p((c * b)b) = (p(c * b))b$, by (A), for some element p of P. Hence $x \in U(\rho_b\rho_c^{-1})$, by P(3), if and only if $x = p(c * b)$ for some element p in P. Thus $U(\rho_b\rho_c^{-1}) = P(c * b)$. On the other hand, $U(\rho_{c*b}^{-1}\rho_{b*c}) = P(c * b)$, since ρ_{c*b}^{-1} is a mapping of $P(c * b)$ onto P. Thus $U(\rho_b\rho_c^{-1}) = U(\rho_{c*b}^{-1}\rho_{b*c}) = U$, say. Then, for any $x \in U$, as above, $xb = p(b \vee c) = p((b * c)c) = (p(b * c))c$ and $xb = p((c * b)b) = (p(c * b))b$. In particular, $x = p(c * b)$ for some element p of P. Then $x\rho_b\rho_c^{-1} = xb\rho_c^{-1} = (p(b * c))c\rho_c^{-1} = p(b * c)$. On the other hand, $x\rho_{c*b}^{-1}\rho_{b*c} = p(c * b)\rho_{c*b}^{-1}\rho_{b*c} = p\rho_{b*c} = p(b * c)$. Thus $\rho_b\rho_c^{-1} = \rho_{c*b}^{-1}\rho_{b*c}$.

Now let $\rho_a^{-1}\rho_b$ and $\rho_c^{-1}\rho_d$ be any two elements of S. Then

$$(\rho_a^{-1}\rho_b)(\rho_c^{-1}\rho_d) = \rho_a^{-1}\rho_{c*b}^{-1}\rho_{b*c}\rho_d = (\rho_{c*b}\rho_a)^{-1}\rho_{(b*c)d} = \rho_{(c*b)a}^{-1}\rho_{(b*c)d}$$

as $b * c$ and $c * b$ are both elements of P. Thus S is a subsemigroup of $\mathscr{I}_R$.

Lemma 2.5. *$S = (R\rho)^{-1}(R\rho)$ is a bisimple inverse subsemigroup of $\mathscr{I}_R$ and E_S is isomorphic with $\mathscr{P}(\mathscr{L}')$. S has an identity if and only if $Pa = R$ for some a in R and then $\rho_a^{-1}\rho_a$ is the identity element of S.*

Proof. We know that S is a subsemigroup of $\mathscr{I}_R$, by Lemma 2.4.

Now, as $\rho_1^{-1}\rho_r = \rho_1\rho_r = \rho_{1\cdot r} = \rho_r$ and $\rho_r^{-1}\rho_1 = \rho_r^{-1}\rho_1^{-1} = (\rho_1\rho_r)^{-1} = \rho_r^{-1}$, for all r in R, we clearly have $(R\rho)^{-1} \cup (R\rho) \subseteq (R\rho)^{-1}(R\rho)$.

Each element $\rho_a^{-1}\rho_b$ of S has an inverse $(\rho_a^{-1}\rho_b)^{-1}$ in the inverse semigroup $\mathscr{I}_R$. However, $(\rho_a^{-1}\rho_b)^{-1} = \rho_b^{-1}\rho_a$, an element of S. Thus S is an inverse subsemigroup of $\mathscr{I}_R$.

Now $\rho_b^{-1}\rho_a$ is an idempotent in S (and so in $\mathscr{I}_R$) if and only if it is the identity transformation of some subset of R. Suppose that $\rho_b^{-1}\rho_a$ is an idempotent. Then $U(\rho_b^{-1}\rho_a)=Pb$ and so, in particular, $b\in U(\rho_b^{-1}\rho_a)$. Consequently $b=b\rho_b^{-1}\rho_a$ $=1\cdot b\rho_b^{-1}\rho_a=1\cdot\rho_a=a$. Thus all idempotents are of the form $\rho_a^{-1}\rho_a$, while any element of this form is clearly an idempotent.

For any elements a, b of R, $\rho_a^{-1}\rho_a$ has domain Pa and $\rho_b^{-1}\rho_b$ has domain Pb. Hence, $\rho_a^{-1}\rho_a\leqq\rho_b^{-1}\rho_b$ if and only if $Pa\subseteq Pb$, that is, $L'_a\leqq L'_b$. Moreover, $\rho_a^{-1}\rho_a$ $=\rho_b^{-1}\rho_b$ if and only if $Pa=Pb$ or, equivalently, if and only if $L'_a=L'_b$. Hence E_S is isomorphic with $\mathscr{P}(\mathscr{L}')$. Now, S has an identity if and only if, for some element a of R, $\rho_b^{-1}\rho_b\leqq\rho_a^{-1}\rho_a$ for all elements b of R; that is, if and only if for some a, $Pa\supseteq Pb$ for all b. Thus $\rho_a^{-1}\rho_a$ is an identity for S if and only if $Pa=R$.

To show that S is bisimple we make use of the following lemma ([4, Lemma 1.1]).

LEMMA 2.6. *An inverse semigroup S is bisimple if and only if for any idempotents e, f of S there exists an element x of S such that $xx^{-1}=e$, $x^{-1}x=f$.*

Let $\rho_a^{-1}\rho_a$ and $\rho_b^{-1}\rho_b$ be any two idempotents of S. Then, for the element $\rho_a^{-1}\rho_b$ of S, we have

$$(\rho_a^{-1}\rho_b)(\rho_a^{-1}\rho_b)^{-1} = \rho_a^{-1}\rho_b\rho_b^{-1}\rho_a = \rho_a^{-1}\rho_a$$

and

$$(\rho_a^{-1}\rho_b)^{-1}(\rho_a^{-1}\rho_b) = \rho_b^{-1}\rho_a\rho_a^{-1}\rho_b = \rho_b^{-1}\rho_b$$

since $\rho_a\rho_a^{-1}=\rho_b\rho_b^{-1}$, the identity transformation of P.

Hence S is a bisimple inverse subsemigroup of $\mathscr{I}_R$.

LEMMA 2.7. *$S=(R\rho)^{-1}(R\rho)$ is isomorphic with $R^{-1}\circ R$.*

Proof. Let $\rho_a^{-1}\rho_b=\rho_c^{-1}\rho_d$. Then

$$\rho_a^{-1}\rho_b\rho_b^{-1}\rho_a = (\rho_a^{-1}\rho_b)(\rho_a^{-1}\rho_b)^{-1} = (\rho_c^{-1}\rho_d)(\rho_c^{-1}\rho_d)^{-1} = \rho_c^{-1}\rho_d\rho_d^{-1}\rho_c.$$

Hence $\rho_a^{-1}\rho_a=\rho_c^{-1}\rho_c$, that is, $Pa=Pc$ and so, by Lemma 2.1, $a=uc$ for some unit u of P. Then

$$\rho_b = \rho_a\rho_a^{-1}\rho_b = \rho_a\rho_c^{-1}\rho_d = \rho_{uc}\rho_c^{-1}\rho_d = \rho_u\rho_c\rho_c^{-1}\rho_d = \rho_u\rho_d = \rho_{ud}$$

since $\rho_a\rho_a^{-1}=\rho_c\rho_c^{-1}$, the identity transformation of P. Hence $b=ud$. Thus $a=uc$ and $b=ud$ for some unit u of P.

Conversely, if $a=uc$ and $b=ud$ for some unit u of P, then

$$\rho_a^{-1}\rho_b = \rho_{uc}^{-1}\rho_{ud} = (\rho_u\rho_c)^{-1}\rho_u\rho_d = \rho_c^{-1}\rho_u^{-1}\rho_u\rho_d = \rho_c^{-1}\rho_d.$$

Hence, on account of Lemma 2.4, the mapping ϕ of S into $R^{-1}\circ R$ defined by $(\rho_a^{-1}\rho_b)\phi=(a, b)$ is clearly an isomorphism.

We complete the first half of Theorem 2.2 with the following lemma.

LEMMA 2.8. *R is isomorphic with the $\mathscr{R}$-class R_{ρ_1} of S and consequently is isomorphic with the $\mathscr{R}$-class of $R^{-1}\circ R$ containing* $(1, 1)$. *The latter is just the set of all elements of the form* $(1, a)$ *with $a\in R$.*

Proof. For any element a of R, $\rho_a\rho_a^{-1}=\rho_1$ and so $\rho_a \in R_{\rho_1}$. Now suppose that $\rho_a^{-1}\rho_b$ is contained in R_{ρ_1}. Then

$$\rho_1 = (\rho_a^{-1}\rho_b)(\rho_a^{-1}\rho_b)^{-1} = \rho_a^{-1}\rho_b\rho_b^{-1}\rho_a = \rho_a^{-1}\rho_a.$$

Thus, Pa=domain of $\rho_a^{-1}\rho_a$=domain of $\rho_1=P$. Hence a is a unit of P and so $\rho_a^{-1}=\rho_{a^{-1}}$. Then $\rho_a^{-1}\rho_b=\rho_{a^{-1}}\rho_b=\rho_{a^{-1}b}$. Thus $R\rho_1=\{\rho_a : a \in R\}$.

Now, as remarked prior to Lemma 2.4, ρ is a one-to-one mapping of R into $\mathscr{I}_R$. Hence ρ is, in fact, a bijection of R onto $R\rho_1$. If $\rho_a\rho_b \in R\rho_1$ then

$$\rho_1 = \rho_a\rho_b\rho_b^{-1}\rho_a^{-1} = \rho_a\rho_1\rho_a^{-1}$$

and so

$$\rho_a^{-1}\rho_a = \rho_a^{-1}\rho_1\rho_a = \rho_a^{-1}\rho_a\rho_1\rho_a^{-1}\rho_a = \rho_a^{-1}\rho_a\rho_1.$$

Thus $\rho_a^{-1}\rho_a \leqq \rho_1$, that is, $Pa \subseteq P$. Hence $a \in P$. Conversely, for any element a of P, we know that $\rho_a\rho_b=\rho_{ab}$, for all $b \in R$. Thus $\rho_a\rho_b$ is defined (when $R\rho_1$ is considered a right partial semigroup, as in §1) if and only if $a \in P$. On the other hand ab is defined if and only if $a \in P$. Then for any elements a, b such that ab is defined, we have $a \in P$ and so $\rho_a\rho_b=\rho_{ab}$. Thus ρ is an isomorphism.

3. In this section we prove the converse part of Theorem 6.

Lemma 3.1. *Let R_e, where e is an idempotent, be any $\mathscr{R}$-class of a bisimple inverse semigroup S. Then $S=R_e^{-1}R_e$.*

Proof. Let a be any element of S and let x be contained in $L_a \cap R_e$. Since $a^{-1}a=x^{-1}x$ we have $a=aa^{-1}a=ax^{-1}x$. Now $(ax^{-1})^{-1}(ax^{-1})=xa^{-1}ax^{-1}$ $=xx^{-1}xx^{-1}=e^2=e$ and so $ax^{-1} \in L_e=R_e^{-1}$.

Lemma 3.2. *For any elements x, y of R_e, $(x, y) \in \mathscr{L}$ if and only if $x=uy$ for some element u of H_e.*

Proof. Let $(x, y) \in L$. Then, since $x^{-1}x=y^{-1}y$, we have $x=xy^{-1}y=(xy^{-1})y$ where xy^{-1} is an element of H_e by [2, Theorem 2.17].

Conversely, if $x=uy$, for some element u of H_e, where x and y are elements of R_e, then $x^{-1}x=y^{-1}u^{-1}uy=y^{-1}ey=y^{-1}y$. Thus $(x, y) \in L$.

We know from §1 that (R_e, P_e) is an RP-system and we now wish to show that S is isomorphic with $R_e^{-1} \circ R_e$. Define the mapping ϕ of $R_e^{-1}R_e$ into $R_e^{-1} \circ R_e$ by $(a^{-1}b)\phi=(a, b)$.

Now $a^{-1}b=c^{-1}d$ implies that $a^{-1}bb^{-1}a=c^{-1}dd^{-1}c$, that is, that $a^{-1}a=c^{-1}c$. Hence, by Lemma 3.2, $a=uc$ for some element u of H_e. Then

$$b = bb^{-1}b = aa^{-1}b = ac^{-1}d = ucc^{-1}d = ud.$$

Conversely, $a=uc$ and $b=ud$ for some element u of H_e implies that

$$a^{-1}b = c^{-1}u^{-1}ud = c^{-1}ed = c^{-1}d.$$

Thus ϕ is well defined and is a bijection.

Now, for all elements b, c of R_e, we have, by Lemma 1.2,

$$R_e \cap S(b \vee c) = P(b \vee c) = Pb \cap Pc = R_e \cap Sb \cap R_e \cap Sc$$
$$= R_e \cap Sb^{-1}b \cap Sc^{-1}c = R_e \cap Sb^{-1}bc^{-1}c.$$

Thus, for $x \in R_e \cap L_{b \vee c}$ and $y \in R_e \cap L_{b^{-1}bc^{-1}c}$ we have

$$Px = R_e \cap Sx = R_e \cap Sb \vee c = R_e \cap Sb^{-1}bc^{-1}c = R_e \cap Sy = Py$$

and so $(x, y) \in \mathscr{L}'$. Hence, by Lemma 3.2 and the note following Lemma 2.1, $x = uy$ for some unit u of P_e, that is, for some element u of H_e. Then

$$x^{-1}x = y^{-1}u^{-1}uy = y^{-1}ey = y^{-1}y.$$

Hence $(b \vee c)^{-1}(b \vee c) = x^{-1}x = y^{-1}y = b^{-1}bc^{-1}c$, as $x \in L_{b \vee c}$ and $y \in L_{b^{-1}bc^{-1}c}$.

Also, as $(b * c)c = b \vee c = (c * b)b$, we have

$$c * b = (c * b)e = (c * b)bb^{-1} = (b \vee c)b^{-1}$$

and so $(c * b)^{-1} = b(b \vee c)^{-1}$. Similarly, $(b * c) = (b \vee c)c^{-1}$. Hence, for any elements $a^{-1}b$, $c^{-1}d$ of $R_e^{-1}R_e$ we have

$$a^{-1}bc^{-1}d = a^{-1}b(b^{-1}b)(c^{-1}c)c^{-1}d = a^{-1}b(b \vee c)^{-1}(b \vee c)c^{-1}d$$
$$= a^{-1}(c * b)^{-1}(b * c)d = ((c * b)a)^{-1}(b * c)d.$$

Hence ϕ is an isomorphism.

4. This section is devoted to the proof of the following theorem.

THEOREM 4.1. *Let* (R_1, P_1) *and* (R_2, P_2) *be two RP-systems. Then* $R_1^{-1} \circ R_1 \cong R_2^{-1} \circ R_2$ *if and only if there exists a bijection* ϕ *of* R_1 *onto* R_2 *and an element* t *of* R_1 *such that*

(a) $(ab)\phi = (at)\phi b\phi$ *for all* $a \in P_1$, $b \in R_1$,

(b) $P_2 = (P_1 t)\phi$.

Moreover, when this is the case then

(c) $L'_a \leqq L'_b$ *if and only if* $L'_{a\phi} \leqq L'_{b\phi}$ *for all* $a, b \in R_1$.

Proof. Suppose that there exists a bijection ϕ of R_1 onto R_2 and an element t of R_1 such that (a) and (b) hold. We first show that this implies condition (c).

Suppose that $L'_a \leqq L'_b$. Then $P_1 a \subseteq P_1 b$ and so $a = xb$ for some x in P_1. Hence $a\phi = (xb)\phi = (xt)\phi b\phi$ and so $P_2(a\phi) \subseteq P_2(b\phi)$. Consequently, $L'_{a\phi} \leqq L'_{b\phi}$. Conversely, if $L'_{a\phi} \leqq L'_{b\phi}$ then $a\phi = y(b\phi)$ for some element y of P_2. By (b), $y = (xt)\phi$ for some element x of P_1 and so

$$a\phi = (xt)\phi b\phi = (xb)\phi.$$

Thus $a = xb$ and $L'_a \leqq L'_b$. Thus condition (c) holds. In particular, it follows that $L'_a = L'_b$ if and only if $L'_{a\phi} = L'_{b\phi}$ and so, for all a, b in R_1, $(a \vee b)\phi = w(a\phi \vee b\phi)$ for some unit w of P_2.

Now let e_1 and e_2 be the identities of P_1 and P_2, respectively. Then

$$t\phi = (e_1t)\phi = (e_1t)\phi t\phi = (t\phi)^2$$

and so, by P(3), $t\phi = e_2$.

We define a mapping Φ of $R_1^{-1} \circ R_1$ into $R_2^{-1} \circ R_2$ by

$$(a, b)\Phi = (a\phi, b\phi).$$

To show that Φ is single valued, let $(a, b) = (c, d)$. Then there exists a unit u of P_1 such that $a = uc$ and $b = ud$. Now u has an inverse u^{-1} in the unit group of P_1 and, consequently, $u(u^{-1}t) = e_1t = t$. Hence

$$e_2 = t\phi = (u(u^{-1}t))\phi = (ut)\phi(u^{-1}t)\phi$$

and so $(ut)\phi$ is a unit of P_2. From $a\phi = (ut)\phi c\phi$ and $b\phi = (ut)\phi\, d\phi$ it then follows that $(a\phi, b\phi) = (c\phi, d\phi)$.

As ϕ is an onto mapping so also is Φ. Now suppose that $(a, b)\Phi = (c, d)\Phi$. Then $(a\phi, b\phi) = (c\phi, d\phi)$ and so $a\phi = u(c\phi)$ and $b\phi = u(d\phi)$ for some unit u of P_2. Hence, $L'_{a\phi} = L'_{c\phi}$ and so, by (c), $L'_a = L'_c$. Consequently, $a = vc$, for some unit v of P_1 and so $u(c\phi) = a\phi = (vc)\phi = (vt)\phi c\phi$. Hence, $u = (vt)\phi$. Then

$$b\phi = u(d\phi) = (vt)\phi(d\phi) = (vd)\phi,$$

i.e., $b = vd$.

Hence $(a, b) = (c, d)$ and Φ is a bijection.

Now

$$((a, b)(c, d))\Phi = ((c * b)a, (b * c)d)\Phi = (((c * b)a)\phi, ((b * c)d)\phi)$$

and

$$(a, b)\Phi(c, d)\Phi = (a\phi, b\phi)(c\phi, d\phi) = ((c\phi * b\phi)a\phi, (b\phi * c\phi)d\phi).$$

However, $(c\phi * b\phi)b\phi = c\phi \vee b\phi = w(c \vee b)\phi = w((c * b)b)\phi = w((c * b)t)\phi b\phi$ for some unit w of P_2 and so $(c\phi * b\phi) = w((c * b)t)\phi$.

Hence, $(c\phi * b\phi)a\phi = w((c * b)t)\phi a\phi = w((c * b)a)\phi$. Similarly,

$$(b\phi * c\phi)d\phi = w((b * c)d)\phi \quad \text{and} \quad (a, b)\Phi(c, d)\Phi = ((a, b)(c, d))\Phi.$$

Thus Φ is an isomorphism.

We now consider the converse.

As in §1, for any idempotent e of a bisimple inverse semigroup S we denote by P_e the right unit subsemigroup of eSe. Let $\mathscr{L}'$ be defined on R_e as in §2.

LEMMA 4.2. *Let R_e and R_f be any two $\mathscr{R}$-classes of a bisimple inverse semigroup S. Then there exists a bijection λ of R_e onto R_f satisfying the conditions* (a), (b) *of Theorem* 4.1.

Proof. Let $s \in L_e \cap R_f$. Then $s^{-1} \in R_e \cap L_f$. Consequently, $ss^{-1} = f$ and $s^{-1}s = e$. Let λ be the mapping of R_e into R_f defined by $a\lambda = sa$, for all a in R_e. Now $s^{-1} \in R_e$ and $s^{-1}\lambda = ss^{-1} = f \in R_f$. Hence, by [2, Lemma 2.2], λ is a one-to-one mapping of R_e onto R_f. Moreover, for $a \in P_e$ and $b \in R_e$

$$(ab)\lambda = sab = s(ae)b = sas^{-1}sb = (as^{-1})\lambda b\lambda$$

where $s^{-1} \in R_e$. Thus λ satisfies condition (a) of Theorem 4.1, with $t = s^{-1}$. Now let $b \in P_2$. Then it is a simple matter to show that $a = s^{-1}bs$ is an element of P_1. Moreover, $(as^{-1})\lambda = ss^{-1}bss^{-1} = fbf = b$. Thus λ satisfies condition (b) of Theorem 4.1.

Let (R_1, P_1) and (R_2, P_2) be RP-systems and θ be an isomorphism of $R_1^{-1} \circ R_1$ onto $R_2^{-1} \circ R_2$. As there is no danger of confusion we denote both the identity of P_1 and the identity of P_2 by 1. Let θ_1 (θ_2) be an isomorphism of R_1 (R_2) onto the $\mathscr{R}$-class $R_g = \{(1, a) : a \in R_1\}(R_f = \{(1, a) : a \in R_2\})$ of $R_1^{-1} \circ R_1$ $(R_2^{-1} \circ R_2)$ where $g = (1, 1)$ $(f = (1, 1))$. Such isomorphisms exist by Lemma 2.8. Then θ will induce an isomorphism of R_g onto some $\mathscr{R}$-class R_e, say, of $R_2^{-1} \circ R_2$. Let λ be a bijection of R_e onto R_f defined as in Lemma 4.2 and put $\phi = \theta_1\theta\lambda\theta_2^{-1}$. As each component of ϕ is a bijection so also is ϕ.

Now for any elements a, b of R_e, with $a \in P_e$, we have $(ab)\lambda = (at')\lambda b\lambda$ for some element t' of R_e. Let $t = t'\theta^{-1}\theta_1^{-1}$. Then, for any elements a, b of P_1 and R_1, respectively, we have

$$\begin{aligned}(ab)\phi &= (ab)\theta_1\theta\lambda\theta_2^{-1} = ((a\theta_1\theta)(b\theta_1\theta))\lambda\theta_2^{-1},\\ &= ((a\theta_1\theta t')\lambda(b\theta_1\theta)\lambda)\theta_2^{-1},\\ &= (a\theta_1\theta t\theta_1\theta)\lambda\theta_2^{-1}b\theta_1\theta\lambda\theta_2^{-1},\\ &= (at)\theta_1\theta\lambda\theta_2^{-1}b\phi,\\ &= (at)\phi b\phi.\end{aligned}$$

Moreover,

$$(P_1t)\phi = (P_1(t'\theta^{-1}\theta_1^{-1}))\theta_1\theta\lambda\theta_2^{-1} = (P_1\theta_1\theta t')\lambda\theta_2^{-1} = (P_et')\lambda\theta_2^{-1} = P_f\theta_2^{-1} = P_2.$$

Hence ϕ satisfies all the conditions of Theorem 4.1.

Corollary 4.3. *If, in Theorem* 4.1, $R_1 = P_1$ *and* $R_2 = P_2$, *then* $P_1^{-1} \circ P_1$ *is isomorphic with* $P_2^{-1} \circ P_2$ *if and only if* P_1 *is isomorphic with* P_2.

Proof. If θ is an isomorphism of P_1 onto P_2 then in Theorem 4.1 we can take $\phi = \theta$ and t to be the identity of P_1 and then all the conditions of that theorem are satisfied.

Conversely, let $P_1^{-1} \circ P_1$ and $P_2^{-1} \circ P_2$ be isomorphic. Then there is a mapping ϕ with the properties stated in Theorem 4.1. In particular, there is an element t of P_1 such that $(ab)\phi = (at)\phi b\phi$, for all elements a, b of P_1 and such that $t\phi$ is the identity of P_2. Hence, by condition (c) of Theorem 4.1, t must be a unit of P_1.

Define the mapping θ of P_1 into P_2 by $a\theta = (at)\phi$. Then, as t is a unit and ϕ is a bijection so also is θ a bijection. Moreover, for all elements a, b of P_1,

$$(ab)\theta = (abt)\phi = (a(bt))\phi = (at)\phi(bt)\phi = a\theta b\theta.$$

Thus θ is an isomorphism.

Note. It follows from Lemma 4.2 and Theorem 4.1 that if R_e and R_f are any two $\mathscr{R}$-classes of a bisimple inverse semigroup S then $R_e^{-1} \circ R_e$ is isomorphic with

$R_f^{-1} \circ R_f$ which ties in with the fact that we already know that both of these are isomorphic with S.

5. Every inverse semigroup has a maximal group homomorphic image which was characterized by Munn [3, Theorem 1] as follows:

LEMMA 5.1. *Let S be an inverse semigroup and let a relation σ be defined on S by the rule that $x\sigma y$ if and only if there is an idempotent e in S such that $ex=ey$ (or, equivalently, $xe=ye$). Then σ is a congruence and S/σ is a group. Further, if τ is any congruence on S with the property that S/τ is a group then $\sigma \subset \tau$ and so S/τ is isomorphic with some quotient group of S/σ.*

We call S/σ the *maximal group homomorphic image* of S and σ the *minimum group congruence* on S.

COROLLARY 5.2. *Let e be any idempotent of an inverse semigroup S. Let σ and τ be the minimum group congruences on S and eSe, respectively. Then S/σ is isomorphic with eSe/τ.*

Proof. By Lemma 1.2, eSe is an inverse subsemigroup of S and so τ is defined on eSe by the rule in Lemma 5.1.

It suffices to show that every σ-class contains one and only one τ-class. Let the σ-class (τ-class) containing the element a be noted by a_σ (a_τ). Then we show that, for any element a of S, $a_\sigma \cap eSe = x_\tau$, for some element x of eSe. Let a be any element of S. Then $a_\sigma = (ea)_\sigma = (eae)_\sigma$. Thus $a_\sigma \cap eSe$ is nonempty. Let x and y belong to $a_\sigma \cap eSe$. Then, for some idempotent f in S, $fx=fy$. Hence, since e is the identity of eSe and idempotents of S commute, we have $(efe)x=(efe)y$; that is, $x\tau y$, completing the proof.

Now let S be a bisimple inverse semigroup, let e be any idempotent of S and let P_e be the right unit subsemigroup of eSe. If, following Warne [6], we define the relation η on P_e by: $a\eta b$ if and only if there exists an element h of P_e such that $ha=hb$, then η is a congruence on P_e and P_e/η is the maximum cancellative homomorphic image of P_e (cf. [2, p. 18]). Now, since (R_e, P_e) satisfies condition P(4) it follows, in particular, that P_e is right reversible, that is, for any elements a, b of P_e, there exist elements x, y of P_e such that $xa=yb$ [2, p. 34]. Then P_e/η must also be right reversible and so can be embedded in a group of left quotients [2, Theorem 1.24]. This group is unique to within isomorphism [2, Theorem 1.25], and is the maximum group homomorphic image of eSe [6, Theorem 2.1] and so, by Corollary 5.2, the maximum group homomorphic image of S.

Now suppose that S is isomorphic with $R^{-1} \circ R$ for some RP-system (R, P). Let R_e, for some idempotent e of S, be the $\mathcal{R}$-class of S which is isomorphic with R. Then the right unit subsemigroup P_e of eSe is isomorphic with P and so the group of left quotients of the maximum cancellative homomorphic image of P is isomorphic with the group of left quotients of the maximum cancellative homomorphic image of P_e. Thus we have

THEOREM 5.3. *Let (R, P) be an RP-system. Then the maximum group homomorphic image of $R^{-1} \circ R$ is isomorphic with the group of left quotients of the maximum cancellative homomorphic image of P.*

Now suppose that (R, P) is an *RP*-system and that R is a group. Then P is a cancellative semigroup and so the maximum group homomorphic image is isomorphic with the group of left quotients of P. However, by property P(4), for any element a of R, $Pa \cap P$ is nonempty. Thus there exist elements p, q of P such that $pa=q$. Hence, in R, we have $a=p^{-1}q$. Thus R is the group of left quotients of P and we have

COROLLARY 5.4. *Let (R, P) be an RP-system and let R be a group. Then the maximum group homomorphic image of $R^{-1} \circ R$ is isomorphic with R.*

6. **Examples.** Let G be a lattice ordered group with the group operation denoted by addition. Write $a^+ = a \vee 0$. Let $P=G^+$, the positive cone of G and let R be any subset of G such that

$$\text{(6.1)} \qquad \text{(i) } P \subseteq R, \qquad \text{(ii) } x \in R, \quad x \leqq y \text{ implies that } y \in R.$$

Then we can consider (R, P) as an *RP*-system as follows: define a partial operation in R, denoted by juxtaposition, by

$$ab \text{ is defined if and only if } a \in P \text{ and then } ab = a+b.$$

Conditions A, P(1)–P(4) are easily shown to be satisfied, so that $R^{-1} \circ R$, as defined in Theorem 2.2, is a bisimple inverse semigroup. Moreover, $Pa=Pb$ if and only if $a=b$ and, for any a, b in R, $Pa \cap Pb = P(a \vee b)$ where $a \vee b$ denotes the lattice join of a and b in G. Thus, $(a * b)b = a \vee b$, that is $a * b + b = a \vee b$ and so

$$a * b = a \vee b - b = (a-b) \vee 0 = (a-b)^+.$$

Hence, in $R^{-1} \circ R$ we have as our multiplication

$$\begin{aligned} (a, b)(c, d) &= ((c * b)a, (b * c)d) \\ &= ((c-d)^+ + a, (b-c)^+ + d). \end{aligned}$$

In particular, for any idempotents (a, a), (b, b) of $R^{-1} \circ R$,

$$\begin{aligned} (a, a)(b, b) &= ((b-a)^+ + a, (a-b)^+ + b) \\ &= ((b-a) \vee 0 + a, (a-b) \vee 0 + b) \\ &= (a \vee b, a \vee b). \end{aligned}$$

Hence $(a, a) \leqq (b, b)$ if and only if $a \vee b = a$, that is, if and only if $a \geqq b$. Thus $E_{R^{-1} \circ R}$ is order anti-isomorphic to the partially ordered set R (where the partial ordering in R is the order induced by the lattice order in G).

EXAMPLE 1. Let I be the additive group of integers under the natural ordering, let P be the set of nonnegative integers and let R be any subset of I satisfying (6.1). Then multiplication in $R^{-1} \circ R$ is defined by

$$\begin{aligned}(m, n)(p, q) &= (m+(p-n)^+, q+(n-p)^+)\\ &= (m+(p-n) \vee 0, q+(n-p) \vee 0)\\ &= (m+p-r, n+q-r), \qquad \text{where } r = \min(n, p).\end{aligned}$$

If $R=P$ then $R^{-1} \circ R$ is just the bicyclic semigroup ([2, §1.12]). If $R=I$ then R is just an I-bisimple semigroup as defined in [8].

EXAMPLE 2. Let $G=I \boxplus I$, the cardinal sum of two copies of the integers under the natural ordering of the integers. Again let $P=G^+$. Let

$$\begin{aligned}R_1 &= \{(a, b) : a+b \geqq 0, b \geqq 0, a \geqq -n+1\}\\ R_2 &= \{(a, b) : a+b \geqq 0\}.\end{aligned}$$

Now R_1 and R_2 both satisfy conditions (6.1). Let $S_1=R_1^{-1} \circ R_1$ and $S_2=R_2^{-1} \circ R_2$. Let $e=((a, b), (a, b))$ be any idempotent of S_2. Then $b \geqq -a$, $(a, b) \geqq (a, -a)$ and so $e \leqq ((a, -a), (a, -a))$. If $e \in S_1$ then either $a, b \geqq 0$ or $-n+1 \leqq a < 0$ and so, either $(0, 0) \leqq (a, b)$ or $-n+1 \leqq a<0$, $-a \leqq b$ and $(a, -a) \leqq (a, b)$. Hence either $e \leqq ((0, 0), (0, 0))$ or $-n+1 \leqq a < 0$ and $e \leqq ((a, -a), (a, -a))$. At the same time, the idempotents of the form $((a, -a), (a, -a))$ are incomparable. Hence S_1 has a finite number (and S_2 has a countably infinite number) of maximal idempotents such that every idempotent of S_1 (S_2) is comparable to at least one of these.

EXAMPLE 3. Let P be a unique factorization domain. Let (a, b) denote the greatest common divisor of a and b (unique to within unit factor). Then $Pa \cap Pb=Pc$ where $c=ab/(a, b)$. Once again, with $R=P$, conditions (A), P(1)–P(4) are satisfied and $P^{-1} \circ P$ is a bisimple inverse semigroup (with an identity). For $a, b \in P$,

$$(a * b)b = a \vee b = ab/(a, b),$$

and so $a * b=a/(a, b)$. Thus, for any $(a, b), (c, d) \in P^{-1} \circ P$,

$$(a, b)(c, d) = ((c * b)a, (b * c)d) = (ac/(b, c), bd/(b, c)).$$

ACKNOWLEDGEMENT. I would like to thank Professor A. H. Clifford for many helpful and illuminating comments.

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