

# ERGODIC THEORY AND BOUNDARIES

BY

M. A. AKCOGLU AND R. W. SHARPE

1. **Introduction.** Let  $T$  be a conservative positive contraction on the  $L_1$  space of a finite measure space  $(X, \mathcal{F}, \mu)$ . A theorem of Chacon [5], [2] shows that  $T$  defines a sub  $\sigma$ -field  $\mathcal{I}$  of  $\mathcal{F}$ , consisting of invariant subsets of  $X$ . The ratio ergodic limits are measurable with respect to  $\mathcal{I}$  [5], [2] and the class of these limits contains  $L_\infty(X, \mathcal{I}, \mu)$ , which can be considered as the invariant functions of the adjoint transformation [2]. The main purpose of the present paper is to show that any positive contraction on  $L_1(X, \mathcal{F}, \mu)$  behaves, asymptotically, like a conservative transformation (Theorems 3 and 4) and that the invariant functions of the adjoint transformation can be approximated by the ratio ergodic limits.

Intuitively, a ratio ergodic limit corresponds to the result of an averaging process of different values of a function. It is then natural to consider these limits as functions that are smooth with respect to the asymptotic behaviour of the transformation. This leads (Theorem 6) to a Martin-Doob type representation [12], [8] of invariant functions as the  $L_\infty$  functions of a compact Hausdorff space  $\mathcal{M}$  with a Baire measure. The topology on  $\mathcal{M}$  is just strong enough to make the ergodic limits to correspond to continuous functions. As an example we consider a transformation of Feller [10] and show that for this case the above representation is identical with the Poisson representation of harmonic functions in the unit disk. We also consider the possibility of joining  $X$  and  $\mathcal{M}$ , convergence of measures to  $\mathcal{M}$  in  $X \cup \mathcal{M}$  (Theorem 7), and a relation (Lemma 9) between the Feller and Martin-Doob type representations, corresponding to a result of Feldman [9].

2. **Preliminaries.** Let  $(X, \mathcal{F}, \mu)$  be a finite measure space and let  $L_p = L_p(X, \mathcal{F}, \mu)$ ,  $1 \leq p \leq \infty$  be the usual Banach spaces, and  $L_p^+$  denote the positive cone of  $L_p$ . Let  $T: L_1 \rightarrow L_1$  be a positive linear contraction and  $U: L_\infty \rightarrow L_\infty$  be its dual. For  $\alpha \in L_\infty$  define  $T_\alpha: L_1 \rightarrow L_1$  as  $T_\alpha f = \alpha f + T(1 - \alpha)f$ ,  $f \in L_1$ , and let  $U_\alpha$  be its dual. If  $\chi_E$  is the characteristic function of  $E \in \mathcal{F}$  we write  $T_E$  and  $U_E$  instead of  $T_{\chi_E}$  and  $U_{\chi_E}$ .

The following partial ordering of  $L_1^+$  is similar to that of Bishop and deLeeuw given in [3].

**DEFINITION 1.** For  $f, g \in L_1^+$ ,  $f < g$  if and only if there exist an integer  $n \geq 1$  and  $\alpha_1, \dots, \alpha_n \in L_\infty$  such that  $0 \leq \alpha_i \leq 1$  for  $i = 1, \dots, n$  and  $g = T_{\alpha_n} \cdots T_{\alpha_1} f$ .

This relation is reflexive and transitive and  $f < g$  implies  $\|f\|_1 \geq \|g\|_1$ . Also, an

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Received by the editors March 13, 1967.

induction argument shows that if  $f < g$  then there exists an integer  $n \geq 1$  such that  $g < T^n f$ . Hence  $\{g \in L_1^+ \mid g > f\}$  is (upward) directed by  $<$ .

DEFINITION 2. For  $E \in \mathcal{F}, f \in L_1^+$  let

$$\Psi_E f = \sup_{g > f} \int_E g \, d\mu, \quad \Theta_E f = \lim_{g > f} \Psi_E g.$$

Note that  $\Theta_E f = \lim_{n \rightarrow \infty} \Psi_E T^n f$ .

LEMMA 1. The limits  $\psi_E = \lim_{n \rightarrow \infty} U_E^n \chi_E$  and  $\theta_E = \lim_{n \rightarrow \infty} U^n \psi_E$  both exist (a.e.) and satisfy

$$\Psi_E f = \int \psi_E f \, d\mu, \quad \Theta_E f = \int \theta_E f \, d\mu.$$

**Proof.** By induction,  $U_E^n \chi_E \uparrow$  and  $U^n \psi_E \downarrow$ , so the limits exist. Now if  $f \in L_\infty^+$  satisfies

$$(*) \quad \chi_E Uf \leq \chi_E f, \quad \chi_{E^c} Uf \geq \chi_{E^c} f$$

with  $E^c = X - E$ , then for all  $\alpha \in L_\infty, 0 \leq \alpha \leq 1$ , we have

$$U_\alpha f = \alpha f + (1 - \alpha)Uf \leq \chi_E f + \chi_{E^c} Uf = U_E f.$$

Since, by induction,  $U_E^n \chi_E$  satisfies (\*) for all  $n \geq 0$ , we get, again by induction,  $U_{\alpha_n} \cdots U_{\alpha_1} \chi_E \leq U_E^n \chi_E$ , and hence  $\Psi_E f = \int \psi_E f \, d\mu$ . The final part follows from the definition (cf. also [4] and [2]).

DEFINITION 3. For  $E, F \in \mathcal{F}$ , let

$$\psi_{EF} = \psi_E + \psi_F - \psi_{E \cup F}, \quad \theta_{EF} = \theta_E + \theta_F - \theta_{E \cup F}.$$

$\Psi_{EF}, \Theta_{EF}$  are the functionals on  $L_1$  defined by the  $L_\infty$  functions  $\psi_{EF}, \theta_{EF}$ .

We note that  $\psi_{EF}$  and  $\theta_{EF}$  are monotone and subadditive in each index. This follows easily from the following general result, which will be useful to obtain other relations between these set functions (cf. [7]).

LEMMA 2. If  $a_i$  is real and  $A_i \in \mathcal{F}$  for  $i = 1, \dots, n$  and  $A = \bigcup_{i=1}^n A_i$ , then  $\chi_A \sum_{i=1}^n a_i \psi_{A_i} \geq 0$  implies  $\sum_{i=1}^n a_i \psi_{A_i} \geq 0$  and  $\sum_{i=1}^n a_i \theta_{A_i} \geq 0$ .

**Proof.** If  $f \in L_1^+$  and  $E \subset F, E, F \in \mathcal{F}$ , then by induction:  $\chi_{F^c} T_F^n f \leq \chi_{F^c} T_E^n f$ . Hence

$$\begin{aligned} 0 &\leq \int_{F^c} \psi_E T_F^n f \, d\mu \leq \int_{F^c} \psi_E T_E^n f \, d\mu \leq \int_{E^c} \psi_E T_E^n f \, d\mu \\ &\leq \int \psi_E T_E^n f \, d\mu - \int_E \psi_E T_E^n f \, d\mu \leq \Psi_E f - \int_E T_E^n f \, d\mu \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ . Now

$$\begin{aligned} \int \psi_E f \, d\mu &= \int \psi_E \chi_F f \, d\mu + \int \psi_E \chi_{F^c} f \, d\mu = \int \psi_E \chi_F f \, d\mu + \int \psi_E T_E \chi_F f \, d\mu \\ &= \int \psi_E \chi_F f \, d\mu + \int \psi_E T \chi_{F^c} f \, d\mu = \int \psi_E T_F f \, d\mu \\ &= \int \psi_E T_F^n f \, d\mu = \int_F \psi_E T_F^n f \, d\mu + \int_{F^c} \psi_E T_F^n f \, d\mu, \end{aligned}$$

and hence  $\lim_{n \rightarrow \infty} \int_F \psi_E T_F^n f \, d\mu = \int \psi_E f \, d\mu$ . Using this for the case of  $A_i \subset A$ ,  $i = 1, \dots, n$ , we get

$$0 \leq \sum_{i=1}^n a_i \int_A \psi_{A_i} T_{A_i}^n f \, d\mu \rightarrow \sum_{i=1}^n a_i \int \psi_{A_i} f \, d\mu$$

as  $n \rightarrow \infty$ , which proves the first assertion. Since  $U$  is positive, the remainder follows.

LEMMA 3. *If  $\chi_E \theta_E \geq \alpha \chi_E$  then  $\theta_E \geq \alpha \psi_E$ .*

**Proof.** From the proof of the previous lemma we have that, for  $f \in L_1^+$ ,

$$\lim_{n \rightarrow \infty} \int_{E^c} \theta_E T_E^n f \, d\mu = 0.$$

Hence

$$\begin{aligned} \int \theta_E f \, d\mu &= \int \theta_E T_E^n f \, d\mu = \lim_{n \rightarrow \infty} \int_E \theta_E T_E^n f \, d\mu \\ &\geq \alpha \lim_{n \rightarrow \infty} \int_E T_E^n f \, d\mu \geq \alpha \int \psi_E f \, d\mu. \end{aligned}$$

Finally we prove the following.

LEMMA 4. *For  $E, F \in \mathcal{F}$ ,  $\|\theta_E\|_\infty = \|\chi_E \theta_E\|_\infty = 0$  or  $1$  and  $\|\theta_{EF}\|_\infty = \|\chi_E \theta_{EF}\|_\infty = \|\chi_F \theta_{EF}\|_\infty = 0$  or  $1$ .*

**Proof.** For  $g \in L_1^+$ , as  $n \rightarrow \infty$ ,  $0 \leq \Theta_E(\chi_{E^c} T_E^n g) \leq \Psi_E(\chi_{E^c} T_E^n g) \rightarrow 0$  as in the proof of Lemma 2. Hence the decomposition  $\Theta_E g = \Theta_E T_E^n g = \Theta_E(\chi_E T_E^n g) + \Theta_E(\chi_{E^c} T_E^n g)$  shows that  $\|\theta_E\|_\infty = \|\chi_E \theta_E\|_\infty$ . Now, for  $n, m \geq 1$ ,

$$\begin{aligned} \Theta_{EG} &= \Theta_E T_E^n T^m g = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \Theta_E(\chi_E T_E^n T^m g) \leq \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \|\theta_E\|_\infty \|\chi_E T_E^n T^m g\|_1 \\ &= \lim_{m \rightarrow \infty} \|\theta_E\|_\infty \Psi_E T^m g = \|\theta_E\|_\infty \Theta_E g \end{aligned}$$

which completes the proof of the first part, since  $\|\theta_E\|_\infty \leq 1$ . For the second part, we have, if  $g \in L_1^+$ ,  $0 \leq \Theta_{EF}(\chi_{E^c} T_E^n g) \leq \Theta_E(\chi_{E^c} T_E^n g) \rightarrow 0$  as  $n \rightarrow \infty$  which shows that  $\|\theta_{EF}\|_\infty = \|\chi_E \theta_{EF}\|_\infty$ . Now

$$\Theta_{EG} - \Theta_E(\chi_{E^c} T_E^n g) = \Theta_E(\chi_E T_E^n g) \leq \Theta_{E \cup F}(\chi_E T_E^n g) \leq \|\chi_E T_E^n g\|_1 \leq \Psi_E g;$$

thus,  $\Theta_{EG} \leq \lim_{n \rightarrow \infty} \Theta_{E \cup F}(\chi_E T_E^n g) \leq \Psi_{EG}$ . Replacing  $g$  by  $T^m g$  and letting  $m \rightarrow \infty$  we get

$$\Theta_{EG} = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \Theta_{E \cup F}(\chi_E T_E^n T^m g).$$

Next, consider

$$\Theta_{EF}(\chi_E T_E^n T^m g) = (\Theta_E + \Theta_F - \Theta_{E \cup F})(\chi_E T_E^n T^m g)$$

and let  $n \rightarrow \infty$  to get

$$\Theta_{EFG} = \Theta_{EG} + \lim_{n \rightarrow \infty} \Theta_F(\chi_E T_E^n T^m g) - \lim_{n \rightarrow \infty} \Theta_{E \cup F}(\chi_E T_E^n T^m g).$$

Now, letting  $m \rightarrow \infty$  we have

$$\Theta_{EFG} = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \Theta_F(\chi_E T_E^n T^m g).$$

But

$$\begin{aligned} \Theta_{EFG} &\leq \|\theta_{EF}\|_\infty \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \|\chi_E T_E^n T^m g\|_1 \\ &\leq \|\theta_{EF}\|_\infty \lim_{m \rightarrow \infty} \Psi_E T^m g \\ &\leq \|\theta_{EF}\|_\infty \Theta_{EG}. \end{aligned}$$

Hence

$$\begin{aligned} \Theta_{EFG} &= \Theta_{EF}(T_E^n T^m g) \\ &= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \Theta_{EF}(\chi_E T_E^n T^m g) \\ &\leq \|\theta_{EF}\|_\infty \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \Theta_F(\chi_E T_E^n T^m g) \\ &\leq \|\theta_{EF}\|_\infty \Theta_{EFG}. \end{aligned}$$

This completes the proof, since  $\|\theta_{EF}\|_\infty \leq 1$ .

DEFINITION 4.  $\Sigma = \{E \in \mathcal{F} \mid \Theta_{EE^c} = 0\}$ .

LEMMA 5.  $\Sigma$  is a field.

**Proof.** Let  $E, F \in \Sigma$  and  $G = E \cap F$ . Then

$$0 \leq \theta_{GG^c} = \theta_{G(E^c \cup F^c)} \leq \theta_{GE^c} + \theta_{GF^c} \leq \theta_{EE^c} + \theta_{FF^c} = 0.$$

Thus  $G \in \Sigma$ .

DEFINITION 5.  $\mathcal{A}$  is the  $L_\infty$ -closure of the class of  $\Sigma$ -simple functions.

We note that  $\mathcal{A}$  is a sub-Banach space of  $L_\infty$ .

THEOREM 1. For a real valued function  $f \in L_\infty$ , the following conditions are equivalent:

- (i)  $f \in \mathcal{A}$ ,
- (ii)  $\lim_{g \rightarrow g_0} \int fg \, d\mu$  exists for all  $g_0 \in L_1^+$ ,
- (iii) for all real numbers  $\alpha$  and  $\varepsilon > 0$ ,

$$\theta_{EF} = 0 \quad \text{where} \quad E = \{x \mid f(x) \leq \alpha\}, \quad F = \{x \mid f(x) \geq \alpha + \varepsilon\}.$$

**Proof.** (i)⇒(ii). If  $E \in \Sigma$  then  $\theta_E + \theta_{E^c} = \theta_X$ ; thus, for a real valued  $g_0 \in L_1^+$ ,

$$\begin{aligned} \limsup_{g > g_0} \int_E g \, d\mu &= \limsup_{g > g_0} \int g \, d\mu - \limsup_{g > g_0} \int_{E^c} g \, d\mu \\ &= \liminf_{g > g_0} \int_E g \, d\mu. \end{aligned}$$

Therefore  $\lim_{g > g_0} \int \chi_E g \, d\mu$  exists for all  $E \in \Sigma$ .

Hence it exists for all  $\Sigma$ -simple functions, and thus for all  $f \in \mathcal{A}$ .

(ii)⇒(iii). Suppose  $E$  and  $F$  are as in (iii) but that  $\theta_{EF} \neq 0$ . Then  $\|\theta_{EF}\|_\infty = 1$  and for all  $\delta > 0$  there exists  $g_0 \in L_1^+$  with  $\|g_0\|_1 = 1$  and  $\int \theta_{EF} g_0 \, d\mu \geq 1 - \delta$ . Hence  $\Theta_{Eg_0} \geq 1 - \delta$  and  $\Theta_{Fg_0} \geq 1 - \delta$ . Thus  $\limsup_{g > g_0} \int fg \, d\mu \geq (1 - \delta)(\alpha + \epsilon)$  and  $\liminf_{g > g_0} \int fg \, d\mu \leq (1 - \delta)\alpha + \delta\|f\|_\infty$ . If  $\delta$  is chosen sufficiently small we see that  $\lim_{g > g_0} \int fg \, d\mu$  does not exist.

(iii)⇒(i). Let  $a_1 < a_2 < \dots < a_n$  be  $n$  numbers and let  $E_i = \{x \mid f(x) \leq a_i\}$ . Now

$$\begin{aligned} \sum_{i=1}^n \theta_{E_i E_i^c} &= \sum_{i=1}^n (\theta_{E_i} + \theta_{E_i^c} - \theta_X) \\ &\leq \sum_{i=2}^n (\theta_{E_{i-1}} + \theta_{E_i^c} - \theta_{E_{i-1} \cup E_i^c}) + (\theta_{E_n} + \theta_{E_1^c} - \theta_X) \\ &\leq 1. \end{aligned}$$

Hence if  $E_a = \{x \mid f(x) \leq a\}$  then  $\theta_{E_a E_a^c} \neq 0$  for only countably many  $a$ 's, and so  $f \in \mathcal{A}$ .

**3. Invariant functions.**

**DEFINITION 6.**  $\mathcal{H} = \{f \mid f \in L_\infty, f = Uf\}$  is the class of invariant functions of  $U$ . We assume  $\mathcal{H} \neq \{0\}$ .

Note that  $\mathcal{H}$  is a sub-Banach space of  $L_\infty$ . Also, if  $h \in \mathcal{H}$  and  $g' > g \in L_1^+$ , then  $\int hg' \, d\mu = \int hg \, d\mu$  and hence  $\lim_{g' > g} \int hg' \, d\mu$  exists. Thus  $\mathcal{H} \subset \mathcal{A}$ .

If  $f \in \mathcal{A}$ , then  $\lim_{n \rightarrow \infty} \int f T^n g \, d\mu = \lim_{n \rightarrow \infty} \int U^n f g \, d\mu$  exists for all  $g \in L_1(X, \mathcal{F}, \mu)$ . Hence the bounded sequence  $U^n f, n = 1, 2, \dots$  has a limit  $\pi(f)$  in the  $w^*$ -topology of  $L_\infty$ . Obviously the limit lies in  $\mathcal{H}$ , so  $\pi: \mathcal{A} \rightarrow \mathcal{H}$  is a positive linear contraction.

**DEFINITION 7.**  $\mathcal{A}_0 = \ker \pi = \{f \in \mathcal{A} \mid w^* - \lim U^n f = 0\}$ . Hence  $\mathcal{A}/\mathcal{A}_0 \cong \mathcal{H}$  is a canonical, isometric isomorphism.

Now  $\mathcal{A}$  is a  $C^*$ -algebra with the usual operations. We show that  $\mathcal{A}_0$  is a closed ideal.

**THEOREM 2.**  $\mathcal{A}_0$  is a closed ideal in  $\mathcal{A}$ .

**Proof.** Let  $f \in \mathcal{A}_0$  and assume that  $f$  is real. Choose  $\epsilon > 0$  and set  $E = \{x \mid f(x) \geq \epsilon\}$ . We may assume  $E \in \Sigma$ . Suppose  $\theta_E \neq 0$ ; then for all  $\delta > 0$ , there is a  $g \in L_1^+$  such that  $\|g\|_1 = 1$  and  $\Theta_E g \geq 1 - \delta$ . Hence:

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \int U^n f \cdot g \, d\mu = \lim_{n \rightarrow \infty} \int f T^n g \, d\mu \\ &\geq \epsilon \lim_{n \rightarrow \infty} \int_E T^n g \, d\mu - \|f\|_\infty \lim_{n \rightarrow \infty} \int_{E^c} T^n g \, d\mu \\ &\geq \epsilon(1 - \delta) - \|f\|_\infty \delta. \end{aligned}$$

Clearly, this fails for small  $\delta$ , and so  $\theta_E=0$ . Thus if  $E=\{x \mid |f(x)| > \epsilon\}$ , we have  $\theta_E=0$ .

Now if  $h \in \mathcal{A}$ ,  $h \neq 0$ , set  $F=\{x \mid |f(x)h(x)| \geq \epsilon\}$ . Since  $F \subset \{x \mid |f(x)| \geq \epsilon/\|h\|_\infty\}$ , we have  $\theta_F=0$ . Hence

$$\begin{aligned} \left| \lim_{n \rightarrow \infty} \int U^n(fh)g \, d\mu \right| &\leq \epsilon \lim_{n \rightarrow \infty} \int_F T^n g \, d\mu + \epsilon \|g\|_1 && \text{if } g \in L_1^+ \\ &\leq \epsilon \|g\|_1 && \text{for all } \epsilon > 0. \end{aligned}$$

Hence  $fh \in \mathcal{A}_0$ .

As a result of the lemma, we have given  $\mathcal{A}/\mathcal{A}_0$ , and hence  $\mathcal{H}$  the structure of a  $C^*$ -algebra. Thus  $\mathcal{H}$  has a representation as the set of complex valued continuous functions on its maximal ideal space. This corresponds to Feller's representation [10] of the invariant functions of certain Markov processes, and we shall refer to  $\mathcal{H}$ 's maximal ideal space as the Feller boundary.

As is known [8], [11], the Feller boundary is larger than it need be. In the next section, we obtain some properties of ratio ergodic limits, and use them to define a sub  $C^*$ -algebra  $\mathcal{G}$  of  $\mathcal{H}$ , with a maximal ideal space  $\mathcal{M}$ , smaller than the Feller boundary, but large enough to represent  $\mathcal{H}$  as a function algebra on  $\mathcal{M}$ . This corresponds to the Martin-Doob representation [12], [8], [11] for some classes of functions, and  $\mathcal{M}$  will be referred to as the Martin-Doob boundary.

**4. Properties of ratio ergodic limits.** In [6] Chacon and Ornstein proved that for any  $f, g \in L_1$ , with  $g > 0$ , the limit:

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n T^k f}{\sum_{k=1}^n T^k g}$$

exists a.e. We denote the limit function by  $(f/g)$ . It is also known [5], [4], [1], that if  $\alpha \leq (f/g) \leq \beta$  a.e. on  $E \in \mathcal{F}$ , then  $\alpha \leq \Psi_E(f)/\Psi_E(g) \leq \beta$ .

**THEOREM 3.** *If  $f, g \in L_1^+$  with  $g > 0$ , and*

$$\begin{aligned} E &= \{x \mid (f/g)(x) \leq a\}, \\ F &= \{x \mid (f/g)(x) \geq a + \epsilon\}, \end{aligned}$$

*then  $\theta_{E,F}=0$ , for all  $a \geq 0$  and  $\epsilon > 0$ .*

**Proof.** If  $\theta_{E,F} \neq 0$  then  $\|\chi_E \theta_{E,F}\|_\infty = 1$ . Let  $\delta > 0$  and set  $E_\delta = \{x \mid \theta_{E,F}(x) \geq 1 - \delta\} \cap E$ , and similarly for  $F_\delta$ . Then  $\|\chi_{E-E_\delta} \theta_{E-E_\delta}, F\|_\infty \leq \|\chi_{E-E_\delta} \theta_{E,F}\|_\infty \leq 1 - \delta$ . Hence  $\theta_{E-E_\delta, F} = 0$ , and so  $\theta_{E_\delta, F} \leq \theta_{E,F} \leq \theta_{E_\delta, F} + \theta_{E-E_\delta, F} = \theta_{E_\delta, F}$  which implies  $\theta_{E_\delta, F} = \theta_{E,F}$ . Now  $\psi_{E_\delta} \geq \theta_{E_\delta, F_\delta} \geq 1 - \delta$  on  $E_\delta \cup F_\delta$ . Hence  $\psi_{E_\delta} \geq (1 - \delta)\psi_{F_\delta}$  on  $E_\delta \cup F_\delta$ , which by Lemma 2 yields  $\psi_{E_\delta} \geq (1 - \delta)\psi_{F_\delta}$ , and  $\psi_{F_\delta} \geq (1 - \delta)\psi_{E_\delta}$ . Now  $(f/g) \leq a$  on  $E_\delta$  yields  $\Psi_{E_\delta} f / \Psi_{E_\delta} g \leq a$ . Similarly  $(f/g) \geq a + \epsilon$  on  $F_\delta$  implies  $\Psi_{F_\delta} f / \Psi_{F_\delta} g \geq a + \epsilon$ . These relations yield  $a \Psi_{E_\delta} g \geq (1 - \delta)^2(a + \epsilon)\Psi_{E_\delta} g$  which is false for small  $\delta$  if  $\Psi_{E_\delta}(g) \neq 0$ . Hence  $\theta_{E,F} = 0$ .

COROLLARY. *If  $f, g \in L_1$  and  $(f/g) \in L_\infty$ , then  $(f/g) \in \mathcal{A}$ .*

REMARK. If  $T$  is conservative, then Theorem 3 corresponds to the fact that  $(f/g)$  is measurable with respect to the  $\sigma$ -field of invariant sets (cf. [5], [2]).

THEOREM 4. *If  $(f/g) \in L_\infty$ , and  $h \in \mathcal{A}$ , then  $\int \pi(h) \cdot f \, d\mu = \int \pi(h(f/g))g \, d\mu$ .*

Proof. Recall that  $\int \pi(h(f/g))g \, d\mu = \lim_{n \rightarrow \infty} \int h(f/g)T^n g \, d\mu$ . We may assume  $f$  and  $h$  are real. Choose  $\varepsilon > 0$ , and let  $E_{ij}$ ,  $1 \leq i, j \leq k$  be a  $\Sigma$  partition of  $X$  such that

$$\left\| h - \sum_{ij} h_i \chi_{E_{ij}} \right\|_\infty < \varepsilon, \quad \left\| (f/g) - \sum_{ij} \alpha_j \chi_{E_{ij}} \right\|_\infty < \varepsilon$$

for suitable real  $h_i, \alpha_j$  with  $|h_i| \leq \|h\|_\infty, |\alpha_j| \leq \|(f/g)\|_\infty$ . Now

$$\begin{aligned} & \left| \lim_{n \rightarrow \infty} \int h(f/g)T^n g \, d\mu - \sum_{ij=1}^k h_i \alpha_j \lim_{n \rightarrow \infty} \int_{E_{ij}} T^n g \, d\mu \right| \\ &= \left| \lim_{n \rightarrow \infty} \int h(f/g)T^n g \, d\mu - \sum_{ij} h_i \alpha_j \Theta_{E_{ij}}(g) \right| \leq \varepsilon \|g\|_1 (\|h\|_\infty + \|(f/g)\|_\infty). \end{aligned}$$

Let  $\delta > 0$  be fixed and set  $E'_{ij} = \{x \mid \theta_{E_{ij}}(x) \geq 1 - \delta\} \cap E_{ij}$ . Then, as before,  $\theta_{E'_{ij}} = \theta_{E_{ij}}$ , and from Lemma 3,  $\theta_{E'_{ij}} \geq (1 - \delta)\psi_{E'_{ij}}$ . Now  $|\alpha_j - (f/g)| \leq \varepsilon$  on  $E'_{ij}$  implies that  $|\alpha_j - \Psi_{E'_{ij}} f / \Psi_{E'_{ij}} g| \leq \varepsilon$ . [Here we consider only those  $E'_{ij}$ 's with  $\theta_{E_{ij}} \neq 0$ .] Hence:

$$\left| \sum_{ij} h_i \alpha_j \Theta_{E_{ij}} g - \sum_{\theta_{E_{ij}} \neq 0} h_i \frac{\Psi_{E'_{ij}} f}{\Psi_{E'_{ij}} g} \Theta_{E_{ij}} g \right| \leq \varepsilon \|h\|_\infty \|g\|_1.$$

Also:

$$\left| \sum_{\theta_{E_{ij}} \neq 0} h_i \frac{\Psi_{E'_{ij}} f}{\Psi_{E'_{ij}} g} \Theta_{E_{ij}} g - \sum_{ij} h_i \Psi_{E'_{ij}} f \right| \leq \|g\|_1 \|h\|_\infty (\|(f/g)\|_\infty + \varepsilon) k^2 \delta.$$

Finally,

$$\left| \sum_{ij} h_i \Psi_{E'_{ij}} f - \sum_{ij} h_i \Theta_{E_{ij}} f \right| \leq \|h\|_\infty \|f\|_1 k^2 \delta$$

and

$$\left| \sum_{ij} h_i \Theta_{E_{ij}} f - \lim_{n \rightarrow \infty} \int h T^n f \, d\mu \right| \leq \varepsilon \|f\|_1.$$

Putting together all these inequalities, we conclude the result.

### 5. A representation for $\mathcal{H}$ .

DEFINITION 8.  $\mathcal{G}$  is the sub- $C^*$ -algebra of  $\mathcal{H}$  generated by the class  $\{\pi(l/1) \mid l \in L_\infty\}$ .

Let  $\mathcal{M} \subset \mathcal{G}^*$  be the maximal ideal space of  $\mathcal{G}$  with the  $w^*$  topology induced from  $\mathcal{G}^*$ . Let  $\mathcal{B}$  be the  $\sigma$ -field of Baire sets of  $\mathcal{M}$ .

Note that  $\mathcal{G}$  contains the unit  $\pi(1)$  of  $\mathcal{H}$ , and that  $g \in \mathcal{G}$  is invertible in  $\mathcal{G}$  if and only if it is invertible in  $\mathcal{H}$ .

The  $C^*$ -algebra  $\mathcal{C}(\mathcal{M})$  of continuous complex valued functions on  $\mathcal{M}$  is isometrically\* isomorphic to  $\mathcal{G}$  under the Gelfand mapping  $\sigma: \mathcal{G} \rightarrow \mathcal{C}(\mathcal{M})$ . This mapping is order preserving. To see this first we need a few lemmas.

LEMMA 6. *If  $f \in \mathcal{H}$  then  $\pi|f|^2 \geq |f|^2$ .*

**Proof.** We can assume that  $f$  is real. Let  $g \in L_1^+$  with  $\|g\|_1 = 1$ . Then

$$\left| \int fg \, d\mu \right| = \left| \int f \cdot Tg \, d\mu \right| \leq \left| \int |f|^2 Tg \, d\mu \right|^{1/2} \left| \int Tg \, d\mu \right|^{1/2}.$$

Hence  $|\int fg \, d\mu|^2 \leq \int U|f|^2 g \, d\mu$ . If  $|f|^2 > U|f|^2$  on a set of positive measure, then there exist  $E \in \mathcal{F}$ ,  $\mu(E) > 0$ ,  $a \geq 0$  and  $\varepsilon > 0$  such that  $|f| \geq a + \varepsilon$  and  $U|f|^2 \leq a^2$  on  $E$ . Take  $g = f\chi_E/|f|\mu(E)$ . Then

$$(a + \varepsilon)^2 \leq \left| \int fg \, d\mu \right|^2 \leq \int U|f|^2 g \, d\mu \leq a^2$$

which is a contradiction. Hence  $U|f|^2 \geq |f|^2$  and  $\pi|f|^2 \geq |f|^2$ .

There is a canonical map  $j: L_1 \rightarrow \mathcal{G}^*$  defined by  $(jf)(g) = \int fg \, d\mu$ ,  $f \in L_1$ ,  $g \in \mathcal{G}$ . We now show that

LEMMA 7.  *$\mathcal{M}$  is contained in the  $w^*$ -closure of  $jL_1^+$  in  $\mathcal{G}^*$ .*

**Proof.** Choose  $m \in \mathcal{M}$  and suppose that the  $w^*$  neighborhood  $\{F \mid |Fg_i - mg_i| < \varepsilon, i = 1, \dots, n\}$  of  $m$  defined by  $g_1, \dots, g_n \in \mathcal{G}$ ,  $\varepsilon > 0$  is disjoint of  $jL_1^+$ . Let

$$u = \sum_{i=1}^n \pi[(g_i - 1mg_i)\overline{(g_i - 1mg_i)}].$$

Now, let  $f \in L_1^+$ ,  $\|f\|_1 = 1$ . Then

$$\begin{aligned} (jf)u &= \sum_{i=1}^n \int \pi|g_i - 1mg_i|^2 f \, d\mu \\ &\geq \sum_{i=1}^n \int |g_i - 1mg_i|^2 f \, d\mu \\ &\geq \sum_{i=1}^n \left| \int (g_i - 1mg_i) f \, d\mu \right| \geq \varepsilon^2. \end{aligned}$$

Hence  $u \geq \varepsilon^2$  a.e. and hence  $u$  is invertible in  $L_\infty$ . This implies that  $u$  is invertible in  $\mathcal{G}$ . But this is impossible since  $mu = 0$ .

COROLLARY.  *$jL_1$  is dense in  $\mathcal{G}^*$  in the  $w^*$ -topology.*

THEOREM 5. *The Gelfand mapping  $\sigma: \mathcal{G} \rightarrow \mathcal{C}(\mathcal{M})$  is positive.*

**Proof.** If  $g \geq 0$  a.e. then  $g^{**} \geq 0$  on  $jL_1^+$  where  $g \rightarrow g^{**}$  is the canonical embedding of  $\mathcal{G}$  into  $\mathcal{G}^{**}$ . Since  $jL_1^+$  is dense in  $\mathcal{M}$  and  $g^{**}$  is continuous,  $g^{**} \geq 0$  on  $\mathcal{M}$ . Hence  $\sigma g = g^{**}|_{\mathcal{M}} \geq 0$ .

Now we would like to extend  $\sigma$  to  $\mathcal{H}$ . First note that, by the Riesz representation theorem, any  $F \in \mathcal{G}^*$  can be represented by a measure  $\mu_F$  on  $(\mathcal{M}, \mathcal{B})$ . In particular, let  $\tilde{\mu} = \mu_{j_1}$ . From the order-preserving property of the Riesz representation one can see that for any  $f \in L_1$ ,  $\mu_{jf}$  is absolutely continuous with respect to  $\tilde{\mu}$ . In fact we can obtain  $d\mu_{jf}/d\tilde{\mu}$  as follows. First, considering only  $L_\infty$  functions we have

LEMMA 8. *If  $f \in L_\infty$  then  $\mu_{jf} \ll \tilde{\mu}$  and  $d\mu_{jf}/d\tilde{\mu} = \sigma\pi(f/1)$ .*

**Proof.** For any  $g \in \mathcal{G}$ ,

$$\begin{aligned} \int_{\mathcal{M}} \sigma g \cdot \sigma\pi(f/1) d\tilde{\mu} &= \int_{\mathcal{M}} \sigma\pi[g \cdot \pi(f/1)] d\tilde{\mu} = \int_X \pi[g\pi(f/1)] d\mu \\ &= \int_X \pi[g(f/1)] d\mu = \int_X gf d\mu, \end{aligned}$$

where the last equality follows from Theorem 4.

DEFINITION 9. Let  $\tau f = \sigma\pi(f/1)$ ,  $f \in L_\infty$ .

Note that the linear mapping  $f \rightarrow \tau f$  defines a positive contraction  $L_\infty(X, \mathcal{F}, \mu) \rightarrow L_\infty(\mathcal{M}, \mathcal{B}, \tilde{\mu})$ . But it is also a contraction for the corresponding  $L_1$  norms; hence it is a contraction for all  $L_p$  norms,  $1 \leq p \leq \infty$ . We can then extend this mapping to  $L_p(X, \mathcal{F}, \mu) \rightarrow L_p(\mathcal{M}, \mathcal{B}, \tilde{\mu})$  with the property that  $\int_X gf d\mu = \int_{\mathcal{M}} \sigma g \tau f d\tilde{\mu}$  for all  $g \in \mathcal{G}$ ,  $f \in L_p$ .

We can now prove a representation theorem for  $\mathcal{H}$ .

THEOREM 6. *There is a positive isometric \* isomorphism between  $\mathcal{H}$  and  $L_\infty(\mathcal{M}, \mathcal{B}, \tilde{\mu})$ .*

**Proof.** Let  $h \in \mathcal{H}$  and define  $\phi_h \in \mathcal{G}^*$  by

$$\phi_h(g) = \int_X \pi(gh) d\mu.$$

Note that if  $h \in \mathcal{G}$  then  $\phi_h$  is represented by the measure  $\sigma(h) \cdot d\tilde{\mu}$  on  $\mathcal{M}$ . Let  $\gamma_h$  be the representing measure of  $\phi_h$ ,  $h \in \mathcal{H}$ . Then, for any nonnegative continuous function  $\sigma g$  ( $g \in \mathcal{G}^+$ ) on  $\mathcal{M}$

$$\left| \int_{\mathcal{M}} \sigma g d\gamma_h \right| = \left| \int_X \pi(gh) d\mu \right| \leq \|h\|_\infty \int_X g d\mu = \|h\|_\infty \int_{\mathcal{M}} \sigma g \cdot d\tilde{\mu}$$

which shows that  $\gamma_h$  is absolutely continuous with respect to  $\tilde{\mu}$  and has a density function bounded by  $\|h\|_\infty$ . We denote this density function by  $\sigma h$ , noting that it is actually an extension of  $\sigma$ , and  $\|\sigma h\|_\infty \leq \|h\|_\infty$ . Furthermore, if  $l \in L_\infty$  then

$$\begin{aligned} \int_{\mathcal{M}} \sigma(h)\tau(l) d\tilde{\mu} &= \int_{\mathcal{M}} \sigma(h)\sigma\pi(l/1) d\tilde{\mu} = \int_X \pi(h \cdot \pi(l/1)) d\mu \\ &= \int_X \pi(h \cdot (l/1)) d\mu = \int_X hl d\mu. \end{aligned}$$

Hence  $|\int_X hl \, d\mu| \leq \|sh\|_\infty \|\tau l\|_1 \leq \|sh\|_\infty \cdot \|l\|_1$ , so  $\|h\|_\infty \leq \|sh\|_\infty$ . Thus the extended  $\sigma$  is also an  $L_\infty$ -norm isometry. To show that  $\sigma\mathcal{H} = L_\infty(\mathcal{M}, \mathcal{B}, \bar{\mu})$ , first note that, if  $h \in \mathcal{H}, l \in L_\infty(X, \mathcal{F}, \mu)$  then

$$\left| \int_X hl \, d\mu \right| = \left| \int_{\mathcal{M}} \sigma(h)\tau(l) \, d\bar{\mu} \right| \leq \|sh\|_1 \|\tau l\|_\infty \leq \|sh\|_1 \|l\|_\infty,$$

hence  $\|h\|_1 \leq \|sh\|_1$ . Thus  $\sigma^{-1}: \sigma\mathcal{H} \rightarrow \mathcal{H}$  is an  $L_1$ -contraction onto  $\mathcal{H}$ . Now if  $sh_n$  is an a.e. monotone sequence in  $\sigma\mathcal{H}$  converging a.e. to a function  $l$  in  $L_\infty(\mathcal{M}, \mathcal{B}, \bar{\mu})$  then  $h_n$  is an a.e. bounded and monotone sequence in  $\mathcal{H}$ . If the limit function is  $g$ , one can easily see that  $g \in \mathcal{H}$  and  $sg = l$ . Since  $\sigma\mathcal{H}$  contains the continuous functions, this shows that  $\sigma\mathcal{H} = L_\infty(\mathcal{M}, \mathcal{B}, \bar{\mu})$ . Now we want to show that

$$\int_X \pi(hf) \, d\mu = \int_{\mathcal{M}} \sigma(h)\sigma(f) \, d\bar{\mu},$$

for all  $h, f \in \mathcal{H}$ . In fact, for a fixed  $h \in \mathcal{H}$ , let  $\mathcal{N} \subset \mathcal{H}$  be the class of functions  $f$  for which this relation holds. Then  $\sigma\mathcal{N}$  contains the continuous functions of  $\mathcal{M}$ , and one can show, as before, that  $\sigma\mathcal{N}$  is closed under a.e. monotone limits. Hence  $\sigma\mathcal{N} = L_\infty(\mathcal{M}, \mathcal{B}, \bar{\mu})$ .

Finally, we show that extended  $\sigma$  is multiplicative, i.e.  $\sigma(h_1) \cdot \sigma(h_2) = \sigma(\pi(h_1h_2))$  for all  $h_1, h_2 \in \mathcal{H}$ . First note that if  $f \in L_\infty(\mathcal{M}, \mathcal{F}, \bar{\mu})$  and  $\int_{\mathcal{M}} f\tau(l) \, d\bar{\mu} = 0$  for all  $l \in L_\infty(X, \mathcal{F}, \mu)$  then  $\sigma^{-1}f = 0$ , hence  $f = 0$ . Now for  $h \in \mathcal{H}, g \in \mathcal{G}, l \in L_\infty(X, \mathcal{F}, \mu)$ ,

$$\begin{aligned} \int_{\mathcal{M}} \sigma(h)\sigma(g)\tau(l) \, d\bar{\mu} &= \int_{\mathcal{M}} \sigma(h)\sigma(g)\sigma(\pi(l/1)) \, d\bar{\mu} \\ &= \int_{\mathcal{M}} \sigma(h)\sigma\pi(g\pi(l/1)) \, d\bar{\mu} = \int_X \pi(h\pi(g\pi(l/1))) \, d\mu \\ &= \int_X \pi(hg(l/1)) \, d\mu = \int_X \pi(hg)l \, d\mu = \int_{\mathcal{M}} \sigma\pi(hg)\tau(l) \, d\bar{\mu}, \end{aligned}$$

hence  $\sigma(h) \cdot \sigma(g) = \sigma\pi(hg)$ .

Now suppose that  $h_1, h_2 \in \mathcal{H}, l \in L_\infty(X, \mathcal{F}, \mu)$ . Then

$$\begin{aligned} \int_{\mathcal{M}} \sigma(h_1)\sigma(h_2)\tau(l) \, d\bar{\mu} &= \int_{\mathcal{M}} \sigma(h_1)\sigma\pi(h_2\pi(l/1)) \, d\bar{\mu} \\ &= \int_X \pi(h_1\pi(h_2\pi(l/1))) \, d\mu = \int_X \pi(h_1h_2)l \, d\mu \\ &= \int_{\mathcal{M}} \sigma\pi(h_1h_2)\tau(l) \, d\bar{\mu} \end{aligned}$$

which shows that  $\sigma(h_1)\sigma(h_2) = \sigma\pi(h_1h_2)$ , and completes the proof of the theorem.

We remark that every  $f \in L_p(\mathcal{M}, \mathcal{B}, \bar{\mu}), 1 \leq p < \infty$ , induces a function  $h \in L_p(X, \mathcal{F}, \mu)$ , defined by  $\int_X hl \, d\mu = \int_{\mathcal{M}} f\tau(l) \, d\bar{\mu}$  for all  $l \in L_q(X, \mathcal{F}, \mu), 1/p + 1/q = 1$ . Since  $\tau$  is an  $L_q$ -contraction the integral on  $\mathcal{M}$  is defined and  $h$  satisfies  $\int_X hl \, d\mu =$

$\int_X hTl \, d\mu$ , for all  $l \in L_0(X, \mathcal{F}, \mu)$ . The case  $p=1$  causes no difficulty. If  $f \in L_1(\mathcal{M}, \mathcal{B}, \bar{\mu})$ ,  $l \in L_\infty(X, \mathcal{F}, \mu)$ ,

$$\begin{aligned} \left| \int_{\mathcal{M}} f\tau(l) \, d\bar{\mu} \right| &\leq \left| \int_{\{|f| \geq n\}} f\tau(l) \, d\bar{\mu} \right| + \left| \int_{\{|f| < n\}} f\tau(l) \, d\bar{\mu} \right| \\ &\leq \|\tau l\|_\infty \int_{\{|f| \geq n\}} |f| \, d\bar{\mu} + n\|\tau l\|_1 \\ &\leq \|l\|_\infty \left[ \int_{\{|f| \geq n\}} |f| \, d\bar{\mu} \right] + n\|l\|_1. \end{aligned}$$

Thus, if  $l_k$  is a sequence in  $L_\infty$  with  $\|l_k\|_1 \rightarrow 0$  and  $\|l_k\|_\infty \leq K$  then

$$\lim_k \left| \int f\tau(l_k) \, d\bar{\mu} \right| \leq K \int_{\{|f| \geq n\}} |f| \, d\bar{\mu} \quad \text{for all } n \geq 1.$$

Hence this limit is zero and the functional  $l \rightarrow \int f\tau(l) \, d\bar{\mu}$  on  $L$  is induced by an  $L_1$ -function  $h$ . In a similar way, any Baire measure on  $(\mathcal{M}, \mathcal{B})$  induces what one might call "an invariant functional" on  $L_\infty(X, \mathcal{F}, \mu)$ .

We also note the following relation between the maximal ideal spaces of  $\mathcal{H}$  and  $\mathcal{G}$ ; that is, between the Feller and Martin boundaries (cf. [9]). Since  $\mathcal{H}$  is isometrically isomorphic to  $L_\infty(\mathcal{M}, \mathcal{B}, \bar{\mu})$ , we state this relation in the following familiar form:

**LEMMA 9.** *Let  $\mathcal{M}$  be a compact Hausdorff space,  $\mathcal{B}$  its Baire sets, and  $\bar{\mu}$  a Baire measure on  $(\mathcal{M}, \mathcal{B})$  with support  $\mathcal{M}$ . Let  $\mathcal{M}'$  be the maximal ideal space of the  $C^*$ -algebra  $L_\infty(\mathcal{M}, \mathcal{B}, \bar{\mu})$ . Then there is a continuous and onto map  $\rho: \mathcal{M}' \rightarrow \mathcal{M}$ .*

**Proof.** Interpret  $\mathcal{M}'$  and  $\mathcal{M}$  as classes of homomorphisms and define  $\rho: \mathcal{M}' \rightarrow \mathcal{M}$  by  $\rho(\phi) = \phi|_{\mathcal{G}(\mathcal{M})}$ . Then  $\rho$  is continuous. We show it is onto. Let  $m \in \mathcal{M}$ , and consider the ideal generated by  $m \cdot L_\infty(\mathcal{M}, \mathcal{B}, \bar{\mu})$ . If it is proper, it can be embedded in a maximal ideal, whose image must then be  $m$  under  $\rho$ . We show it is proper. If not, then  $1 = \sum_1^n f_i g_i$  where  $f_i \in m$ ,  $g_i \in L_\infty(\mathcal{M}, \mathcal{B}, \bar{\mu})$ . Since  $m$  is a maximal ideal,  $\exists x_0$  such that  $f_i(x_0) = 0 \quad i = 1, \dots, n$ . Hence  $|f_i| \leq \epsilon/h \sup |g_i|$  on some neighborhood  $U$  of  $x_0$ , such that  $\mu(U) \neq 0$ . Hence  $1 = |\sum f_i g_i| \leq \epsilon$  on  $U$ , which is a contradiction.

**COROLLARY.**  $\mathcal{M}$  is homeomorphic to the quotient space  $\mathcal{M}'/\rho$ .

We finish this section by considering the possibility of joining  $X$  and  $\mathcal{M}$ . In general, this cannot be done. If, however,  $T$  is induced by a Markov kernel, such that the transform of every point measure is absolutely continuous with respect to  $\mu$ , then the members of  $\mathcal{H}$  can be considered as actual functions on  $X$ , and the evaluations of these functions at points of  $X$  induce bounded linear functionals on  $\mathcal{H}$ . Hence  $X$  can be embedded in  $\mathcal{G}^*$  (possibly in a many to one fashion). We shall denote the image of  $X$  under this mapping as  $X$  also. Hence  $X \subset j(L_1^+(X, \mathcal{F}, \mu))$ . Using the method of Lemma 7,  $X$  is dense in  $\mathcal{M}$ , in the  $w^*$ -topology of  $\mathcal{G}^*$ .

Let  $\bar{X}$  be the  $w^*$ -closure of  $X$  in  $\mathcal{G}^*$ . Then  $\bar{X}$  is a compact Hausdorff space. The following result, stated for the Martin-Doob boundary, is also true for the Feller boundary.

**THEOREM 7.** *For any  $g \in L_1(X, \mathcal{F}, \mu)$ ,  $T^n g \, d\mu \rightarrow \tau(g) \, d\bar{\mu}$  in the  $w^*$ -topology of Baire measures on  $\bar{X}$ .*

**Proof.** Let  $\mathcal{A}_1$  = the sub- $C^*$ -algebra of  $\mathcal{A}$ , consisting of functions  $g' \in \mathcal{A}$  such that  $\pi(g') \in \mathcal{G}$ .

Let  $\mathcal{C} = \{f \in \mathcal{C}(\bar{X}) \mid f|_X \in \mathcal{A}_1\}$ ,  $\mathcal{C}_0 = \{f \in \mathcal{C}(\bar{X}) \mid f|_X \in \mathcal{A}_0\}$ . By the Stone-Weierstrass theorem  $\mathcal{C} = \mathcal{C}(\bar{X})$ . Also,  $\mathcal{C}_0$  is a closed ideal in  $\mathcal{C}$ . Let  $\mathcal{N} \subset \bar{X}$  be the closed subset such that  $\mathcal{C}_0 = \{f \in \mathcal{C} \mid f(\mathcal{N}) = 0\}$ . Then we have

$$\mathcal{C}(\mathcal{N}) \cong^* \mathcal{C}(\bar{X})/\mathcal{C}_0 \cong^* \mathcal{A}_1/\mathcal{A}_0 \cong^* \mathcal{G} \cong^* \mathcal{C}(\mathcal{M}).$$

Hence  $\mathcal{C}(\mathcal{N}) \cong^* \mathcal{C}(\mathcal{M})$  is induced by a homeomorphism  $\phi: \mathcal{N} \rightarrow \mathcal{M}$ . Hence  $g(s) = g(\phi(s))$  under the above sequence of isomorphisms. But  $\mathcal{G}$  separates the points of  $\mathcal{G}^*$ , so  $\phi = \text{identity}$  and  $\mathcal{N} = \mathcal{M}$ .

In other words,

$$\{f \in \mathcal{C}(\bar{X}) \mid f|_X \in \mathcal{A}_0\} = \{f \in \mathcal{C}(\bar{X}) \mid f(\mathcal{M}) = 0\}.$$

Thus if  $f \in \mathcal{C}(\bar{X})$ ,  $g \in L_1(\bar{X}, \mathcal{F}, \mu)$ , then:

$$\int_X f T^n g \, d\mu = \int_X U^n(f|_X)g \, d\mu \rightarrow \int_X \pi(f|_X)g \, d\mu = \int_{\mathcal{M}} \sigma\pi(f|_X)\tau(g) \, d\bar{\mu}$$

and

$$\int_{\mathcal{M}} \sigma\pi(f|_X) \cdot \tau(g) \, d\bar{\mu} = \int_{\mathcal{M}} f|_{\mathcal{M}} \cdot \tau(g) \, d\bar{\mu} = \int_X f\tau(g) \, d\bar{\mu}.$$

Thus  $T^n g \, d\mu \rightarrow \tau(g) \, d\bar{\mu}$ .

**6. Harmonic functions in the unit disk.** As an example we consider a transformation suggested by Feller in [10].

Let  $D = \{z = re^{i\phi} \mid 0 \leq r < 1, -\pi \leq \phi \leq \pi\}$  be the unit disk with the (geometric) boundary  $C$ . Let  $\mathcal{F}$  and  $\mu$  be the  $\sigma$ -field of Borel subsets and the Lebesgue measure. For every  $z \in D$ ,  $E \in \mathcal{F}$ , let

$$P(z, E) = \mu(Q_z \cap E) / \mu(Q_z)$$

where  $Q_z = \{Z \mid |Z - z| < 1 - |z|\}$ . Then  $P$  defines a Markov kernel, such that the transformation of a unit mass at  $z \in D$  is given by the measure  $P(z, \cdot) \ll \mu$ . We let  $T$  be the induced transformation on  $L_1(D, \mathcal{F}, \mu)$ . The adjoint  $U$  of  $T$  is given by

$$(Uf)(z) = \iint f(Z)P(z, dZ), \quad f \in L_\infty, z \in D.$$

It is clear that any bounded harmonic function  $h$  belongs to  $\mathcal{H}$ . The converse is also true, but it seems that no explicit proof of it has been given and we would like to indicate an outline for this proof.

If  $R$  is a Borel subset of  $[0, 1)$  let  $C_R = \{z \mid |z| \in R\}$ . One can then obtain the following

LEMMA 10. Let  $\frac{1}{2} \leq K < 1$  and  $R$  be a Borel subset of  $[K, 1)$ . Then for all  $z \in D$ ,  $\frac{1}{2} \leq |z| \leq K$ ,

$$\frac{\mu(Q_z \cap C_R)}{\mu(Q_z \cap C_{[K, 1)})} \geq \frac{1}{16} \left[ \frac{\lambda(R)}{1-K} \right]^{3/2}$$

where  $\lambda$  is the one dimensional Lebesgue measure.

COROLLARY. Let  $E = C_{[0, 1/2)} \cup [K, 1)$  and let  $f \in L^1_+$ ,  $f=0$  a.e. on  $C_{[K, 1)}$ . Then

$$\int_{C_R} T^n_E f d\mu \geq \frac{1}{16} \left[ \frac{\lambda(R)}{1-K} \right]^{3/2} \int_{C_{[K, 1)}} T^n_E f d\mu$$

for all  $n \geq 0$ .

Using this corollary one can see that if a function  $h \in \mathcal{H}$  (which is necessarily continuous) has the form  $h(re^{i\phi}) = f(r)g(\phi)$  then  $\lim_{r \uparrow 1} f(r)$  exists, and that this implies the harmonicity of  $h$ .

Now if  $h$  is any function in  $\mathcal{H}$ , let  $t$  be an irrational number and consider, for a fixed  $n$ ,  $-\infty < n < \infty$ ,

$$\lim_{m \rightarrow \infty} \frac{1}{m} \sum_{k=1}^m (\tau^k \cdot z)^{-n} h(\tau^k \cdot z) = F_n$$

where  $\tau: D \rightarrow D$  is given by  $\tau z = e^{i2\pi t} z$ . This limit  $F_n$  exists for all nonzero  $z \in D$ , depends only on  $r = |z|$ , and satisfies

$$r^n F_n(r) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{-in\phi} h(re^{in\phi}) d\phi.$$

But, it is clear that

$$e^{in\phi} r^n F_n(r) = \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{k=1}^m e^{-i2\pi knt} h(re^{i\phi} e^{i2\pi kt})$$

is a function in  $\mathcal{H}$ , hence  $e^{in\phi} r^n F_n(r)$  must be harmonic, which shows that  $r^n F_n(r) = C_n r^{|n|}$  and completes the proof of the following

LEMMA 11. A bounded function belongs to  $\mathcal{H}$  if and only if it is a harmonic function.

One then shows that the  $C^*$ -algebra  $\mathcal{H}$  is isometrically  $*$ -isomorphic to  $L_\infty$  of the unit circle. For any bounded measurable function  $l$  on  $D$ , let  $\lambda_l$  be the measure on the unit circle obtained by sweeping out  $l d\mu$  by the Poisson kernel. The harmonic function  $\pi(l/1)$  corresponds to  $d\lambda_l/d\lambda$ , which is continuous. It then follows that the maximal ideal space  $\mathcal{M}$  of  $\mathcal{G}$  is homeomorphic to the unit circle. Since  $T$  is induced by a Markov kernel,  $D$  can be imbedded into  $\mathcal{G}^*$ . Then  $D \cup \mathcal{M}$  is homeomorphic to the closed unit disk.

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UNIVERSITY OF TORONTO,  
TORONTO, CANADA