ERGODIC THEORY AND BOUNDARIES

BY

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1. Introduction. Let $T$ be a conservative positive contraction on the $L_1$ space of a finite measure space $(X, \mathcal{F}, \mu)$. A theorem of Chacon [5], [2] shows that $T$ defines a sub $\sigma$-field $\mathcal{I}$ of $\mathcal{F}$, consisting of invariant subsets of $X$. The ratio ergodic limits are measurable with respect to $\mathcal{I}$ [5], [2] and the class of these limits contains $L_\infty(X, \mathcal{I}, \mu)$, which can be considered as the invariant functions of the adjoint transformation [2].

The main purpose of the present paper is to show that any positive contraction on $L_1(X, \mathcal{F}, \mu)$ behaves, asymptotically, like a conservative transformation (Theorems 3 and 4) and that the invariant functions of the adjoint transformation can be approximated by the ratio ergodic limits.

Intuitively, a ratio ergodic limit corresponds to the result of an averaging process of different values of a function. It is then natural to consider these limits as functions that are smooth with respect to the asymptotic behaviour of the transformation. This leads (Theorem 6) to a Martin-Doob type representation [12], [8] of invariant functions as the $L_\infty$ functions of a compact Hausdorff space $\mathcal{M}$ with a Baire measure. The topology on $\mathcal{M}$ is just strong enough to make the ergodic limits to correspond to continuous functions. As an example we consider a transformation of Feller [10] and show that for this case the above representation is identical with the Poisson representation of harmonic functions in the unit disk.

We also consider the possibility of joining $X$ and $\mathcal{M}$, convergence of measures to $\mathcal{M}$ in $X \cup \mathcal{M}$ (Theorem 7), and a relation (Lemma 9) between the Feller and Martin-Doob type representations, corresponding to a result of Feldman [9].

2. Preliminaries. Let $(X, \mathcal{F}, \mu)$ be a finite measure space and let $L_p = L_p(X, \mathcal{F}, \mu)$, $1 \leq p \leq \infty$ be the usual Banach spaces, and $L_\infty^\ast$ denote the positive cone of $L_\infty$. Let $T: L_1 \to L_1$ be a positive linear contraction and $U: L_\infty \to L_\infty$ be its dual. For $\alpha \in L_\infty$ define $T_\alpha: L_1 \to L_1$ as $T_\alpha f = \alpha f + T((1-\alpha)f, f \in L_1$, and let $U_\alpha$ be its dual. If $\chi_E$ is the characteristic function of $E \in \mathcal{F}$ we write $T_E$ and $U_E$ instead of $T_{\chi_E}$ and $U_{\chi_E}$.

The following partial ordering of $L_1^\ast$ is similar to that of Bishop and deLeeuw given in [3].

Definition 1. For $f, g \in L_1^\ast$, $f \leq g$ if and only if there exist an integer $n \geq 1$ and $\alpha_1, \ldots, \alpha_n \in L_\infty$ such that $0 \leq \alpha_i \leq 1$ for $i = 1, \ldots, n$ and $g = T_{\alpha_n} \cdots T_{\alpha_1} f$.

This relation is reflexive and transitive and $f \leq g$ implies $\|f\|_1 \geq \|g\|_1$. Also, an
induction argument shows that if \( f < g \) then there exists an integer \( n \geq 1 \) such that \( g < T^n f \). Hence \( \{ g \in L^+ \mid g > f \} \) is (upward) directed by \( < \).

**Definition 2.** For \( E \in \mathcal{F}, f \in L^+_1 \) let

\[
\Psi_E f = \sup_{g > f} \int_E g \, d\mu, \quad \Theta_E f = \lim_{g \nearrow f} \Psi_E g.
\]

Note that \( \Theta_E f = \lim_{n \to \infty} \Psi_E T^n f \).

**Lemma 1.** The limits \( \psi_E = \lim_{n \to \infty} U^{\infty}_E f \) and \( \theta_E = \lim_{n \to \infty} U^n \psi_E \) both exist (a.e.) and satisfy

\[
\Psi_E f = \int \psi_E f \, d\mu, \quad \Theta_E f = \int \theta_E f \, d\mu.
\]

**Proof.** By induction, \( U^{\infty}_E \uparrow \) and \( U^n \psi_E \downarrow \), so the limits exist. Now if \( f \in L^+_\infty \) satisfies

\[
(*) \quad \chi_E Uf \leq \chi_E f, \quad \chi_E^c Uf \geq \chi_E^c f
\]

with \( E^c = X - E \), then for all \( \alpha \in \mathbb{L}_0, 0 \leq \alpha \leq 1 \), we have

\[
U_{\alpha}f = \alpha f + (1 - \alpha) Uf \leq \chi_E f + \chi_E^c Uf = U_E f.
\]

Since, by induction, \( U^{\infty}_E \) satisfies \((*)\) for all \( n \geq 0 \), we get, again by induction, \( U_{a_n} \cdots U_{a_1} \chi_E \leq U^{\infty}_E \), and hence \( \Psi_E f = \int \psi_E f \, d\mu \). The final part follows from the definition (cf. also [4] and [2]).

**Definition 3.** For \( E, F \in \mathcal{F} \), let

\[
\psi_{EF} = \psi_E + \psi_F - \psi_{E \cup F}, \quad \theta_{EF} = \theta_E + \theta_F - \theta_{E \cup F}.
\]

\( \Psi_{EF}, \Theta_{EF} \) are the functionals on \( L^+_1 \) defined by the \( L^\infty \) functions \( \psi_{EF}, \theta_{EF} \).

We note that \( \psi_{EF} \) and \( \theta_{EF} \) are monotone and subadditive in each index. This follows easily from the following general result, which will be useful to obtain other relations between these set functions (cf. [7]).

**Lemma 2.** If \( a_i \) is real and \( A_i \in \mathcal{F} \) for \( i = 1, \ldots, n \) and \( A = \bigcup_{i=1}^n A_i \), then \( \chi_A \sum_{i=1}^n a_i \psi_{A_i} \geq 0 \) implies \( \sum_{i=1}^n a_i \psi_{A_i} \geq 0 \) and \( \sum_{i=1}^n a_i \theta_{A_i} \geq 0 \).

**Proof.** If \( f \in L^+_1 \) and \( E \subseteq F, E, F \in \mathcal{F} \), then by induction: \( \chi_{E^c} T^f f \leq \chi_{E^c} T^E f \).

Hence

\[
0 \leq \int_{E^c} \psi_{E} T^f f \, d\mu \leq \int_{E^c} \psi_{E} T^E f \, d\mu \leq \int_{E^c} \psi_{E} T^E f \, d\mu \leq \int_{E^c} \psi_{E} T^E f \, d\mu \leq \int_{E^c} \psi_{E} T^E f \, d\mu \leq \int_{E^c} \psi_{E} T^E f \, d\mu \rightarrow 0
\]
as \( n \to \infty \). Now

\[
\int \psi E f \, d\mu = \int \psi E X F f \, d\mu + \int \psi E X F^* f \, d\mu = \int \psi E X F f \, d\mu + \int \psi E T E X F f \, d\mu = \int \psi E f \, d\mu
\]

and hence \( \lim_{n \to \infty} \int \psi E T E f \, d\mu = \int \psi E f \, d\mu \). Using this for the case of \( A_i \subseteq A \), \( i = 1, \ldots, n \), we get

\[
0 \leq \sum_{i=1}^{n} a_i \int_{A} \psi E A_i f \, d\mu \to \sum_{i=1}^{n} a_i \int_{A} \psi E f \, d\mu
\]
as \( n \to \infty \), which proves the first assertion. Since \( U \) is positive, the remainder follows.

**Lemma 3.** If \( \psi E \leq a \psi E \) then \( \theta E \leq a \theta E \).

**Proof.** From the proof of the previous lemma we have that, for \( f \in L^1 \),

\[
\lim_{n \to \infty} \int_{E} \theta E T E f \, d\mu = 0.
\]

Hence

\[
\int \theta E f \, d\mu = \int \theta E T E f \, d\mu = \lim_{n \to \infty} \int_{E} \theta E T E f \, d\mu = \alpha \lim_{n \to \infty} \int_{E} T E f \, d\mu \geq \alpha \int \psi E f \, d\mu.
\]

Finally we prove the following.

**Lemma 4.** For \( E, F \in \mathcal{F} \), \( \|\psi E\|_\infty = \|\psi E \psi E\|_\infty = 0 \) or \( 1 \) and \( \|\theta E\|_\infty = \|\psi E \psi E\|_\infty = \|\xi E \psi E\|_\infty = 0 \) or \( 1 \).

**Proof.** For \( g \in L^1 \), as \( n \to \infty \), \( 0 \leq \Theta E (\psi E T E g) \leq \psi E T E g \to 0 \) as in the proof of Lemma 2. Hence the decomposition \( \Theta E F = \Theta E T E F = \Theta E (\psi E T E g) + \Theta E (\psi E T E g) \) shows that \( \|\theta E\|_\infty = \|\psi E \psi E\|_\infty \). Now, for \( n, m \geq 1 \),

\[
\Theta E F = \Theta E T E T E^* F = \lim_{m \to \infty} \lim_{n \to \infty} \|\theta E\|_\infty \|\psi E T E T E^* F\|_1 \leq \|\psi E T E T E^* F\|_1 \leq \|\psi E T E T E^* F\|_1 \leq \psi E T E T E^* F
\]

which completes the proof of the first part, since \( \|\theta E\|_\infty \leq 1 \). For the second part, we have, if \( g \in L^1 \), \( 0 \leq \Theta E F (\psi E T E g) \leq \Theta E (\psi E T E g) \to 0 \) as \( n \to \infty \) which shows that \( \|\theta E\|_\infty = \|\psi E \psi E\|_\infty \). Now

\[
\Theta E F - \Theta E F (\psi E T E g) = \Theta E (\psi E T E g) \leq \Theta E \vee F (\psi E T E g) \leq \|\psi E T E T E^* F\|_1 \leq \psi E T E T E^* F
\]
thus, \( \Theta_E^g \leq \lim_{n \to \infty} \Theta_{E \cup F}(T_E^g) \leq Y^g_E. \) Replacing \( g \) by \( T_n^g \) and letting \( m \to \infty \) we get
\[
\Theta_E^g = \lim_{m \to \infty} \lim_{n \to \infty} \Theta_{E \cup F}(T_{E^m}^g).
\]
Next, consider
\[
\Theta_E^f(T_{E^m}^g) = (\Theta_E + \Theta_F - \Theta_{E \cup F})(T_{E^m}^g)
\]
and let \( n \to \infty \) to get
\[
\Theta_E^g = \Theta_E^g + \lim_{n \to \infty} \Theta_F(T_{E^m}^g) - \lim_{n \to \infty} \Theta_{E \cup F}(T_{E^m}^g).
\]
Now, letting \( m \to \infty \) we have
\[
\Theta_E^g = \lim_{m \to \infty} \lim_{n \to \infty} \Theta_F(T_{E^m}^g).
\]
But
\[
\Theta_E^g \leq \|\Theta_E\| \lim_{m \to \infty} \lim_{n \to \infty} \|T_{E^m}^g\|
\leq \|\Theta_E\| \lim_{m \to \infty} \|W^g_E\|
\leq \|\Theta_E\| \Theta_E^g.
\]
Hence
\[
\Theta_E^g = \Theta_E^g(T_{E^m}^g)
= \lim_{m \to \infty} \lim_{n \to \infty} \Theta_E^f(T_{E^m}^g)
\leq \|\Theta_E\| \lim_{m \to \infty} \lim_{n \to \infty} \Theta_F(T_{E^m}^g)
\leq \|\Theta_E\| \Theta_E^g.
\]
This completes the proof, since \( \|\Theta_E\| \leq 1. \)

Definition 4. \( \Sigma = \{ E \in \mathcal{F} \mid \Theta_{E \cap F} = 0 \}. \)

Lemma 5. \( \Sigma \) is a field.

Proof. Let \( E, F \in \Sigma \) and \( G = E \cap F. \) Then
\[
0 \leq \Theta_{E \cap F} = \Theta_{E \cap F \cup F} \leq \Theta_{E \cap F} + \Theta_{E \cap F} \leq \Theta_{E \cap F} + \Theta_{E \cap F} = 0.
\]
Thus \( G \in \Sigma. \)

Definition 5. \( \mathcal{A} \) is the \( L_\infty \)-closure of the class of \( \Sigma \)-simple functions.
We note that \( \mathcal{A} \) is a sub-Banach space of \( L_\infty. \)

Theorem 1. For a real valued function \( f \in L_\infty, \) the following conditions are equivalent:
(i) \( f \in \mathcal{A}, \)
(ii) \( \lim_{g \to g_0} \int fg \, d\mu \) exists for all \( g_0 \in L_\infty^2, \)
(iii) for all real numbers \( \alpha \) and \( \epsilon > 0, \)
\[
\Theta_{E \cap F} = 0 \quad \text{where} \quad E = \{ x \mid f(x) \leq \alpha \}, \quad F = \{ x \mid f(x) \geq \alpha + \epsilon \}.
\]
Proof. (i)⇒(ii). If \( E \in \Sigma \) then \( \theta_E + \theta_{\overline{E}} = \theta_X \); thus, for a real valued \( g_0 \in L_1^+ \),

\[
\limsup_{\theta \to \theta_0} \int_E g \, d\mu = \limsup_{\theta \to \theta_0} \int g \, d\mu - \limsup_{\theta \to \theta_0} \int_{\overline{E}} g \, d\mu = \liminf_{\theta \to \theta_0} \int_{\overline{E}} g \, d\mu.
\]

Therefore \( \lim_{\theta \to \theta_0} \int \chi_E g \, d\mu \) exists for all \( E \in \Sigma \).

Hence it exists for all \( \Sigma \)-simple functions, and thus for all \( f \in \mathcal{A} \).

(ii)⇒(iii). Suppose \( E \) and \( F \) are as in (iii) but that \( \theta_{EF} \neq 0 \). Then \( \| \theta_{EF} \|_\infty = 1 \) and for all \( \delta > 0 \) there exists \( g_0 \in L_1^+ \) with \( \| g_0 \|_1 = 1 \) and \( \int \theta_{EF} g \, d\mu \geq 1 - \delta \). Hence \( \Theta_\delta g_0 \geq 1 - \delta \) and \( \Theta_\delta g_0 \geq 1 - \delta \). Thus \( \limsup_{\theta \to \theta_0} \int_f g \, d\mu \leq (1 - \delta)(a + \epsilon) \) and \( \liminf_{\theta \to \theta_0} \int_f g \, d\mu \leq (1 - \delta)a + \delta \| f \|_\infty \). If \( \delta \) is chosen sufficiently small we see that \( \lim_{\theta \to \theta_0} \int_f g \, d\mu \) does not exist.

(iii)⇒(i). Let \( a_1 < a_2 < \cdots < a_n \) be \( n \) numbers and let \( E_i = \{ x \mid f(x) \leq a_i \} \). Now

\[
\sum_{i=1}^n \theta_{E_i} \leq \sum_{i=1}^n (\theta_{E_i} + \theta_{E_i} - \theta_X) = 1.
\]

Hence if \( E_a = \{ x \mid f(x) \leq a \} \) then \( \theta_{E_a} \neq 0 \) for only countably many \( a \)'s, and so \( f \in \mathcal{A} \).

3. Invariant functions.

Definition 6. \( \mathcal{H} = \{ f \mid f \in L_\infty, f = Uf \} \) is the class of invariant functions of \( U \).

We assume \( \mathcal{H} \neq \{0\} \).

Note that \( \mathcal{H} \) is a sub-Banach space of \( L_\infty \). Also, if \( h \in \mathcal{H} \) and \( g' \geq g \in L_1^+ \), then \( \int h g' \, d\mu = \int h g \, d\mu \) and hence \( \lim_{\delta \to 0} \int h g' \, d\mu \) exists. Thus \( \mathcal{H} \subseteq \mathcal{A} \).

If \( f \in \mathcal{A} \), then \( \lim_{n \to \infty} \int T^n g \, d\mu = \lim_{n \to \infty} \int U^* g \, d\mu \) exists for all \( g \in L_1(X, \mathcal{F}, \mu) \).

Hence the bounded sequence \( U^* f, n = 1, 2, \ldots \) has a limit \( \pi(f) \) in the \( w^* \)-topology of \( L_\infty \). Obviously the limit lies in \( \mathcal{H} \), so \( \pi : \mathcal{A} \to \mathcal{H} \) is a positive linear contraction.

Definition 7. \( \mathcal{A}_0 = \ker \pi = \{ f \in \mathcal{A} \mid w^* \lim \, U^* f = 0 \} \). Hence \( \mathcal{A}/\mathcal{A}_0 \cong \mathcal{H} \) is a canonical, isometric isomorphism.

Now \( \mathcal{A} \) is a \( C^* \)-algebra with the usual operations. We show that \( \mathcal{A}_0 \) is a closed ideal.

Theorem 2. \( \mathcal{A}_0 \) is a closed ideal in \( \mathcal{A} \).

Proof. Let \( f \in \mathcal{A}_0 \) and assume that \( f \) is real. Choose \( \epsilon > 0 \) and set \( E = \{ x \mid f(x) \geq \epsilon \} \).

We may assume \( E \in \Sigma \). Suppose \( \theta_E \neq 0 \); then for all \( \delta > 0 \), there is a \( g \in L_1^+ \) such that \( \| g \|_1 = 1 \) and \( \Theta_\delta g \geq 1 - \delta \). Hence:

\[
0 = \lim_{n \to \infty} \int_U T^n f \, g \, d\mu = \lim_{n \to \infty} \int T^n g \, d\mu \geq \epsilon \lim_{n \to \infty} \int T^n g \, d\mu - \| f \|_\infty \lim_{n \to \infty} \int T^n g \, d\mu \geq \epsilon(1 - \delta) - \| f \|_\infty \delta.
\]
Clearly, this fails for small \( \delta \), and so \( \theta_E = 0 \). Thus if \( E = \{ x \mid |f(x)| > \epsilon \} \), we have \( \theta_E = 0 \).

Now if \( h \in \mathcal{A} \), \( h \neq 0 \), set \( F = \{ x \mid |f(x)h(x)| \geq \epsilon \} \). Since \( F \subset \{ x \mid |f(x)| \geq \epsilon / \| h \|_\infty \} \), we have \( \theta_F = 0 \). Hence

\[
\lim_{n \to \infty} \int \frac{U^n(fh)g}{\mu} \leq \epsilon \lim_{n \to \infty} \int_T T^n g \, d\mu + \epsilon \| g \|_1 \quad \text{if } g \in L^+_1
\]

\[
\leq \epsilon \| g \|_1 \quad \text{for all } \epsilon > 0.
\]

Hence \( fh \in \mathcal{A}_0 \).

As a result of the lemma, we have given \( \mathcal{A}_1 \subseteq \mathcal{A}_0 \), and hence \( \mathcal{H} \) the structure of a \( C^* \)-algebra. Thus \( \mathcal{H} \) has a representation as the set of complex valued continuous functions on its maximal ideal space. This corresponds to Feller’s representation [10] of the invariant functions of certain Markov processes, and we shall refer to \( \mathcal{H}'s \) maximal ideal space as the Feller boundary.

As is known [8], [11], the Feller boundary is larger than it need be. In the next section, we obtain some properties of ratio ergodic limits, and use them to define a sub \( C^* \)-algebra \( \mathcal{F} \) of \( \mathcal{H} \), with a maximal ideal space \( \mathcal{M} \), smaller than the Feller boundary, but large enough to represent \( \mathcal{H} \) as a function algebra on \( \mathcal{M} \). This corresponds to the Martin-Doob representation [12], [8], [11] for some classes of functions, and \( \mathcal{M} \) will be referred to as the Martin-Doob boundary.

4. Properties of ratio ergodic limits. In [6] Chacon and Ornstein proved that for any \( f, g \in L^+_1 \), with \( g > 0 \), the limit:

\[
\lim_{n \to \infty} \sum_{k=1}^n \frac{T^k f}{T^k g}
\]

exists a.e. We denote the limit function by \( (f/g) \). It is also known [5], [4], [1], that if \( \alpha \leq (f/g) \leq \beta \) a.e. on \( E \in \mathcal{H} \), then \( \alpha \leq \Psi_E(f) / \Psi_E(g) \leq \beta \).

**Theorem 3.** If \( f, g \in L^+_1 \) with \( g > 0 \), and

\[
E = \{ x \mid (f/g)(x) \leq \alpha \},
\]

\[
F = \{ x \mid (f/g)(x) \geq \alpha + \epsilon \},
\]

then \( \theta_{E,F} = 0 \), for all \( \alpha \geq 0 \) and \( \epsilon > 0 \).

**Proof.** If \( \theta_{E,F} \neq 0 \) then \( \| \chi_E \theta_{E,F} \|_\infty = 1 \). Let \( \delta > 0 \) and set \( E_\delta = \{ x \mid \theta_{E,F}(x) \geq 1 - \delta \} \cap E \), and similarly for \( F_\delta \). Then \( \| \chi_{E_\delta} \theta_{E,F} \|_\infty \leq \| \chi_{E_\delta} \theta_{E,F} \|_\infty \leq 1 - \delta \). Hence \( \theta_{E_\delta,F} = 0 \), and so \( \theta_{E_\delta,F} \leq \theta_{E,F} \leq \theta_{E_\delta,F} \). Now \( \Psi_{E_\delta} \geq \theta_{E_\delta,F} \geq 1 - \delta \) on \( E_\delta \cup F_\delta \). Hence \( \Psi_{E_\delta} \geq (1 - \delta) \Psi_{E_\delta} \), which by Lemma 2 yields \( \Psi_{E_\delta} \geq (1 - \delta) \Psi_{E_\delta} \). Now \( (f/g) \leq \alpha \) on \( E_\delta \) yields \( \Psi_{E_\delta} f / \Psi_{E_\delta} g \leq \alpha \).

Similarly \( (f/g) \geq \alpha + \epsilon \) on \( F_\delta \) implies \( \Psi_{E_\delta} f / \Psi_{E_\delta} g \geq \alpha + \epsilon \). These relations yield \( \Psi_{E_\delta} g \geq (1 - \delta)^2 (\alpha + \epsilon) \Psi_{E_\delta} g \) which is false for small \( \delta \) if \( \Psi_{E_\delta}(g) \neq 0 \). Hence \( \theta_{E,F} = 0 \).
Corollary. If \( f, g \in L_1 \) and \( (f/g) \in L_\infty \), then \( (f/g) \in \mathcal{A} \).

Remark. If \( T \) is conservative, then Theorem 3 corresponds to the fact that \( (f/g) \) is measurable with respect to the \( \sigma \)-field of invariant sets (cf. [5], [2]).

Theorem 4. If \( (f/g) \in L_\infty \), and \( h \in \mathcal{A} \), then \( \int \pi(h) f \, d\mu = \int \pi(h/(f/g)) g \, d\mu \).

Proof. Recall that \( \int \pi(h/(f/g)) g \, d\mu = \lim_{n \to \infty} \int h(f/g) T^n g \, d\mu \). We may assume \( f \) and \( h \) are real. Choose \( \varepsilon > 0 \), and let \( E_{ij} \), \( 1 \leq i, j \leq k \) be a \( \Sigma \) partition of \( X \) such that

\[
\left\| h - \sum_{ij} h_{ij} \chi_{E_{ij}} \right\|_\infty < \varepsilon, \quad \left\| (f/g) - \sum_{ij} \alpha_{ij} \chi_{E_{ij}} \right\|_\infty < \varepsilon
\]

for suitable real \( h_{ij}, \alpha_{ij} \) with \( |h_{ij}| \leq \|h\|_\infty, |\alpha_{ij}| \leq \|(f/g)\|_\infty \). Now

\[
\lim_{n \to \infty} \int h(f/g) T^n g \, d\mu - \sum_{ij=1}^k h_{ij} \lim_{n \to \infty} \int E_{ij} T^n g \, d\mu = \left| \lim_{n \to \infty} \int h(f/g) T^n g \, d\mu - \sum_{ij} h_{ij} \alpha_{ij} \Theta_{E_{ij}}(g) \right| \leq \varepsilon \|g\|_1 (\|h\|_\infty + \|(f/g)\|_\infty).
\]

Let \( \delta > 0 \) be fixed and set \( E_{ij}^\prime = \{x \mid \theta_{E_{ij}}(x) \geq 1 - \delta\} \cap E_{ij} \). Then, as before, \( \theta_{E_{ij}^\prime} = \theta_{E_{ij}} \), and from Lemma 3, \( \theta_{E_{ij}^\prime} \geq (1 - \delta) \theta_{E_{ij}} \). Now \( |\alpha_{ij} - (f/g)| \leq \varepsilon \) on \( E_{ij}^\prime \) implies that \( |\alpha_{ij} - \Psi_{E_{ij}^\prime} f/\Psi_{E_{ij}^\prime} g| \leq \varepsilon \). [Here we consider only those \( E_{ij} \)'s with \( \theta_{E_{ij}} \neq 0 \).] Hence:

\[
\left| \sum_{ij} h_{ij} \alpha_{ij} \Theta_{E_{ij}} g - \sum_{\theta_{E_{ij}} \neq 0} h_{ij} \frac{\Psi_{E_{ij}^\prime} f}{\Psi_{E_{ij}^\prime} g} \Theta_{E_{ij}} g \right| \leq \varepsilon \|g\|_1 \|h\|_\infty \|f\|_1.
\]

Also:

\[
\left| \sum_{\theta_{E_{ij}} \neq 0} h_{ij} \frac{\Psi_{E_{ij}^\prime} f}{\Psi_{E_{ij}^\prime} g} \Theta_{E_{ij}} g - \sum_{ij} h_{ij} \Psi_{E_{ij}^\prime} f \right| \leq \|g\|_1 \|h\|_\infty (\|(f/g)\|_\infty + \varepsilon) k^2 \delta.
\]

Finally,

\[
\left| \sum_{ij} h_{ij} \Psi_{E_{ij}^\prime} f - \sum_{ij} h_{ij} \Theta_{E_{ij}} f \right| \leq \|h\|_\infty \|f\|_1 k^2 \delta
\]

and

\[
\left| \sum_{ij} h_{ij} \Theta_{E_{ij}} f - \lim_{n \to \infty} \int h T^n f \, d\mu \right| \leq \varepsilon \|f\|_1.
\]

Putting together all these inequalities, we conclude the result.

5. A representation for \( \mathcal{H} \).

**Definition 8.** \( \mathcal{B} \) is the sub-\( C^* \)-algebra of \( \mathcal{H} \) generated by the class \( \{\pi(l) \mid l \in L_\infty\} \).

Let \( \mathcal{M} \subset \mathcal{B}^* \) be the maximal ideal space of \( \mathcal{B} \) with the \( w^* \) topology induced from \( \mathcal{B}^* \). Let \( \mathcal{B} \) be the \( \sigma \)-field of Baire sets of \( \mathcal{M} \).

Note that \( \mathcal{B} \) contains the unit \( \pi(1) \) of \( \mathcal{H} \), and that \( g \in \mathcal{B} \) is invertible in \( \mathcal{B} \) if and only if it is invertible in \( \mathcal{H} \).
The C*-algebra $\mathcal{C}(\mathcal{M})$ of continuous complex valued functions on $\mathcal{M}$ is isometrically isomorphic to $\mathcal{G}$ under the Gelfand mapping $\sigma: \mathcal{G} \to \mathcal{C}(\mathcal{M})$. This mapping is order preserving. To see this first we need a few lemmas.

**Lemma 6.** If $f \in \mathcal{M}$ then $\|f\|^2 \geq |f|^2$.

**Proof.** We can assume that $f$ is real. Let $g \in L^1_\mathcal{G}$ with $\|g\|_1 = 1$. Then

$$|\int fg \, d\mu| = \left|\int f \cdot T_g \, d\mu\right| \leq \left(\int |f|^2 T_g \, d\mu\right)^{1/2} \left(\int T_g \, d\mu\right)^{1/2}.$$  

Hence $|\int fg \, d\mu|^2 \leq \int |U|f|^2 g \, d\mu$. If $|f|^2 \geq |Uf|^2$ on a set of positive measure, then there exist $E \in \mathcal{F}$, $\mu(E) > 0$, $a \geq 0$ and $\varepsilon > 0$ such that $|f| \geq a + \varepsilon$ and $U|f|^2 \leq a^2$ on $E$. Take $g = f_{\mathcal{E}}|f|\mu(E)$. Then

$$(a + \varepsilon)^2 \leq \left(\int fg \, d\mu\right)^2 \leq \int U|f|^2 g \, d\mu \leq a^2$$

which is a contradiction. Hence $U|f|^2 \geq |f|^2$ and $\pi|f|^2 \geq |f|^2$.

There is a canonical map $j: L_1 \to S^*$ defined by $(jf)(g) = \int fg \, d\mu, f \in L_1, g \in \mathcal{G}$. We now show that

**Lemma 7.** $\mathcal{M}$ is contained in the $w^*$-closure of $jL_1^+$ in $S^*$.

**Proof.** Choose $m \in \mathcal{M}$ and suppose that the $w^*$ neighborhood $\{F \mid |Fg_i - mg_i| < \varepsilon, i = 1, \ldots, n\}$ of $m$ defined by $g_1, \ldots, g_n \in \mathcal{G}$, $\varepsilon > 0$ is disjoint of $jL_1^+$. Let

$$u = \sum_{i=1}^n \pi[(g_i - 1mg_i)(g_i - 1mg_i)].$$

Now, let $f \in L_1^+$, $\|f\|_1 = 1$. Then

$$(jf)u = \sum_{i=1}^n \int \pi|g_i - 1mg_i|^2f \, d\mu$$

$$\geq \sum_{i=1}^n \int |g_i - 1mg_i|^2f \, d\mu$$

$$\geq \sum_{i=1}^n \int (g_i - 1mg_i)f \, d\mu \geq \varepsilon^2.$$  

Hence $u \geq \varepsilon^2$ a.e. and hence $u$ is invertible in $L_\infty$. This implies that $u$ is invertible in $\mathcal{G}$. But this is impossible since $mu = 0$.

**Corollary.** $jL_1$ is dense in $\mathcal{G}$ in the $w^*$-topology.

**Theorem 5.** The Gelfand mapping $\sigma: \mathcal{G} \to \mathcal{C}(\mathcal{M})$ is positive.

**Proof.** If $g \geq 0$ a.e. then $g^{**} \geq 0$ on $jL_1^+$ where $g \to g^{**}$ is the canonical embedding of $\mathcal{G}$ into $\mathcal{G}^{**}$. Since $jL_1^+$ is dense in $\mathcal{M}$, and $g^{**}$ is continuous, $g^{**} \geq 0$ on $\mathcal{M}$. Hence $\sigma g = g^{**}|_\mathcal{M} \geq 0$. 

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Now we would like to extend $\sigma$ to $\mathcal{H}$. First note that, by the Riesz representation theorem, any $F \in \mathcal{D}^*$ can be represented by a measure $\mu_F$ on $(\mathcal{M}, \mathcal{B})$. In particular, let $\bar{\mu} = \mu_{11}$. From the order-preserving property of the Riesz representation one can see that for any $f \in L_1$, $\mu_{11}$ is absolutely continuous with respect to $\bar{\mu}$. In fact we can obtain $d\mu_{11}/d\bar{\mu}$ as follows. First, considering only $L_\infty$ functions we have

**Lemma 8.** If $f \in L_\infty$ then $\mu_{11} \ll \bar{\mu}$ and $d\mu_{11}/d\bar{\mu} = \sigma(\tau f)$. 

**Proof.** For any $g \in \mathcal{B}$,

$$\int_{\mathcal{M}} g \cdot \sigma(\tau f) \, d\bar{\mu} = \int_{\mathcal{M}} \sigma(g \cdot \tau f) \, d\bar{\mu} = \int_X \pi(g \cdot \tau f) \, d\mu = \int_X g \, d\bar{\mu},$$

where the last equality follows from Theorem 4.

**Definition 9.** Let $T_f = \sigma(\tau f)$, $f \in L_\infty$.

Note that the linear mapping $f \mapsto \tau f$ defines a positive contraction $L_\infty(X, \mathcal{F}, \mu) \to L_\infty(\mathcal{M}, \mathcal{B}, \bar{\mu})$. But it is also a contraction for the corresponding $L_1$ norms; hence it is a contraction for all $L_p$ norms, $1 \leq p \leq \infty$. We can then extend this mapping to $L_p(X, \mathcal{F}, \mu) \to L_p(\mathcal{M}, \mathcal{B}, \bar{\mu})$ with the property that $\int_X g \, d\mu = \int_{\mathcal{M}} \sigma(\tau f) \, d\bar{\mu}$ for all $g \in \mathcal{B}, f \in L_p$.

We can now prove a representation theorem for $\mathcal{H}$.

**Theorem 6.** There is a positive isometric $*$ isomorphism between $\mathcal{H}$ and $L_\infty(\mathcal{M}, \mathcal{B}, \bar{\mu})$.

**Proof.** Let $h \in \mathcal{H}$ and define $\phi_h \in \mathcal{D}^*$ by

$$\phi_h(g) = \int_X \pi(gh) \, d\mu.$$ 

Note that if $h \in \mathcal{B}$ then $\phi_h$ is represented by the measure $\sigma(gh) \cdot d\bar{\mu}$ on $\mathcal{M}$. Let $\gamma_h$ be the representing measure of $\phi_h$, $h \in \mathcal{H}$. Then, for any nonnegative continuous function $og (g \in \mathcal{D}^+)$ on $\mathcal{M}$

$$\left| \int_{\mathcal{M}} og \, d\gamma_h \right| = \left| \int_X \pi(gh) \, d\mu \right| \leq \|h\|_\infty \int_X g \, d\mu = \|h\|_\infty \int_{\mathcal{M}} og \cdot d\bar{\mu}$$

which shows that $\gamma_h$ is absolutely continuous with respect to $\bar{\mu}$ and has a density function bounded by $\|h\|_\infty$. We denote this density function by $oh$, noting that it is actually an extension of $\sigma$, and $\|oh\|_\infty \leq \|h\|_\infty$. Furthermore, if $l \in L_\infty$ then

$$\int_{\mathcal{M}} o(h) \tau(l) \, d\bar{\mu} = \int_{\mathcal{M}} o(h) \sigma(l/1) \, d\bar{\mu} = \int_X \pi(h \cdot \tau(l/1)) \, d\mu = \int_X \pi(h \cdot (l/1)) \, d\mu = \int_X h \cdot (l/1) \, d\mu.$$
Hence \( \left| \int_X h l \, d\mu \right| \leq \| oh \|_\infty \| \tau l \|_1 \leq \| oh \|_\infty \cdot \| l \|_1 \), so \( \| h \|_\infty \leq \| oh \|_\infty \). Thus the extended \( \sigma \) is also an \( L_\infty \)-norm isometry. To show that \( \sigma \mathcal{H} = L_\sigma (\mathcal{M}, \mathcal{B}, \mu) \), first note that, if \( h \in \mathcal{H}, l \in L_\sigma (X, \mathcal{F}, \mu) \) then

\[
\int_X h l \, d\mu = \int_\mathcal{M} \sigma(h) \tau(l) \, d\bar{\mu} \leq \| oh \|_1 \| \tau l \|_\infty \leq \| oh \|_1 \| l \|_\infty ,
\]

hence \( \| h \|_1 \leq \| oh \|_1 \). Thus \( \sigma^{-1} : \sigma \mathcal{H} \to \mathcal{H} \) is an \( L_1 \)-contraction onto \( \mathcal{H} \). Now if \( o\h_n \) is an a.e. monotone sequence in \( \sigma \mathcal{H} \) converging a.e. to a function \( l \) in \( L_\sigma (\mathcal{M}, \mathcal{B}, \mu) \) then \( h_n \) is an a.e. bounded and monotone sequence in \( \mathcal{H} \). If the limit function is \( g \), one can easily see that \( g \in \mathcal{H} \) and \( og = l \). Since \( \sigma \mathcal{H} \) contains the continuous functions, this shows that \( \sigma \mathcal{H} = L_\sigma (\mathcal{M}, \mathcal{B}, \mu) \). Now we want to show that

\[
\int_X \sigma(h) \sigma(g) \tau(l) \, d\mu = \int_\mathcal{M} \sigma(h) \sigma(f) \, d\bar{\mu},
\]

for all \( h, f \in \mathcal{H} \). In fact, for a fixed \( h \in \mathcal{H} \), let \( \mathcal{N} \subset \mathcal{H} \) be the class of functions \( f \) for which this relation holds. Then \( \sigma \mathcal{N} \) contains the continuous functions of \( \mathcal{M} \), and one can show, as before, that \( \sigma \mathcal{N} \) is closed under a.e. monotone limits. Hence \( \sigma \mathcal{N} = L_\sigma (\mathcal{M}, \mathcal{B}, \mu) \).

Finally, we show that extended \( \sigma \) is multiplicative, i.e. \( \sigma(h_1) \cdot \sigma(h_2) = \sigma(\tau(h_1 h_2)) \) for all \( h_1, h_2 \in \mathcal{H} \). First note that if \( f \in L_\sigma (\mathcal{M}, \mathcal{F}, \mu) \) and \( \int_\mathcal{M} f r(l) \, d\bar{\mu} = 0 \) for all \( l \in L_\sigma (X, \mathcal{F}, \mu) \) then \( \sigma^{-1} f = 0 \), hence \( f = 0 \). Now for \( h \in \mathcal{H}, g \in \mathcal{G}, l \in L_\sigma (X, \mathcal{F}, \mu) \),

\[
\int_\mathcal{M} \sigma(h) \sigma(g) \tau(l) \, d\mu = \int_\mathcal{M} \sigma(h) \sigma(g) \sigma(\tau(l)) \, d\mu = \int_\mathcal{M} \sigma(h) \sigma(g \tau(l)) \, d\mu = \int_X \sigma(h g \tau(l)) \, d\mu = \int_X \sigma(h) \sigma(g \tau(l)) \, d\mu = \int_X \sigma(h) \sigma(g) \tau(l) \, d\mu,
\]

hence \( \sigma(h) \cdot \sigma(g) = \sigma(h g) \).

Now suppose that \( h_1, h_2 \in \mathcal{H}, l \in L_\sigma (X, \mathcal{F}, \mu) \). Then

\[
\int_\mathcal{M} \sigma(h_1) \sigma(h_2) \tau(l) \, d\bar{\mu} = \int_\mathcal{M} \sigma(h_1) \sigma(h_2 \tau(l)) \, d\mu = \int_X \sigma(h_1) \sigma(h_2 \tau(l)) \, d\mu = \int_X \sigma(h_1) \tau(l) \, d\mu,
\]

which shows that \( \sigma(h_1) \sigma(h_2) = \sigma(h_1 h_2) \), and completes the proof of the theorem.

We remark that every \( f \in L_p (\mathcal{M}, \mathcal{B}, \mu), 1 \leq p < \infty \), induces a function \( h \in L_p (X, \mathcal{F}, \mu) \), defined by \( \int_X h l \, d\mu = \int_\mathcal{M} f r(l) \, d\bar{\mu} \) for all \( l \in L_\sigma (X, \mathcal{F}, \mu) \), \( 1/p + 1/q = 1 \). Since \( \tau \) is an \( L_q \)-contraction the integral on \( \mathcal{M} \) is defined and \( h \) satisfies \( \int_X h l \, d\mu = \int_\mathcal{M} \sigma(h) \tau(l) \, d\bar{\mu} \).
\[ \int_X |hTl| \, d\mu, \text{ for all } l \in L_\alpha(X, \mathcal{F}, \mu). \] The case \( p=1 \) causes no difficulty. If \( f \in L_1(\mathcal{M}, \mathcal{B}, \bar{\mu}), l \in L_\alpha(X, \mathcal{F}, \mu), \)

\[
\left| \int_{\mathcal{M}} f \tau(l) \, d\bar{\mu} \right| \leq \left| \int_{\{|l| \leq n\}} f \tau(l) \, d\bar{\mu} \right| + \left| \int_{\{|l| > n\}} f \tau(l) \, d\bar{\mu} \right| 
\]

\[
\leq \| f \| \| \tau \| \int \{|l| \leq n\} |f| \, d\bar{\mu} + n\| f \|_{\infty} 
\]

\[
\leq \| f \| \| \tau \| \int \{|l| \leq n\} |f| \, d\bar{\mu} + n\| f \|_{l_1}. 
\]

Thus, if \( l_k \) is a sequence in \( L_\alpha \) with \( \| l_k \| \to 0 \) and \( \| l_k \|_{\infty} \leq K \) then

\[
\lim_k \left| \int \tau(l_k) \, d\bar{\mu} \right| \leq K \int \{|l| \geq n\} |f| \, d\bar{\mu} \quad \text{for all } n \geq 1. 
\]

Hence this limit is zero and the functional \( l \to \int f \tau(l) \, d\bar{\mu} \) on \( L \) is induced by an \( L_1 \)-function \( h \). In a similar way, any Baire measure on \((\mathcal{M}, \mathcal{B})\) induces what one might call "an invariant functional" on \( L_\alpha(X, \mathcal{F}, \mu) \).

We also note the following relation between the maximal ideal spaces of \( \mathcal{M} \) and \( \mathcal{B} \); that is, between the Feller and Martin boundaries (cf. [9]). Since \( \mathcal{M} \) is isometrically isomorphic to \( L_\alpha(X, \mathcal{F}, \mu) \), we state this relation in the following familiar form:

**Lemma 9.** Let \( M \) be a compact Hausdorff space, \( \mathcal{B} \) its Baire sets, and \( \bar{\mu} \) a Baire measure on \((M, \mathcal{B})\) with support \( M \). Let \( M' \) be the maximal ideal space of the \( C^* \)-algebra \( L_\alpha(M, \mathcal{B}, \bar{\mu}) \). Then there is a continuous and onto map \( \rho: M' \to M \).

**Proof.** Interpret \( M' \) and \( M \) as classes of homomorphisms and define \( \rho: M' \to M \) by \( \rho(\phi) = \phi|_{\mathcal{C}(M)} \). Then \( \rho \) is continuous. We show it is onto. Let \( m \in M \), and consider the ideal generated by \( m \cdot L_\alpha(M, \mathcal{B}, \bar{\mu}) \). If it is proper, it can be embedded in a maximal ideal, whose image must then be \( m \) under \( \rho \). We show it is proper. If not, then \( 1 = \sum f_i g_i \) where \( f_i \in m, g_i \in L_\alpha(M, \mathcal{B}, \bar{\mu}) \). Since \( m \) is a maximal ideal, \( \exists x_0 \) such that \( f_i(x_0) = 0 \) for \( i = 1, \ldots, n \). Hence \( |f_i| \leq e/h \sup |g_i| \) on some neighborhood \( U \) of \( x_0 \), such that \( \mu(U) \neq 0 \). Hence \( 1 = \sum f_i g_i \leq e \) on \( U \), which is a contradiction.

**Corollary.** \( M \) is homeomorphic to the quotient space \( M'/\rho \).

We finish this section by considering the possibility of joining \( X \) and \( M \). In general, this cannot be done. If, however, \( T \) is induced by a Markov kernel, such that the transform of every point measure is absolutely continuous with respect to \( \mu \), then the members of \( \mathcal{H} \) can be considered as actual functions on \( X \), and the evaluations of these functions at points of \( X \) induce bounded linear functionals on \( \mathcal{H} \). Hence \( X \) can be embedded in \( \mathcal{B}^* \) (possibly in a many to one fashion). We shall denote the image of \( X \) under this mapping as \( X \) also. Hence \( X \in j(L_1^+(X, \mathcal{F}, \mu)) \).

Using the method of Lemma 7, \( X \) is dense in \( M \), in the \( \omega^* \)-topology of \( \mathcal{B}^* \).
Let $\bar{X}$ be the $w^*$-closure of $X$ in $\mathcal{B}$. Then $\bar{X}$ is a compact Hausdorff space. The following result, stated for the Martin-Doob boundary, is also true for the Feller boundary.

**Theorem 7.** For any $g \in L_1(X, \mathcal{F}, \mu)$, $T^*g \, d\mu \rightarrow \tau(g) \, d\mu$ in the $w^*$-topology of Baire measures on $\bar{X}$.

**Proof.** Let $\mathcal{A}_1$ is the sub-$C^*$-algebra of $\mathcal{A}$, consisting of functions $g' \in \mathcal{A}$ such that $\pi(g') \in \mathcal{G}$.

Let $\mathcal{C} = \{f \in C(\bar{X}) \mid f|_X \in \mathcal{A}_1\}$, $\mathcal{C}_0 = \{f \in C(\bar{X}) \mid f|_X \in \mathcal{A}_0\}$. By the Stone-Weierstrass theorem $\mathcal{C} = C(\bar{X})$. Also, $\mathcal{C}_0$ is a closed ideal in $\mathcal{C}$. Let $\mathcal{N} \subset \bar{X}$ be the closed subset such that $\mathcal{C}_0 = \{f \in \mathcal{C} \mid f(\mathcal{N}) = 0\}$. Then we have

$$C(\mathcal{N}) \cong^* C(\bar{X})/C_0 \cong^* A_1/A_0 \cong^* \mathcal{A} \cong^* \mathcal{M}.$$

Hence $C(\mathcal{N}) \cong^* C(\mathcal{M})$ is induced by a homeomorphism $\phi : \mathcal{N} \rightarrow \mathcal{M}$. Hence $g(s) = g(\phi(s))$ under the above sequence of isomorphisms. But $\mathcal{G}$ separates the points of $\mathcal{B}^*$, so $\phi = \text{identity}$ and $\mathcal{N} = \mathcal{M}$.

In other words,

$$\{f \in C(\bar{X}) \mid f|_X \in \mathcal{A}_0\} = \{f \in C(\bar{X}) \mid f(\mathcal{M}) = 0\}.$$

Thus if $f \in C(\bar{X})$, $g \in L_1(\bar{X}, \mathcal{F}, \mu)$, then:

$$\int_X f T^*g \, d\mu = \int_X U^*(f|_X)g \, d\mu \Rightarrow \int_X \pi(f|_X)g \, d\mu = \int_{\mathcal{M}} \pi(f|_X)\tau(g) \, d\mu$$

and

$$\int_{\mathcal{M}} \pi(f|_X) \cdot \tau(g) \, d\mu = \int_{\mathcal{M}} f|_X \cdot \tau(g) \, d\mu = \int_X \pi(f|_X) \cdot \tau(g) \, d\mu.$$

Thus $T^*g \, d\mu \rightarrow \tau(g) \, d\mu$.

6. **Harmonic functions in the unit disk.** As an example we consider a transformation suggested by Feller in [10].

Let $D = \{z = re^{i\phi} \mid 0 \leq r < 1, -\pi \leq \phi \leq \pi\}$ be the unit disk with the (geometric) boundary $C$. Let $\mathcal{F}$ and $\mu$ be the $\sigma$-field of Borel subsets and the Lebesque measure. For every $z \in D$, $E \in \mathcal{F}$, let

$$P(z, E) = \mu(Q_z \cap E)/\mu(Q_z)$$

where $Q_z = \{Z \mid |Z - z| < 1 - |z|\}$. Then $P$ defines a Markov kernel, such that the transformation of a unit mass at $z \in D$ is given by the measure $P(z, \cdot) \ll \mu$. We let $T$ be the induced transformation on $L_1(D, \mathcal{F}, \mu)$. The adjoint $U$ of $T$ is given by

$$(Uf)(z) = \int f(Z)P(z, dZ), \quad f \in L_\infty, z \in D.$$
It is clear that any bounded harmonic function $h$ belongs to $\mathcal{H}$. The converse is also true, but it seems that no explicit proof of it has been given and we would like to indicate an outline for this proof.

If $R$ is a Borel subset of $[0, 1)$ let $C_R = \{ z \mid |z| \in R \}$. One can then obtain the following

**Lemma 10.** Let $\frac{1}{2} \leq K < 1$ and $R$ be a Borel subset of $[K, 1)$. Then for all $z \in D$, $\frac{1}{2} \leq |z| \leq K$,

\[
\frac{\mu(Q_z \cap C_R)}{\mu(Q_z \cap C_{(K,1)})} \geq \frac{1}{16} \left[ \frac{\lambda(R)}{1-K} \right]^{3/2}
\]

where $\lambda$ is the one dimensional Lebesgue measure.

**Corollary.** Let $E = C_{(0,1/2)} \cup [K, 1)$ and let $f \in L^1_{+}, f=0$ a.e. on $C_{(K,1)}$. Then

\[
\int_{C_R} T^k_b f d\mu \geq \frac{1}{16} \left[ \frac{\lambda(R)}{1-K} \right]^{3/2} \int_{C_{(K,1)}} T^k_b f d\mu
\]

for all $n \geq 0$.

Using this corollary one can see that if a function $h \in \mathcal{H}$ (which is necessarily continuous) has the form $h(re^{i\phi}) = f(r)g(\phi)$ then $\lim_{n \to \infty} f(r)$ exists, and that this implies the harmonicity of $h$.

Now if $h$ is any function in $\mathcal{H}$, let $t$ be an irrational number and consider, for a fixed $n$, $-\infty < n < \infty$,

\[
\lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} (\tau^k \cdot z)^{-n} h(\tau^k \cdot z) = F_n
\]

where $\tau : D \to D$ is given by $\tau z = e^{2\pi i t} z$. This limit $F_n$ exists for all nonzero $z \in D$, depends only on $r = |z|$, and satisfies

\[
r^n F_n(r) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{-i n \phi} h(re^{i \phi}) d\phi.
\]

But, it is clear that

\[
e^{i n \phi} r^n F_n(r) = \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} e^{-i2\pi n k t} h(re^{i2\pi k t})
\]

is a function in $\mathcal{H}$, hence $e^{i n \phi} r^n F_n(r)$ must be harmonic, which shows that $r^n F_n(r) = C_n r^{\lfloor n \rfloor}$ and completes the proof of the following

**Lemma 11.** A bounded function belongs to $\mathcal{H}$ if and only if it is a harmonic function.

One then shows that the $C^*$-algebra $\mathcal{H}$ is isometrically $*$-isomorphic to $L_{\infty}$ of the unit circle. For any bounded measurable function $l$ on $D$, let $\lambda_l$ be the measure on the unit circle obtained by sweeping out $l d\mu$ by the Poisson kernel. The harmonic function $\pi(l/l)$ corresponds to $d\lambda_l/d\lambda$, which is continuous. It then follows that the maximal ideal space $\mathcal{M}$ of $\mathcal{H}$ is homeomorphic to the unit circle. Since $T$ is induced by a Markov kernel, $D$ can be imbedded into $\mathcal{M}$, hence $D \cup \mathcal{M}$ is homeomorphic to the closed unit disk.
References


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