1. Introduction. Let $T$ be a conservative positive contraction on the $L_1$ space of a finite measure space $(X, \mathcal{F}, \mu)$. A theorem of Chacon [5], [2] shows that $T$ defines a sub $\sigma$-field $\mathcal{J}$ of $\mathcal{F}$, consisting of invariant subsets of $X$. The ratio ergodic limits are measurable with respect to $\mathcal{J}$ [5], [2] and the class of these limits contains $L_\sigma(X, \mathcal{J}, \mu)$, which can be considered as the invariant functions of the adjoint transformation [2]. The main purpose of the present paper is to show that any positive contraction on $L_1(X, \mathcal{F}, \mu)$ behaves, asymptotically, like a conservative transformation (Theorems 3 and 4) and that the invariant functions of the adjoint transformation can be approximated by the ratio ergodic limits.

Intuitively, a ratio ergodic limit corresponds to the result of an averaging process of different values of a function. It is then natural to consider these limits as functions that are smooth with respect to the asymptotic behaviour of the transformation. This leads (Theorem 6) to a Martin-Doob type representation [12], [8] of invariant functions as the $L_\infty$ functions of a compact Hausdorff space $\mathcal{M}$ with a Baire measure. The topology on $\mathcal{M}$ is just strong enough to make the ergodic limits to correspond to continuous functions. As an example we consider a transformation of Feller [10] and show that for this case the above representation is identical with the Poisson representation of harmonic functions in the unit disk. We also consider the possibility of joining $X$ and $\mathcal{M}$, convergence of measures to $\mathcal{M}$ in $X \cup \mathcal{M}$ (Theorem 7), and a relation (Lemma 9) between the Feller and Martin-Doob type representations, corresponding to a result of Feldman [9].

2. Preliminaries. Let $(X, \mathcal{F}, \mu)$ be a finite measure space and let $L_p = L_p(X, \mathcal{F}, \mu)$, $1 \leq p \leq \infty$ be the usual Banach spaces, and $L_\infty^+$ denote the positive cone of $L_\infty$. Let $T: L_1 \to L_1$ be a positive linear contraction and $U: L_\infty^+ \to L_\infty$ be its dual. For $\alpha \in L_\infty$, define $T_\alpha: L_1 \to L_1$ as $T_\alpha f = \alpha f + T(1-\alpha)f$, $f \in L_1$, and let $U_\alpha$ be its dual. If $\chi_E$ is the characteristic function of $E \in \mathcal{F}$ we write $T_E$ and $U_E$ instead of $T_{\chi_E}$ and $U_{\chi_E}$.

The following partial ordering of $L_1^+$ is similar to that of Bishop and deLeeuw given in [3].

**Definition 1.** For $f, g \in L_1^+$, $f \preceq g$ if and only if there exist an integer $n \geq 1$ and $\alpha_1, \ldots, \alpha_n \in L_\infty$ such that $0 \leq \alpha_i \leq 1$ for $i = 1, \ldots, n$ and $g = T_{\alpha_n} \cdots T_{\alpha_2} f$.

This relation is reflexive and transitive and $f \preceq g$ implies $\|f\|_1 \geq \|g\|_1$. Also, an

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induction argument shows that if $f < g$ then there exists an integer $n \geq 1$ such that $g < T^n f$. Hence $\{g \in L_1^+ \mid g > f\}$ is (upward) directed by $<$.  

**Definition 2.** For $E \in \mathcal{F}, f \in L_1^+$ let 

$$
\Psi_E f = \sup_{g > f} \int_E g \, d\mu, \quad \Theta_E f = \lim_{g > f} \Psi_E g.
$$

Note that $\Theta_E f = \lim_{n \to \infty} \Psi_E T^n f$.

**Lemma 1.** The limits $\psi_E = \lim_{n \to \infty} U^*_E x$ and $\theta_E = \lim_{n \to \infty} U^n\psi_E$ both exist (a.e.) and satisfy 

$$
\Psi_E f = \int \psi_E f \, d\mu, \quad \Theta_E f = \int \theta_E f \, d\mu.
$$

**Proof.** By induction, $U^*_E x \uparrow$ and $U^n\psi_E \downarrow$, so the limits exist. Now if $f \in L_1^+$ satisfies 

(*)

$$
\chi_{E^c} Uf \leq \chi_{E'} f, \quad \chi_{E^c} Uf \geq \chi_{E'} f
$$

with $E^c = X - E$, then for all $\alpha \in L_1^+$, $0 \leq \alpha \leq 1$, we have 

$$
U_{\alpha} f = \alpha f + (1 - \alpha) Uf \leq \chi_{E'} Uf + \chi_{E^c} Uf = U_{E'} f.
$$

Since, by induction, $U^*_E x \uparrow$ satisfies (*) for all $n \geq 0$, we get, again by induction, $U_{a_1} \cdots U_{a_n} \chi_{E} \leq U^*_E x$, and hence $\Psi_E f = \int \psi_E f \, d\mu$. The final part follows from the definition (cf. also [4] and [2]).

**Definition 3.** For $E, F \in \mathcal{F}$, let 

$$
\psi_{EF} = \psi_E - \psi_{E \cup F}, \quad \theta_{EF} = \theta_E - \theta_{E \cup F}.
$$

$\Psi_{EF}, \Theta_{EF}$ are the functionals on $L_1$ defined by the $L_1$ functions $\psi_{EF}, \theta_{EF}$.

We note that $\psi_{EF}$ and $\theta_{EF}$ are monotone and subadditive in each index. This follows easily from the following general result, which will be useful to obtain other relations between these set functions (cf. [7]).

**Lemma 2.** If $a_i$ is real and $A_i \in \mathcal{F}$ for $i = 1, \ldots, n$ and $A = \bigcup_{i=1}^n A_i$, then 

$$
\chi_A \sum_{i=1}^n a_i \psi_{A_i} \geq 0 \text{ implies } \sum_{i=1}^n a_i \psi_{A_i} \geq 0 \text{ and } \sum_{i=1}^n a_i \theta_{A_i} \leq 0.
$$

**Proof.** If $f \in L_1^+$ and $E \subset F$, $E, F \in \mathcal{F}$, then by induction: 

$$
\chi_{E^c} f \leq \chi_{F^c} f
$$

Hence 

$$
0 \leq \int_{E^c} \psi_{E} T^+_f \, d\mu \leq \int_{E^c} \psi_{E} T^-_f \, d\mu \leq \int_{E^c} \psi_{E} T^+_f \, d\mu 
$$

$$
\leq \int_{E^c} \psi_{E} T^-_f \, d\mu - \int_{E} \psi_{E} T^-_f \, d\mu \leq \Psi_E f - \int_{E} T^-_f \, d\mu \to 0
$$
as \( n \to \infty \). Now

\[
\int \psi_E f \, d\mu = \int \psi_{E\chi_F} f \, d\mu + \int \psi_{E\chi_F} f \, d\mu = \int \psi_{E\chi_F} f \, d\mu + \int \psi_{E\chi_F} f \, d\mu
\]

\[
= \int \psi_{E\chi_F} f \, d\mu + \int \psi_{E\chi_F} f \, d\mu = \int \psi_{E\chi_F} f \, d\mu
\]

\[
= \int \psi_{E\chi_F} f \, d\mu + \int \psi_{E\chi_F} f \, d\mu = \int \psi_{E\chi_F} f \, d\mu
\]

and hence \( \lim_{n \to \infty} \int \psi_{E\chi_F} f \, d\mu = \int \psi_{E} f \, d\mu \). Using this for the case of \( A_i \subset A \), \( i = 1, \ldots, n \), we get

\[
0 \leq \sum_{i=1}^{n} a_i \int_A \psi_{A_i} T f \, d\mu \to \sum_{i=1}^{n} a_i \int \psi_{A_i} f \, d\mu
\]

as \( n \to \infty \), which proves the first assertion. Since \( U \) is positive, the remainder follows.

**Lemma 3.** If \( \chi_E \theta_E \geq \alpha \chi_E \) then \( \theta_E \geq \alpha \psi_E \).

**Proof.** From the proof of the previous lemma we have that, for \( f \in L_1^+ \),

\[
\lim_{n \to \infty} \int_E \theta_E T^n f \, d\mu = 0.
\]

Hence

\[
\int \theta_E f \, d\mu = \int \theta_E T^n f \, d\mu = \lim_{n \to \infty} \int_E \theta_E T^n f \, d\mu
\]

\[
\geq \alpha \lim_{n \to \infty} \int_E T^n f \, d\mu \geq \alpha \int \psi_E f \, d\mu.
\]

Finally we prove the following.

**Lemma 4.** For \( E, F \in \mathcal{F} \), \( \| \theta_E \|_\infty = \| \chi_E \theta_E \|_\infty = 0 \) or \( 1 \) and \( \| \theta_{EF} \|_\infty = \| \chi_E \theta_{EF} \|_\infty = \| \chi_F \theta_{EF} \|_\infty = 0 \) or \( 1 \).

**Proof.** For \( g \in L_1^+ \), as \( n \to \infty \), \( 0 \leq \Theta_E(\chi_E T^n g) \leq \psi_E(\chi_E T^n g) \to 0 \) as in the proof of Lemma 2. Hence the decomposition \( \Theta_E g = \Theta_E T^n g = \Theta_E(\chi_E T^n g) + \Theta_E(\chi_E T^n g) \) shows that \( \| \theta_E \|_\infty = \| \chi_E \theta_E \|_\infty \). Now, for \( n, m \geq 1 \),

\[
\Theta_E g = \Theta_E T^n g \leq \lim_{n \to \infty} \Theta_E(\chi_E T^n g) \leq \lim_{m \to \infty} \lim_{n \to \infty} \| \chi_E T^n g \|_1
\]

\[
= \lim_{n \to \infty} \| \chi_E T^n g \|_1 = \| \theta_E \|_\infty = \Theta_E g
\]

which completes the proof of the first part, since \( \| \theta_E \|_\infty \leq 1 \). For the second part, we have, if \( g \in L_1^+ \), \( 0 \leq \Theta_{EF}(\chi_E T^n g) \leq \Theta_E(\chi_E T^n g) \to 0 \) as \( n \to \infty \) which shows that \( \| \theta_{EF} \|_\infty = \| \chi_E \theta_{EF} \|_\infty \). Now

\[
\Theta_E g - \Theta_E(\chi_E T^n g) = \Theta_E(\chi_E T^n g) \leq \Theta_E(\chi_E T^n g) \leq \| \chi_E T^n g \|_1 \leq \psi_E g;
\]
thus, $\Theta_{EG} \leq \lim_{n \to \infty} \Theta_{E \cup F}(X_E T^*_{Eg}) \leq \|Eg\|$. Replacing $g$ by $T^m g$ and letting $m \to \infty$ we get

$$\Theta_{Eg} = \lim_{m \to \infty} \lim_{n \to \infty} \Theta_{E \cup F}(X_E T^*_E T^m g).$$

Next, consider

$$\Theta_{EF}(X_E T^*_E T^m g) = (\Theta_E + \Theta_F - \Theta_{E \cup F})(X_E T^*_E T^m g)$$

and let $n \to \infty$ to get

$$\Theta_{EFg} = \Theta_{Eg} + \lim_{n \to \infty} \Theta_F(X_E T^*_E T^m g) - \lim_{n \to \infty} \Theta_{E \cup F}(X_E T^*_E T^m g).$$

Now, letting $m \to \infty$ we have

$$\Theta_{EFg} = \lim_{m \to \infty} \lim_{n \to \infty} \Theta_F(X_E T^*_E T^m g).$$

But

$$\Theta_{EFg} \leq \|\Theta_{EF}\| \lim_{m \to \infty} \lim_{n \to \infty} \|X_E T^*_E T^m g\|_1 \leq \|\Theta_{EF}\| \lim_{m \to \infty} \|T^*_E T^m g\| \leq \|\Theta_{EF}\| \Theta_{Eg}.$$  

Hence

$$\Theta_{EFg} = \Theta_{EF}(T^*_E T^m g) = \lim_{m \to \infty} \lim_{n \to \infty} \Theta_{EF}(X_E T^*_E T^m g) \leq \|\Theta_{EF}\| \lim_{m \to \infty} \lim_{n \to \infty} \Theta_F(X_E T^*_E T^m g) \leq \|\Theta_{EF}\| \Theta_{EFg}.$$  

This completes the proof, since $\|\Theta_{EF}\| \leq 1$.

**Definition 4.** $\Sigma = \{E \in \mathcal{F} \mid \Theta_{Eg} = 0\}.$

**Lemma 5.** $\Sigma$ is a field.

**Proof.** Let $E, F \in \Sigma$ and $G = E \cap F$. Then

$$0 \leq \theta_{GF} = \theta_{G \cup \text{F} \cap F} \leq \theta_{GF} + \theta_{GF} \leq \Theta_{EF} + \Theta_{EF} = 0.$$  

Thus $G \in \Sigma$.

**Definition 5.** $\mathcal{A}$ is the $L_\infty$-closure of the class of $\Sigma$-simple functions.

We note that $\mathcal{A}$ is a sub-Banach space of $L_\infty$.

**Theorem 1.** For a real valued function $f \in L_\infty$, the following conditions are equivalent:

(i) $f \in \mathcal{A}$,

(ii) $\lim_{\mu \to \mu_0} \int fg \, d\mu$ exists for all $g_0 \in L^+_\infty$,

(iii) for all real numbers $\alpha$ and $\epsilon > 0$,

$$\theta_{EF} = 0 \quad \text{where} \quad E = \{x \mid f(x) \leq \alpha\}, \quad F = \{x \mid f(x) \geq \alpha + \epsilon\}.$$
Proof. (i)⇒(ii). If $E \in \Sigma$ then $\theta_E + \theta_{E^c} = \theta_X$; thus, for a real valued $g_0 \in L^+_1$, 
\[
\limsup_{\varrho \to \varrho_0} \int_E g \, d\mu = \limsup_{\varrho \to \varrho_0} \int g \, d\mu - \limsup_{\varrho \to \varrho_0} \int_{E^c} g \, d\mu \\
= \liminf_{\varrho \to \varrho_0} \int_E g \, d\mu.
\]
Therefore $\lim_{\varrho \to \varrho_0} \int_\Sigma \chi_E g \, d\mu$ exists for all $E \in \Sigma$.

Hence it exists for all $\Sigma$-simple functions, and thus for all $f \in \mathcal{A}$.

(ii)⇒(iii). Suppose $E$ and $F$ are as in (iii) but that $\theta_{EF} \neq 0$. Then $\|\theta_{EF}\|_\infty = 1$ and for all $\delta > 0$ there exists $g_0 \in L^+_1$ with $\|g_0\|_1 = 1$ and $\int \theta_{EF} g \, d\mu \geq 1 - \delta$. Hence $\Theta_{EF} g_0 \geq 1 - \delta$ and $\Theta_{F} g_0 \geq 1 - \delta$. Thus $\limsup_{\varrho \to \varrho_0} \int fg \, d\mu \geq (1 - \delta)(x + \epsilon)$ and $\liminf_{\varrho \to \varrho_0} \int fg \, d\mu \leq (1 - \delta)x + \delta \|f\|_\infty$. If $\delta$ is chosen sufficiently small we see that $\lim_{\varrho \to \varrho_0} \int fg \, d\mu$ does not exist.

(iii)⇒(i). Let $a_1 < a_2 < \cdots < a_n$ be $n$ numbers and let $E_i = \{x | f(x) \leq a_i\}$. Now 
\[
\sum_{i=1}^n \theta_{E_i} = \sum_{i=1}^n \left( \theta_{E_i} + \theta_{E_i^c} - \theta_X \right) \\
\leq \sum_{i=2}^n \left( \theta_{E_{i-1}} + \theta_{E_i^c} - \theta_{E_{i-1}} \cup E_i \right) + \left( \theta_{E_n} + \theta_{E_1^c} - \theta_X \right) \\
\leq 1.
\]
Hence if $E_a = \{x | f(x) \leq a\}$ then $\theta_{E_a E_i} \neq 0$ for only countably many $a_i$'s, and so $f \in \mathcal{A}$.

3. Invariant functions.

Definition 6. $\mathcal{H} = \{f | f \in L_\infty, f = Uf\}$ is the class of invariant functions of $U$.

We assume $\mathcal{H} \neq \{0\}$.

Note that $\mathcal{H}$ is a sub-Banach space of $L_\infty$. Also, if $h \in \mathcal{H}$ and $g' \geq g \in L^+_1$, then $\int h g' \, d\mu = \int h g \, d\mu$ and hence $\limsup_{g \to g'} \int h g \, d\mu$ exists. Thus $\mathcal{H} \subset \mathcal{A}$.

If $f \in \mathcal{A}$, then $\lim_{n \to \infty} \int U^n g \, d\mu = \int U^\infty g \, d\mu$ exists for all $g \in L_1(X, \mathcal{F}, \mu)$. Hence the bounded sequence $U^n f$, $n = 1, 2, \ldots$ has a limit $\pi(f)$ in the $w^*$-topology of $L_\infty$. Obviously the limit lies in $\mathcal{H}$, so $\pi: \mathcal{A} \to \mathcal{H}$ is a positive linear contraction.

Definition 7. $\mathcal{A}_0 = \ker \pi = \{f \in \mathcal{A} | w^* \lim U^\infty f = 0\}$. Hence $\mathcal{A}/\mathcal{A}_0 \cong \mathcal{H}$ is a canonical, isometric isomorphism.

Now $\mathcal{A}$ is a $C^*$-algebra with the usual operations. We show that $\mathcal{A}_0$ is a closed ideal.

Theorem 2. $\mathcal{A}_0$ is a closed ideal in $\mathcal{A}$.

Proof. Let $f \in \mathcal{A}_0$ and assume that $f$ is real. Choose $\epsilon > 0$ and set $E = \{x | f(x) \geq \epsilon\}$.

We may assume $E \in \Sigma$. Suppose $\theta_E \neq 0$; then for all $\delta > 0$, there is a $g \in L^+_1$ such that $\|g\|_1 = 1$ and $\Theta_{E} g \geq 1 - \delta$. Hence:
\[
0 = \lim_{n \to \infty} \int U^nf \cdot g \, d\mu = \lim_{n \to \infty} \int T^n g \, d\mu \\
\geq \epsilon \lim_{n \to \infty} \int E T^n g \, d\mu - \|f\|_\infty \lim_{n \to \infty} \int_{E^c} T^n g \, d\mu \\
\geq \epsilon(1 - \delta) - \|f\|_\infty \delta.
\]
Clearly, this fails for small \( \delta \), and so \( \theta_\delta = 0 \). Thus if \( E = \{x \mid |f(x)| > \epsilon\} \), we have \( \theta_\delta = 0 \).

Now if \( h \in A, h \neq 0 \), set \( F = \{x \mid |f(x)h(x)| \geq \epsilon\} \). Since \( F \subseteq \{x \mid |f(x)| \geq \epsilon/\|h\|_\infty\} \), we have \( \theta_F = 0 \). Hence

\[
\lim_{n \to \infty} \left| \frac{1}{n} \sum_{k=1}^{n} U^n(fh)g \mu \right| \leq \epsilon \lim_{n \to \infty} \int_T T^n g d\mu + \epsilon \|g\|_1 \quad \text{if } g \in L^+_1
\]

for all \( \epsilon > 0 \).

Hence \( fh \in \mathcal{A}_0 \).

As a result of the lemma, we have given \( \mathcal{A} \mid \mathcal{A}_0 \), and hence \( \mathcal{H} \) the structure of a \( C^* \)-algebra. Thus \( \mathcal{H} \) has a representation as the set of complex valued continuous functions on its maximal ideal space. This corresponds to Feller's representation [10] of the invariant functions of certain Markov processes, and we shall refer to \( \mathcal{H} \)'s maximal ideal space as the Feller boundary.

As is known [8], [11], the Feller boundary is larger than it need be. In the next section, we obtain some properties of ratio ergodic limits, and use them to define a sub \( C^* \)-algebra \( \mathcal{I} \) of \( \mathcal{H} \), with a maximal ideal space \( \mathcal{M} \), smaller than the Feller boundary, but large enough to represent \( \mathcal{H} \) as a function algebra on \( \mathcal{M} \). This corresponds to the Martin-Doob representation [12], [8], [11] for some classes of functions, and \( \mathcal{M} \) will be referred to as the Martin-Doob boundary.

### 4. Properties of ratio ergodic limits

In [6] Chacon and Ornstein proved that for any \( f, g \in L^+_1 \), with \( g > 0 \), the limit:

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} T^k f
\]

exists a.e. We denote the limit function by \((f/g)\). It is also known [5], [4], [1], that if \( \alpha \leq (f/g) \leq \beta \) a.e. on \( E \in \mathcal{A} \), then \( \alpha \leq \Psi_E(f)/\Psi_E(g) \leq \beta \).

**Theorem 3.** If \( f, g \in L^+_1 \) with \( g > 0 \), and

\[
E = \{x \mid (f/g)(x) \leq a\},
\]

\[
F = \{x \mid (f/g)(x) \geq a + \epsilon\},
\]

then \( \theta_{E,F} = 0 \), for all \( a \geq 0 \) and \( \epsilon > 0 \).

**Proof.** If \( \theta_{E,F} \neq 0 \) then \( \|x_E \theta_{E,F}\|_\infty = 1 \). Let \( \delta > 0 \) and set \( E_\delta = \{x \mid \theta_{E,F}(x) \geq 1 - \delta\} \cap E \), and similarly for \( F_\delta \). Then \( \|x_E - x_E \theta_{E,F}\|_\infty \leq \|x_E - x_E \theta_{E,F}\|_\infty = 1 - \delta \). Hence \( \theta_{E - E_\delta,F} = 0 \), and so \( \theta_{E_\delta,F} \leq \theta_{E,F} \leq \theta_{E_\delta,F} + \theta_{E - E_\delta,F} = \theta_{E_\delta,F} \) which implies \( \theta_{E_\delta,F} = \theta_{E,F} \). Now \( \psi_{E_\delta} \geq \theta_{E_\delta,F} \geq 1 - \delta \) on \( E_\delta \cup F_\delta \). Hence \( \psi_{E_\delta} \geq (1 - \delta) \psi_{F_\delta} \), and \( \psi_{F_\delta} \geq (1 - \delta) \psi_{E_\delta} \). Now \( (f/g) \leq a \) on \( E_\delta \) yields \( \Psi_{E_\delta}/\Psi_{E_\delta} \leq a \). Similarly \( (f/g) \geq a + \epsilon \) on \( F_\delta \) implies \( \Psi_{F_\delta}/\Psi_{F_\delta} \geq a + \epsilon \). These relations yield \( a \Psi_{E_\delta} \geq (1 - \delta)^2 (a + \epsilon) \Psi_{E_\delta} \) which is false for small \( \delta \) if \( \Psi_{E_\delta}(g) \neq 0 \). Hence \( \theta_{E,F} = 0 \).
COROLLARY. If \( f, g \in L_1 \) and \( (f/g) \in L_{\infty} \), then \( (f/g) \in \mathcal{A} \).

REMARK. If \( T \) is conservative, then Theorem 3 corresponds to the fact that \( (f/g) \) is measurable with respect to the \( \sigma \)-field of invariant sets (cf. [5], [2]).

THEOREM 4. If \( (f/g) \in L_{\infty} \), and \( h \in \mathcal{A} \), then \( \int \pi(h) \cdot f \, d\mu = \int \pi(h(f/g))g \, d\mu. \)

Proof. Recall that \( \int \pi(h(f/g))g \, d\mu = \lim_{n \to \infty} \int h(f/g)T^n g \, d\mu \). We may assume \( f \) and \( h \) are real. Choose \( \varepsilon > 0 \), and let \( E_{ij}, 1 \leq i, j \leq k \) be a \( \Sigma \) partition of \( X \) such that

\[
\left| h - \sum_{ij} h_{ij} X_{ij} \right|_\infty < \varepsilon, \quad \left| (f/g) - \sum_{ij} \alpha_{ij} X_{ij} \right|_\infty < \varepsilon
\]

for suitable real \( h_{ij}, \alpha_{ij} \) with \( \left| h_{ij} \right| \leq \| h \|_\infty, \left| \alpha_{ij} \right| \leq \| (f/g) \|_\infty \). Now

\[
\left| \lim_{n \to \infty} \int h(f/g)T^n g \, d\mu - \sum_{i=1}^k h_{ij} \alpha_{ij} \lim_{n \to \infty} \int_{E_{ij}} T^n g \, d\mu \right|
\]

\[
= \left| \lim_{n \to \infty} \int h(f/g)T^n g \, d\mu - \sum_{ij} h_{ij} \alpha_{ij} \Theta_{E_{ij}}(g) \right| \leq \varepsilon \| g \|_1 (\| h \|_\infty + \| (f/g) \|_\infty).
\]

Let \( \delta > 0 \) be fixed and set \( E_{ij}' = \{ x \mid \theta_{E_{ij}}(x) \geq 1 - \delta \} \cap E_{ij} \). Then, as before, \( \theta_{E_{ij}'} = \theta_{E_{ij}} \), and from Lemma 3, \( \theta_{E_{ij}'} \geq (1 - \delta) \theta_{E_{ij}} \). Now \( |\alpha_{ij} - (f/g)| \leq \varepsilon \) on \( E_{ij}' \) implies that

\[
|\alpha_{ij} - \Psi_{E_{ij}'}f/\Psi_{E_{ij}'}g| \leq \varepsilon. \]  

Here we consider only those \( E_{ij}'s \) with \( \theta_{E_{ij}'} \neq 0 \). Hence:

\[
\left| \sum_{ij} h_{ij} \alpha_{ij} \Theta_{E_{ij}} g - \sum_{\theta_{E_{ij}} \neq 0} h_{ij} \Psi_{E_{ij}'}f \Theta_{E_{ij}} g \right| \leq \varepsilon \| g \|_1 (\| h \|_\infty + \| (f/g) \|_\infty + \varepsilon)k^2 \delta.
\]

Also:

\[
\left| \sum_{\theta_{E_{ij}} \neq 0} h_{ij} \Psi_{E_{ij}'}f \Theta_{E_{ij}} g - \sum_{ij} h_{ij} \Psi_{E_{ij}'}f \right| \leq \| g \|_1 \| h \|_\infty (\| (f/g) \|_\infty + \varepsilon)k^2 \delta.
\]

Finally,

\[
\left| \sum_{ij} h_{ij} \Psi_{E_{ij}'}f - \sum_{ij} h_{ij} \Theta_{E_{ij}} f \right| \leq \| h \|_\infty \| f \|_1 k^2 \delta
\]

and

\[
\left| \sum_{ij} h_{ij} \Theta_{E_{ij}} f - \lim_{n \to \infty} \int h T^n f \, d\mu \right| \leq \varepsilon \| f \|_1.
\]

Putting together all these inequalities, we conclude the result.

5. A representation for \( \mathcal{H} \).

DEFINITION 8. \( \mathcal{B} \) is the sub-C*-algebra of \( \mathcal{H} \) generated by the class \( \{\pi(l) \mid l \in L_{\infty}\} \).

Let \( \mathcal{M} \subset \mathcal{B}^* \) be the maximal ideal space of \( \mathcal{B} \) with the \( w^* \) topology induced from \( \mathcal{B}^* \). Let \( \mathcal{B} \) be the \( \sigma \)-field of Baire sets of \( \mathcal{M} \).

Note that \( \mathcal{B} \) contains the unit \( \pi(l) \) of \( \mathcal{H} \), and that \( g \in \mathcal{B} \) is invertible in \( \mathcal{B} \) if and only if it is invertible in \( \mathcal{H} \).

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The C*-algebra $\mathcal{C}(\mathcal{M})$ of continuous complex valued functions on $\mathcal{M}$ is isometrically isomorphic to $\mathcal{G}$ under the Gelfand mapping $\sigma: \mathcal{G} \to \mathcal{C}(\mathcal{M})$. This mapping is order preserving. To see this first we need a few lemmas.

**Lemma 6.** If $f \in \mathcal{G}$ then $\pi |f|^2 \geq |f|^2$.

**Proof.** We can assume that $f$ is real. Let $g \in L^1$ with $\|g\|_1 = 1$. Then

$$\left| \int fg \, d\mu \right| = \left| \int f - Tg \, d\mu \right| \leq \left| \int |f|^2 Tg \, d\mu \right|^{1/2} \left| \int Tg \, d\mu \right|^{1/2}.$$ 

Hence $\left| \int fg \, d\mu \right|^2 \leq \int |f|^2 g \, d\mu$. If $|f|^2 > |Uf|^2$ on a set of positive measure, then there exist $E \in \mathcal{F}$, $\mu(E) > 0$, $\alpha > 0$ and $\epsilon > 0$ such that $|f|^2 > \alpha + \epsilon$ and $U |f|^2 \leq \alpha^2$ on $E$. Take $g = f_{X_E} / |f| \mu(E)$. Then

$$(\alpha + \epsilon)^2 \leq \left| \int fg \, d\mu \right|^2 \leq \int |f|^2 g \, d\mu \leq \alpha^2$$

which is a contradiction. Hence $U |f|^2 \geq |f|^2$ and $\pi |f|^2 \geq |f|^2$.

There is a canonical map $j: L_1 \to \mathcal{G}^*$ defined by $(jf)(g) = \int fg \, d\mu, f \in L_1, g \in \mathcal{G}$. We now show that

**Lemma 7.** $\mathcal{M}$ is contained in the $w^*$-closure of $jL_1^+$ in $\mathcal{G}^*$.

**Proof.** Choose $m \in \mathcal{M}$ and suppose that the $w^*$ neighborhood $\{ F \mid |Fg_i - mg_i| < \epsilon, i = 1, \ldots, n \}$ of $m$ defined by $g_1, \ldots, g_n \in \mathcal{G}$, $\epsilon > 0$ is disjoint of $jL_1^+$. Let

$$u = \sum_{i=1}^n \pi [(g_i - 1mg_i)(g_i - 1mg_i)].$$

Now, let $f \in L_1^+$, $\|f\|_1 = 1$. Then

$$(jf)u = \sum_{i=1}^n \pi |g_i - 1mg_i|^2 f \, d\mu$$

$$\geq \sum_{i=1}^n |g_i - 1mg_i|^2 f \, d\mu$$

$$\geq \sum_{i=1}^n \left| \int (g_i - 1mg_i) f \, d\mu \right| \geq \epsilon^2.$$ 

Hence $u \geq \epsilon^2$ a.e. and hence $u$ is invertible in $L_\infty$. This implies that $u$ is invertible in $\mathcal{G}$. But this is impossible since $mu = 0$.

**Corollary.** $jL_1$ is dense in $\mathcal{G}^*$ in the $w^*$-topology.

**Theorem 5.** The Gelfand mapping $\sigma: \mathcal{G} \to \mathcal{C}(\mathcal{M})$ is positive.

**Proof.** If $g \geq 0$ a.e. then $g^{**} \geq 0$ on $jL_1^+$ where $g \to g^{**}$ is the canonical embedding of $\mathcal{G}$ into $\mathcal{G}^{**}$. Since $jL_1^+$ is dense in $\mathcal{M}$ and $g^{**}$ is continuous, $g^{**} \geq 0$ on $\mathcal{M}$. Hence $\sigma g = g^{**} |_{\mathcal{M}} \geq 0$. 

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Now we would like to extend $\sigma$ to $\mathcal{H}$. First note that, by the Riesz representation theorem, any $F \in \mathcal{B}^*$ can be represented by a measure $\mu_F$ on $(\mathcal{M}, \mathcal{B})$. In particular, let $\bar{\mu} = \mu_{11}$. From the order-preserving property of the Riesz representation one can see that for any $f \in L_1$, $\mu_{11}$ is absolutely continuous with respect to $\bar{\mu}$. In fact we can obtain $d\mu_{11}/d\bar{\mu}$ as follows. First, considering only $L_\infty$ functions we have

**Lemma 8.** If $f \in L_\infty$ then $\mu_{11} \ll \bar{\mu}$ and $d\mu_{11}/d\bar{\mu} = \sigma(f/1)$.

**Proof.** For any $g \in \mathcal{G}$,

$$\int_{\mathcal{M}} \sigma g \cdot \sigma f/1 \, d\bar{\mu} = \int_{\mathcal{M}} \sigma \{g \cdot \sigma f/1\} \, d\bar{\mu} = \int_{\mathcal{X}} \sigma \{g(f/1)\} \, d\mu$$

$$= \int_{\mathcal{X}} \sigma \{g(f/1)\} \, d\mu = \int_{\mathcal{X}} g f \, d\mu,$$

where the last equality follows from Theorem 4.

**Definition 9.** Let $T_f = \sigma f/1$, $f \in L_\infty$. Note that the linear mapping $f \mapsto \sigma f$ defines a positive contraction $L_\infty(\mathcal{M}, \mathcal{B}, \mu) \to L_\infty(\mathcal{M}, \mathcal{B}, \bar{\mu})$. But it is also a contraction for the corresponding $L_1$ norms; hence it is a contraction for all $L_p$ norms, $1 \leq p \leq \infty$. We can then extend this mapping to $L_p(\mathcal{M}, \mathcal{B}, \mu) \to L_p(\mathcal{M}, \mathcal{B}, \bar{\mu})$ with the property that $\int_{\mathcal{M}} g f \, d\mu = \int_{\mathcal{M}} \sigma g f \, d\bar{\mu}$ for all $g \in \mathcal{G}, f \in L_p$.

We can now prove a representation theorem for $\mathcal{H}$.

**Theorem 6.** There is a positive isometric $*$-isomorphism between $\mathcal{H}$ and $L_\infty(\mathcal{M}, \mathcal{B}, \mu)$.

**Proof.** Let $h \in \mathcal{H}$ and define $\phi_h \in \mathcal{B}^*$ by

$$\phi_h(g) = \int_{\mathcal{X}} \pi(gh) \, d\mu.$$ 

Note that if $h \in \mathcal{B}$ then $\phi_h$ is represented by the measure $\sigma(h) \cdot d\bar{\mu}$ on $\mathcal{M}$. Let $\gamma_h$ be the representing measure of $\phi_h$, $h \in \mathcal{H}$. Then, for any nonnegative continuous function $\sigma g (g \in \mathcal{G}^+)$ on $\mathcal{M}$

$$\left| \int_{\mathcal{M}} \sigma g \, d\gamma_h \right| = \left| \int_{\mathcal{X}} \pi(gh) \, d\mu \right| \leq \|h\|_\infty \int_{\mathcal{X}} g \, d\mu = \|h\|_\infty \int_{\mathcal{M}} \sigma g \, d\bar{\mu},$$

which shows that $\gamma_h$ is absolutely continuous with respect to $\bar{\mu}$ and has a density function bounded by $\|h\|_\infty$. We denote this density function by $\sigma_h$, noting that it is actually an extension of $\sigma$, and $\sigma_h \leq \|h\|_\infty$. Furthermore, if $l \in L_\infty$ then

$$\int_{\mathcal{M}} \sigma(h \cdot l) \, d\bar{\mu} = \int_{\mathcal{M}} \sigma(h) \sigma \pi(l) \, d\mu = \int_{\mathcal{X}} \pi(h \cdot \pi(l)) \, d\mu$$

$$= \int_{\mathcal{X}} \pi(l) \, d\mu = \int_{\mathcal{X}} hl \, d\mu.$$
Hence \( \left| \int_X hl \, d\mu \right| \leq \|oh\|_\infty \|l\|_1 \leq \|oh\|_\infty \cdot \|l\|_1 \), so \( \|h\|_\infty \leq \|oh\|_\infty \). Thus the extended \( \sigma \) is also an \( L_\infty \)-norm isometry. To show that \( \sigma \mathcal{H} = L_\sigma(\mathcal{M}, \mathcal{B}, \hat{\mu}) \), first note that, if \( h \in \mathcal{H}, l \in L_\sigma(X, \mathcal{F}, \mu) \) then

\[
\left| \int_X hl \, d\mu \right| = \left| \int_\mathcal{M} \sigma(h) \tau(l) \, d\hat{\mu} \right| \leq \|oh\|_1 \|l\|_\infty \leq \|oh\|_1 \|l\|_\infty,
\]

hence \( \|h\|_1 \leq \|oh\|_1 \). Thus \( \sigma^{-1} : \mathcal{H} \rightarrow \mathcal{H} \) is an \( L_1 \)-contraction onto \( \mathcal{H} \). Now if \( oh_n \) is an a.e. monotone sequence in \( \mathcal{H} \) converging a.e. to a function \( l \) in \( L_\sigma(\mathcal{M}, \mathcal{B}, \hat{\mu}) \) then \( h_n \) is an a.e. bounded and monotone sequence in \( \mathcal{H} \). If the limit function is \( g \), one can easily see that \( g \in \mathcal{H} \) and \( \sigma g = l \). Since \( \mathcal{H} \) contains the continuous functions, this shows that \( \sigma \mathcal{H} = L_\sigma(\mathcal{M}, \mathcal{B}, \hat{\mu}) \). Now we want to show that

\[
\int_X \pi(hf) \, d\mu = \int_\mathcal{M} \sigma(h) \sigma(f) \, d\hat{\mu},
\]

for all \( h, f \in \mathcal{H} \). In fact, for a fixed \( h \in \mathcal{H} \), let \( \mathcal{N} \subset \mathcal{H} \) be the class of functions \( f \) for which this relation holds. Then \( \sigma \mathcal{N} \) contains the continuous functions of \( \mathcal{M} \), and one can show, as before, that \( \sigma \mathcal{N} \) is closed under a.e. monotone limits. Hence \( \sigma \mathcal{N} = L_\sigma(\mathcal{M}, \mathcal{B}, \hat{\mu}) \).

Finally, we show that extended \( \sigma \) is multiplicative, i.e. \( \sigma(h_1) \cdot \sigma(h_2) = \sigma(\pi(h_1 h_2)) \) for all \( h_1, h_2 \in \mathcal{H} \). First note that if \( f \in L_\sigma(\mathcal{M}, \mathcal{F}, \hat{\mu}) \) and \( \int_\mathcal{M} \sigma(f) \, d\hat{\mu} = 0 \) for all \( l \in L_\sigma(X, \mathcal{F}, \mu) \) then \( \sigma^{-1} f = 0 \), hence \( f = 0 \). Now for \( h \in \mathcal{H}, g \in \mathcal{G}, l \in L_\sigma(X, \mathcal{F}, \mu) \),

\[
\int_\mathcal{M} \sigma(h) \sigma(g) \tau(l) \, d\hat{\mu} = \int_\mathcal{M} \sigma(h) \sigma(g) \sigma(\pi(l/1)) \, d\hat{\mu}
\]

\[
= \int_\mathcal{M} \sigma(h) \sigma(g) \pi(l/1) \, d\hat{\mu} = \int_X \pi(h\pi(g(l/1))) \, d\mu
\]

\[
= \int_X \pi(hg(l/1)) \, d\mu = \int_X \pi(hg) \, d\mu = \int_\mathcal{M} \sigma(hg) \tau(l) \, d\hat{\mu},
\]

hence \( \sigma(h) \cdot \sigma(g) = \sigma(hg) \).

Now suppose that \( h_1, h_2 \in \mathcal{H} \), \( l \in L_\sigma(X, \mathcal{F}, \mu) \). Then

\[
\int_\mathcal{M} \sigma(h_1) \sigma(h_2) \tau(l) \, d\hat{\mu} = \int_\mathcal{M} \sigma(h_1) \sigma(h_2) \pi(l/1) \, d\hat{\mu}
\]

\[
= \int_X \pi(h_1 \pi(h_2 \pi(l/1))) \, d\mu = \int_X \pi(h_1 h_2) l \, d\mu
\]

\[
= \int_\mathcal{M} \sigma(h_1 h_2) \tau(l) \, d\hat{\mu}
\]

which shows that \( \sigma(h_1) \sigma(h_2) = \sigma(h_1 h_2) \), and completes the proof of the theorem.

We remark that every \( f \in L_p(\mathcal{M}, \mathcal{B}, \hat{\mu}), 1 \leq p < \infty \), induces a function \( h \in L_p(X, \mathcal{F}, \mu) \), defined by \( \int_X hl \, d\mu = \int_\mathcal{M} \sigma(f) \, d\hat{\mu} \) for all \( l \in L_\sigma(X, \mathcal{F}, \mu) \), \( 1/p + 1/q = 1 \). Since \( \tau \) is an \( L_\sigma \)-contraction the integral on \( \mathcal{M} \) is defined and \( h \) satisfies \( \int_X hl \, d\mu = \).
\[ \int_X hT_l \, d\mu, \text{ for all } l \in L_0(X, \mathcal{F}, \mu). \] The case \( p=1 \) causes no difficulty. If \( f \in L_1(\mathcal{M}, \mathcal{B}, \tilde{\mu}), l \in L_\infty(X, \mathcal{F}, \mu), \)

\[
\left| \int_{(l)} f(\tau(l)) \, d\tilde{\mu} \right| \leq \left| \int_{(l) \geq n} f(\tau(l)) \, d\tilde{\mu} \right| + \left| \int_{(l) < n} f(\tau(l)) \, d\tilde{\mu} \right| \\
\leq \| f \|_\infty \int_{(l) \geq n} |f| \, d\tilde{\mu} \quad + \quad n \| f \|_1 \\
\leq \| f \|_\infty \left( \int_{(l) \geq n} |f| \, d\tilde{\mu} \right) + n \| f \|_1.
\]

Thus, if \( l_k \) is a sequence in \( L_\infty \) with \( \| l_k \|_1 \to 0 \) and \( \| l_k \|_\infty \leq K \) then

\[
\lim_k \left| \int f \tau(l_k) \right| \, d\tilde{\mu} \leq K \int_{(l) \geq n} |f| \, d\tilde{\mu} \quad \text{for all } n \geq 1.
\]

Hence this limit is zero and the functional \( l \to \int f \tau(l) \, d\tilde{\mu} \) on \( L \) is induced by an \( L_1 \)-function \( h \). In a similar way, any Baire measure on \((\mathcal{M}, \mathcal{B})\) induces what one might call "an invariant functional" on \( L_\infty(X, \mathcal{F}, \mu) \).

We also note the following relation between the maximal ideal spaces of \( \mathcal{H} \) and \( \mathcal{B} \); that is, between the Feller and Martin boundaries (cf. [9]). Since \( \mathcal{H} \) is isometrically isomorphic to \( L_\infty(\mathcal{M}, \mathcal{B}, \mu) \), we state this relation in the following familiar form:

**Lemma 9.** Let \( \mathcal{M} \) be a compact Hausdorff space, \( \mathcal{B} \) its Baire sets, and \( \tilde{\mu} \) a Baire measure on \((\mathcal{M}, \mathcal{B})\) with support \( \mathcal{M} \). Let \( \mathcal{M}' \) be the maximal ideal space of the \( C^* \)-algebra \( L_\infty(\mathcal{M}, \mathcal{B}, \tilde{\mu}) \). Then there is a continuous and onto map \( \rho: \mathcal{M}' \to \mathcal{M} \).

**Proof.** Interpret \( \mathcal{M}' \) and \( \mathcal{M} \) as classes of homomorphisms and define \( \rho: \mathcal{M}' \to \mathcal{M} \) by \( \rho(\phi) = \phi|_{\mathcal{H}(\mathcal{M})} \). Then \( \rho \) is continuous. We show it is onto. Let \( m \in \mathcal{M} \), and consider the ideal generated by \( m \cdot L_\infty(\mathcal{M}, \mathcal{B}, \tilde{\mu}) \). If it is proper, it can be embedded in a maximal ideal, whose image must then be \( m \) under \( \rho \). We show it is proper. If not, then \( 1 = \sum f_i g_i \) where \( f_i \in m, g_i \in L_\infty(\mathcal{M}, \mathcal{B}, \tilde{\mu}) \). Since \( m \) is a maximal ideal, \( \exists x_0 \) such that \( f_i(x_0) = 0 \) for all \( i = 1, \ldots, n \). Hence \( |f_i| \leq \epsilon \sup |g_i| \) on some neighborhood \( U \) of \( x_0 \), such that \( \mu(U) \neq 0 \). Hence \( 1 = \sum f_i g_i \leq \epsilon \) on \( U \), which is a contradiction.

**Corollary.** \( \mathcal{M} \) is homeomorphic to the quotient space \( \mathcal{M}'/\rho \).

We finish this section by considering the possibility of joining \( X \) and \( \mathcal{M} \). In general, this cannot be done. If, however, \( T \) is induced by a Markov kernel, such that the transform of any point measure is absolutely continuous with respect to \( \mu \), then the members of \( \mathcal{H} \) can be considered as actual functions on \( X \), and the evaluations of these functions at points of \( X \) induce bounded linear functionals on \( \mathcal{H} \). Hence \( X \) can be embedded in \( \mathcal{B}^* \) (possibly in a many to one fashion). We shall denote the image of \( X \) under this mapping as \( X \) also. Hence \( X \subseteq j(L_1^+(X, \mathcal{F}, \mu)) \). Using the method of Lemma 7, \( X \) is dense in \( \mathcal{M} \), in the \( w^* \)-topology of \( \mathcal{B}^* \).
Let $\overline{X}$ be the $w^*$-closure of $X$ in $\mathcal{B}$. Then $\overline{X}$ is a compact Hausdorff space. The following result, stated for the Martin-Doob boundary, is also true for the Feller boundary.

**Theorem 7.** For any $g \in L_1(X, \mathcal{F}, \mu)$, $T^*g \, d\mu \to \tau(g) \, d\bar{\mu}$ in the $w^*$-topology of Baire measures on $\overline{X}$.

**Proof.** Let $\mathcal{A}_1 = \text{the sub-C*-algebra of } \mathcal{A}$, consisting of functions $g' \in \mathcal{A}$ such that $\pi(g') \in \mathcal{G}$.

Let $\mathcal{G} = \{ f \in \mathcal{G}(X) \mid |f| \in \mathcal{A}_1 \}$, $\mathcal{G}_0 = \{ f \in \mathcal{G}(X) \mid |f|_X \leq 0 \}$. By the Stone-Weierstrass theorem $\mathcal{G} = \mathcal{G}(X)$. Also, $\mathcal{G}_0$ is a closed ideal in $\mathcal{G}$. Let $\mathcal{N} \subset X$ be the closed subset such that $\mathcal{G}_0 = \{ f \in \mathcal{G} \mid f(\mathcal{N}) = 0 \}$. Then we have $\mathcal{G}(\mathcal{N}) \cong \mathcal{G}(X)/\mathcal{G}_0 \cong \mathcal{A}/\mathcal{A}_0 \cong \mathcal{A} \cong \mathcal{M}$.

Hence $\mathcal{G}(\mathcal{N}) \cong \mathcal{M}$ is induced by a homeomorphism $\phi : \mathcal{N} \to \mathcal{M}$. Hence $g(s) = g(\phi(s))$ under the above sequence of isomorphisms. But $\mathcal{G}$ separates the points of $\mathcal{B}$, so $\phi$ = identity and $\mathcal{N} = \mathcal{M}$.

In other words,

$$\{ f \in \mathcal{G}(X) \mid |f|_X \in \mathcal{A}_0 \} = \{ f \in \mathcal{G}(X) \mid f(\mathcal{M}) = 0 \}.$$ 

Thus if $f \in \mathcal{G}(X)$, $g \in L_1(\overline{X}, \mathcal{F}, \mu)$, then:

$$\int_X fT^*g \, d\mu = \int_X U^*(f|_X)g \, d\mu \to \int_X \pi(f|_X)g \, d\mu = \int_{\mathcal{M}} \sigma\pi(f|_X)\tau(g) \, d\bar{\mu}$$

and

$$\int_{\mathcal{M}} \sigma\pi(f|_X) \cdot \tau(g) \, d\bar{\mu} = \int_{\mathcal{M}} f|_{\mathcal{M}} \cdot \tau(g) \, d\bar{\mu} = \int_X f \tau(g) \, d\bar{\mu}.$$ 

Thus $T^*g \, d\mu \to \tau(g) \, d\bar{\mu}$.

6. **Harmonic functions in the unit disk.** As an example we consider a transformation suggested by Feller in [10].

Let $D = \{ z = re^{i\phi} \mid 0 \leq r < 1, -\pi \leq \phi \leq \pi \}$ be the unit disk with the (geometric) boundary $C$. Let $\mathcal{F}$ and $\mu$ be the $\sigma$-field of Borel subsets and the Lebesgue measure. For every $z \in D$, $E \in \mathcal{F}$, let

$$P(z, E) = \mu(Q_z \cap E)/\mu(Q_z)$$

where $Q_z = \{ Z \mid |Z - z| < 1 - |z| \}$. Then $P$ defines a Markov kernel, such that the transformation of a unit mass at $z \in D$ is given by the measure $P(z, \cdot) \ll \mu$. We let $T$ be the induced transformation on $L_1(D, \mathcal{F}, \mu)$. The adjoint $U$ of $T$ is given by

$$(Uf)(z) = \int f(Z) P(z, dZ), \quad f \in L_\infty, z \in D.$$
It is clear that any bounded harmonic function $h$ belongs to $\mathcal{H}$. The converse is also true, but it seems that no explicit proof of it has been given and we would like to indicate an outline for this proof.

If $R$ is a Borel subset of $[0, 1)$ let $C_R = \{ z \mid |z| \in R \}$. One can then obtain the following

**Lemma 10.** Let $\frac{1}{2} \leq K < 1$ and $R$ be a Borel subset of $[K, 1)$. Then for all $z \in D$, $\frac{1}{2} \leq |z| \leq K$,

$$\frac{\mu(Q_z \cap C_R)}{\mu(Q_z \cap C_{(K,1)})} \geq \frac{1}{16} \left[ \frac{\lambda(R)}{1-K} \right]^{3/2}$$

where $\lambda$ is the one dimensional Lebesgue measure.

**Corollary.** Let $E = C_{(0,1/2)} \cup [K, 1)$ and let $f \in L^1_+, f=0$ a.e. on $C_{[K,1)}$. Then

$$\int_{C_R} T^*_n f \, d\mu \geq \frac{1}{16} \left[ \frac{\lambda(R)}{1-K} \right]^{3/2} \int_{C_{(K,1)}} T^*_n f \, d\mu$$

for all $n \geq 0$.

Using this corollary one can see that if a function $h \in \mathcal{H}$ (which is necessarily continuous) has the form $h(re^{in\theta}) = f(r)g(\phi)$ then $\lim_{n \to \infty} f(r)$ exists, and that this implies the harmonicity of $h$.

Now if $h$ is any function in $\mathcal{H}$, let $t$ be an irrational number and consider, for a fixed $n$, $-\infty < n < \infty$,

$$\lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} (\tau^k \cdot z)^{-n} h(\tau^k \cdot z) = F_n$$

where $\tau : D \to D$ is given by $\tau z = e^{2\pi it}$. This limit $F_n$ exists for all nonzero $z \in D$, depends only on $r = |z|$, and satisfies

$$r^n F_n(r) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{-in\phi} h(re^{in\phi}) \, d\phi.$$  

But, it is clear that

$$e^{in\phi} r^n F_n(r) = \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} e^{-i2\pi nk\phi} h(re^{i2\pi nk\phi})$$

is a function in $\mathcal{H}$, hence $e^{in\phi} r^n F_n(r)$ must be harmonic, which shows that $r^n F_n(r) = C r^{2n|\phi|}$ and completes the proof of the following

**Lemma 11.** A bounded function belongs to $\mathcal{H}$ if and only if it is a harmonic function.

One then shows that the $C^*$-algebra $\mathcal{H}$ is isometrically *-isomorphic to $L_\infty$ of the unit circle. For any bounded measurable function $\lambda$ on $D$, let $\lambda$ be the measure on the unit circle obtained by sweeping out $l \, d\mu$ by the Poisson kernel. The harmonic function $\pi(l/1)$ corresponds to $d\lambda_l/d\lambda$, which is continuous. It then follows that the maximal ideal space $\mathcal{M}$ of $\mathcal{G}$ is homeomorphic to the unit circle. Since $T$ is induced by a Markov kernel, $D$ can be imbedded into $\mathcal{G}$. Then $D \cup \mathcal{M}$ is homeomorphic to the closed unit disk.
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