THE $C^k$-CLASSIFICATION OF CERTAIN OPERATORS IN $L_p(1)$

BY

SHMUEL KANTOROVITZ

Introduction. We investigate in this paper the one-parameter family of operators

$$T_a = M + \alpha J$$

acting in $L_p(0, 1)$ ($1 \leq p < \infty$), where $\alpha \in C$ (the complex field), $M: f(x) \to xf(x)$ and $J: f(x) \to \int_0^x f(t) \, dt$.

Extending a result of Sakhnovich [8] from the case $p=2$ to the case $1 < p < \infty$, Kalisch [4] established recently that $T_a$ is similar to $M$ if $\Re \alpha = 0$.

The situation becomes quite different if $\Re \alpha \neq 0$; thus, $T_a$ is not similar to $M$ if $|\Re \alpha| \geq 1$, and more generally, $T_a$ is not similar to $T_\beta$ if $|\Re \alpha| \neq |\Re \beta|$ (Proposition 13). This result will follow from our discussion of the $C^k$-operational calculus (both "global" and "local", in the sense of [5], [6], [7]) for the operators $T_a$. We recall briefly the terminology, restricting ourselves to bounded operators $T: X \to X$ (a Banach space) with real spectrum $\sigma(T)$.

Fix a compact interval $\Delta \subseteq \mathbb{R}$ (the real line) which contains $\sigma(T)$. For $n = 0, 1, 2, \ldots$, let $C^n(\Delta)$ denote the Banach algebra of all complex valued functions of class $C^n$ on $\Delta$, with the norm

$$\|\varphi\|_{n, \Delta} = \sum_{j=0}^n \sup_{\Delta} |\varphi^{(j)}|.$$ (We shall write $\|\varphi\|_n$ when $\Delta = [0, 1]$.) We say that $T$ is of class $C^n$ (and we write $T \in (C^n)$ if there exists a continuous representation $\varphi \mapsto T(\varphi)$ of $C^n(\Delta)$ on $X$ such that $T(1) = I$ (the identity operator) for $\varphi(t) \equiv 1$ and $T(\varphi) = T$ for $\varphi(t) \equiv t$. Such a representation is unique when it exists, and is called the $C^n$-operational calculus for $T$. For example, $T_0 = M$ is of class $C (= C^0)$, and its $C$-operational calculus is $\varphi \mapsto M(\varphi)$, where

$$M(\varphi): f(x) \to \varphi(x)f(x), f \in L_p(0, 1).$$

If $W \subseteq X$ is a linear manifold, we denote by $T(W)$ the algebra of all linear transformations of $X$ with domain $W$ and range contained in $W$. If $W$ is invariant for $T$, a $C^n$-operational calculus for $T$ on $W$ is an algebra homomorphism $\varphi \mapsto T(\varphi)$ of $C^n(\Delta)$ into $T(W)$ with the following properties:

(i) $T(1) = I/W$ for $\varphi(t) \equiv 1$;

(ii) $T(\varphi) = T/W$ for $\varphi(t) \equiv t$;

(iii) for each $x \in W$, the mapping $\varphi \mapsto T(\varphi)x$ of $C^n(\Delta)$ into $X$ is continuous.

Received by the editors March 9, 1967.

(1) This research was partially supported by NSF-GP-5493.

323
For each \( n = 0, 1, 2, \ldots \), there exists a maximal invariant linear manifold \( W_n(T) \) on which \( T \) admits a \( C^n \)-operational calculus, and the latter is uniquely determined on \( W_n(T) \). In fact, \( W_n(T) \) is the set of all \( x \in X \) for which

\[
|\langle x, \cdot \rangle| = \sup \{ \| \varphi(T)x \| : \varphi \text{ a polynomial with } \| \varphi \|_{n,A} \leq 1 \} \text{ is finite.}
\]

The “semisimplicity manifold” \( W_0(T) \) is particularly important (cf. Theorem 2.1 in [6]); it contains trivially the eigenvectors of \( T \). We shall write \( W_n(T_\alpha; p) \) instead of \( W_n(T_\alpha) \) when it will be necessary to specify the \( L_p \) space under consideration.

We may now describe the main results of this paper (for \( 1 < p < \infty \)):

1°. \( T_\alpha \) is of class \( C^n \) if \( |\Re \alpha| \leq n \) and only if \( |\Re \alpha| < n + 1 \) (Theorem 6); the \( C^n \)-operational calculus for \( T_\alpha \) (\( |\Re \alpha| \leq n \)) is given in Theorems 8 and 9.

2°. The \( W_k \) manifolds of \( T_\alpha \) are dense for \( \Re \alpha < 0 \) and trivial for \( \Re \alpha \geq 1 \) and \( k < \lfloor \Re \alpha \rfloor \) (Theorems 10 and 11).

3°. \( T_\alpha \) is not spectral (in Dunford’s sense) for \( |\Re \alpha| \geq 1 \) (it is of course spectral for \( \Re \alpha = 0 \), by the Kalish-Sakhnović result).

1. **Five lemmas.** Let \( \{ T^\zeta ; \Re \zeta > 0 \} \) be the Riemann-Liouville holomorphic semigroup, acting in \( L_p(0, 1) \) (\( 1 \leq p < \infty \)):

\[
(J^\zeta f)(x) = (1/\Gamma(\zeta)) \int_0^x (x-t)^{\zeta-1} f(t) \, dt
\]

(\( f \in L_p(0, 1), x \in [0, 1], \Re \zeta > 0 \)). It is known (cf. [3] for \( p = 2 \), and [4] for \( 1 < p < \infty \)) that if \( 1 < p < \infty \), the semigroup \( \{ J^\zeta \} \) admits a strongly continuous boundary group \( \{ J^\gamma \}; \gamma \in \mathbb{R} \) of bounded operators, and

\[
\| J^\gamma \| \leq e^{\pi|\gamma|/2} \quad (\gamma \in \mathbb{R})
\]

(see also the estimates at the end of Kalisch’s paper [4]). The operator \( J^\zeta (\Re \zeta > 0) \) is one-to-one in \( L_p(0, 1) (p \geq 1) \); its inverse, with domain \( \mathcal{D}_{-\zeta} = \mathcal{R}(J^\zeta) \) (the range of \( J^\zeta \)), is a closed operator, which we denote by \( J^{-\zeta} \).

For \( 1 < p < \infty \), note that \( \mathcal{R}(J^{\beta+\iota \gamma}) = \mathcal{R}(J^\beta) \) (\( \beta > 0, \gamma \in \mathbb{R} \)), since \( J^{\beta+\iota \gamma} = J^\beta J^\iota \gamma \) and \( J^\iota \gamma \) is nonsingular.

For \( p = 1 \) and \( \gamma \in \mathbb{R} \), we define \( J^\iota \gamma \) as follows. Its domain is \( \mathcal{D}_{\iota \gamma} = U(\mathcal{R}(J^\zeta); \Re \zeta > 0) \); if \( f \in \mathcal{D}_{\iota \gamma} \), say \( f = J^\iota \gamma h \) for some \( h \in L_1(0, 1) \) and \( \zeta \in \mathbb{C} \) with \( \Re \zeta > 0 \), then \( J^\iota \gamma f = J^{\zeta+\iota \gamma} h \). One verifies easily that \( J^\iota \gamma \) is well defined.

We shall use the notation \( \mathcal{D}_\zeta \) also for \( \Re \zeta > 0 \), in which case \( \mathcal{D}_\zeta = L_\infty(0, 1) \) (the domain of \( J^\zeta \)); similarly \( \mathcal{D}_{\iota \gamma} = L_p(0, 1) \) for \( 1 < p < \infty \).

**Lemma 1.** For any \( \zeta \in \mathbb{C} \), \( \mathcal{D}_\zeta \) is invariant under \( M \), and the following (trivially equivalent) identities are valid on \( \mathcal{D}_\zeta \):

1. \( MJ^\zeta = J^\zeta M = \zeta J^{\zeta+1} \),
2. \( J^\zeta M = T_\zeta J^\zeta \),
3. \( MJ^\zeta = J^\iota \zeta T_\zeta \).
Proof. If Re $\zeta > 0$ (Re $\zeta \geq 0$ if $1 < p < \infty$), the first statement is trivial, since $\mathcal{D}_\zeta = L_p(0, 1)$. For $g \in L_p(0, 1)$ and Re $\zeta > 0$,

$$(MJ^\zeta - J^\zeta M)g(x) = \Gamma(\zeta)^{-1}\int_0^\infty (x-t)^{\zeta-1}(x-t) g(t) \, dt$$

$$= \Gamma(\zeta)^{-1}\Gamma(\zeta+1)J^{\zeta+1}g(x) = \zeta J^{\zeta+1}g(x).$$

This proves the first, and hence all three identities of the lemma (for $1 \leq p < \infty$ and Re $\zeta > 0$). By (3), $M\mathcal{R}(J^\zeta) \subset \mathcal{R}(J^\zeta)$, i.e. $\mathcal{D}_{-\zeta}$ is invariant under $M$. Apply $J^{-\zeta}$ to both sides of (3):

$$J^{-\zeta}MJ^\zeta = T_\zeta,$$

i.e., $J^{-\zeta}M = T_\zeta J^{-\zeta}$ on $\mathcal{D}_{-\zeta}$. This proves (2) (and hence the lemma) for Re $\zeta < 0$.

Let $\mathcal{D} = U(\mathcal{D}_\zeta; \text{Re } \zeta < 0)$. $\mathcal{D}$ is $M$-invariant, and dense in $L_p(0, 1)$ ($1 \leq p < \infty$). Let $g \in \mathcal{D}$; then $g = J_\gamma h$ for some $\zeta \in \mathbb{C}$ with Re $\zeta > 0$ and some $h \in L_p(0, 1)$. Using (3) twice (for Re $\zeta > 0$), we obtain (for $\gamma \in \mathbb{R}$):

$$MJ_\gamma g = MJ_\gamma J_\gamma h = J_\gamma M_J h = J_\gamma (M_J + i\gamma J^{\gamma+1})h$$

$$= J_\gamma (M + i\gamma J)J_\gamma h = J_\gamma T_\gamma g.$$

If $p = 1$, this finishes the proof of the lemma for Re $\zeta = 0$. If $1 < p < \infty$, this shows that (3) is valid on the dense subset $\mathcal{D}$ of $L_p(0, 1)$ (for $\zeta = i\gamma$); since $J_\gamma$ is a bounded operator, (3) is valid everywhere on $L_p(0, 1)$.

Lemma 2. For $\beta, \gamma \in \mathbb{R}$ arbitrary, the operators $T_{\beta+i\gamma}$ and $T_\beta$ acting in $L_p(0, 1)$ ($1 < p < \infty$) are similar, with $J_\gamma$ implementing the similarity:

$$J_\gamma T_{\beta+i\gamma} J^{-\gamma} = T_\beta,$$

(for $\beta = 0$ and $p = 2$, this is due to Sakhnovic [8]).

Proof. By (3),

$$J_\gamma T_{\beta+i\gamma} J^{-\gamma} = J_\gamma (T_\gamma + i\gamma J) J^{-\gamma} = M + \beta J = T_\beta.$$
Remark. (1) $(C^{-1}) = \Phi$ by convention. (2) $T_n(\varphi)$ is well defined, since $\mathcal{R}(J^n)$ is a $C^n[0, 1]$-module.

Proof. By Leibnitz' formula,

$$T_n(\varphi) = \sum_{j=0}^{n} \binom{n}{j} M(\varphi^{(j)}) J^j, \quad \varphi \in C^n[0, 1].$$

Thus $T_n(\varphi)$ is a bounded operator on $L_p(0, 1)$. In fact, since $\|J^j\| \leq 1/j!$ (cf. [3, p. 664]), we have

$$\|T_n(\varphi)\| \leq \left( \binom{n}{[n/2]} \right) \|\varphi\|, \quad \varphi \in C^n[0, 1].$$

The map $\varphi \rightarrow T_n(\varphi)$ is clearly an algebra homomorphism of $C^n[0, 1]$ into the bounded operators on $L_p(0, 1)$, which is continuous by (6). If $\varphi(x) \equiv 1$, $T_n(\varphi)$ is trivially the identity operator. If $\varphi(x) \equiv x$, $T_n(\varphi) = T_n$ by (3). Thus, $T_n \in (C^n)$ and its $C^n$-operational calculus is given by (4). Finally, apply (5) to the functions

$$\varphi_t(x) = e^{itx} \quad (x, t \in R).$$

Thus

$$(e^{itT_n} g)(x) = \sum_{j=0}^{n} \binom{n}{j} e^{itx}(J^j g)(x),$$

and consequently $\|e^{itT_n}\| \neq O(|t|^{n-1}) (n = 1, 2, \ldots)$. Therefore $T_n \notin (C^{n-1}) (n \geq 1)$ by Lemma 2.11 in [5].

Lemma 4. Let $n \geq 0$ be an integer. Then, for $\varphi$ of class $C^n$ and $g \in \mathcal{R}(J^n)$ (or vice versa, or for both $\varphi$ and $g$ in $\mathcal{R}(J^n)$), the following identity is valid:

$$J^n(\varphi J^{-n} g) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j J^j (g J^{-j} \varphi).$$

Remark. Here, as usual, $\mathcal{R}(J^n)$ denotes the range of $J^n$ in $L_p(0, 1)$, with $p$ arbitrary ($1 \leq p < \infty$). Note the analogy with Leibnitz' formula (in fact, the latter may be used to prove the lemma).

Proof. We use induction on $n$. The lemma is trivial for $n = 0$. Suppose it is true for $n = k$. Let $\varphi$ and $g$ be as required for $n = k + 1$. Write $g' = J^{-1} g$. Then $\varphi$ and $g'$ satisfy the hypothesis for $n = k$. Therefore

$$J^{k+1}(\varphi J^{-k-1} g) = JJ^k(\varphi J^{-k} g')$$

$$= \sum_{j=0}^{k} (-1)^j \binom{k}{j} J^j (g' J^{-j} \varphi).$$

An integration by parts shows that

$$J(g' J^{-1} \varphi) = g J^{-1} \varphi - J(g J^{-1} \varphi).$$
Thus
\[ J^{k+1}(fJ^{-(k+1)}g) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} [J^j(gJ^{-j}f) - J^{j+1}(gJ^{-(j+1)}f)] \]
\[ = g\varphi + \sum_{j=1}^{k} (-1)^j \left[ \binom{k}{j} + \binom{k}{j-1} \right] J^j(gJ^{-j}\varphi) \]
\[ + (-1)^{k+1} J^{k+1}(gJ^{-(k+1)}f). \]

Since
\[ \binom{k}{j} + \binom{k}{j-1} = \binom{k+1}{j}, \]
we obtain the correct identity for \( n = k + 1 \), Q.E.D.

**Lemma 5.** For any integer \( n \geq 0 \), and for \( 1 \leq p < \infty \), the operator \( T_{-n} \) acting in \( L_p(0, 1) \) belongs to \( (C^n) - (C^{n-1}) \), and the \( C^n \)-operational calculus for \( T_{-n} \) is given by
\[ T_{-n}(\varphi) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j M(\varphi^{(j)}), \quad \varphi \in C^n[0, 1]. \]

**Proof.** The map \( \varphi \rightarrow T_{-n}(\varphi) \) of \( C^n[0, 1] \) into the bounded operators on \( L_p(0, 1) \) is clearly linear and continuous, in fact,
\[ \|T_{-n}(\varphi)\| \leq \binom{n}{\lfloor n/2 \rfloor} \|\varphi\|_n. \]

For \( g \in \mathcal{H}(J^n) \), we have by Lemma 4:
\[ T_{-n}(\varphi)g = J^n(\varphi J^{-n}g). \]
Therefore \( \mathcal{H}(J^n) \) is invariant under \( T_{-n}(\varphi) \) (for all \( \varphi \in C^n[0, 1] \)) and \( T_{-n}(\varphi \psi) = T_{-n}(\varphi)T_{-n}(\psi) \) on \( \mathcal{H}(J^n) \) (for all \( \varphi, \psi \in C^n[0, 1] \)). Since \( \mathcal{H}(J^n) \) is dense in \( L_p(0, 1) \) and \( T_{-n}(\varphi) \) is continuous, it follows that \( T_{-n}(\cdot) \) is multiplicative on \( C^n[0, 1] \). The relations \( T_{-n}(\varphi) = I(T_{-n})(\varphi(x) = 1 \equiv x) \) are trivial on \( \mathcal{H}(J^n) \) by (8) and Lemma 1; by density, they are true throughout \( L_p(0, 1) \).

Finally, one verifies that \( T_{-n} \notin (C^{n-1}) \) just as in Lemma 3.

2. **Global classification** (\( 1 < p < \infty \)).

**Theorem 6.** The operator \( T_\alpha \) acting in \( L_p(0, 1) \) (\( 1 < p < \infty \)) is of class \( C^n \) (\( n = 0, 1, 2, \ldots \)) if \( |\text{Re} \\alpha| \leq n \) and only if \( |\text{Re} \\alpha| < n + 1 \).

In other words, \( T_\alpha \) is of class \( C^n \) in the strip \( |\text{Re} \\alpha| \leq n \) and is not of class \( C^n \) outside the strip \( |\text{Re} \\alpha| < n + 1 \).

The theorem is an immediate corollary of Lemmas 3 and 5, together with the following

**Lemma 7.** Suppose that, for some integer \( n \geq 0 \) and some \( \alpha_0 \in C \), the operator \( T_{\alpha_0} \) is of class \( C^n \) (when acting in \( L_p(0, 1) \), \( 1 < p < \infty \)). Then \( T_\alpha \) is of class \( C^n \) for all \( \alpha \) in the strip \( -n \leq \text{Re} \alpha \leq \text{Re} \alpha_0 \) if \( \text{Re} \alpha \leq 0 \) (Re \( \alpha_0 \leq \text{Re} \alpha \leq n \) if \( \text{Re} \alpha_0 \leq 0 \)).
Proof. To fix the ideas, suppose \( \Re \omega_0 = \beta_0 \geq 0 \). Write \( \alpha = \beta + i\gamma \) (\( \beta, \gamma \in \mathbb{R} \)). Fixing a polynomial \( \varphi \) and elements \( f \in L_\alpha(0, 1), \ g \in L_\varphi(0, 1) \) (\( p^{-1} + q^{-1} = 1 \)), we define

\[
\Phi(\alpha) = \langle e^{i\alpha} \varphi(T_\alpha) f, g \rangle, \quad \alpha \in \mathbb{C}.
\]

Since \( |e^{i\alpha^2}| \leq e^{i\beta^2} \), and since \( \varphi(T_\alpha) \) is a polynomial in \( \alpha \) (with operator coefficients), we have \( |\Phi(\alpha)| = O(e^{i\beta^2}) \) (for \( |\gamma| \to \infty \)) in the strip \( -n \leq \beta \leq \beta_0 \), for any \( \varepsilon > 0 \).

By Lemma 2 and the estimate \( \|J^\nu\| \leq e^{\pi \nu/2} \), we have:

\[
|\Phi(\beta + i\gamma)| \leq \exp \pi(\beta^2 + \gamma^2 + |\gamma|) \cdot \|f\|_p \cdot \|g\|_q \cdot \|\varphi(T_\beta)\|
\]

\[
\leq \exp \pi(\beta^2 + 1/4) \cdot \|f\|_p \cdot \|g\|_q \cdot \|\varphi(T_\beta)\|
\]

for all \( \beta, \gamma \in \mathbb{R} \).

Since \( T_{-n} \) and \( T_{\beta_0} \) are of class \( C^n \) (by Lemma 5, the hypothesis and Lemma 2), there exists a constant \( K \) (depending only on \( n, \beta_0 \) and \( p \)) such that

\[
\|\varphi(T_{-n})\| \leq K\|\varphi\|_n \quad \text{and} \quad \|\varphi(T_{\beta_0})\| \leq K\|\varphi\|_n.
\]

Hence

\[
|\Phi(-n + i\gamma)| \leq M \|f\|_p \cdot \|g\|_q \cdot \|\varphi\|_n
\]

and

\[
|\Phi(\beta_0 + i\gamma)| \leq M \|f\|_p \cdot \|g\|_q \cdot \|\varphi\|_n
\]

for all \( \gamma \), where \( M = K \exp \pi(\delta^2 + 1/4) \) and \( \delta = \max (n, \beta_0) \). By the Phragmèn-Lindelöf principle (cf. [9, p. 180]), it follows that \( |\Phi(\alpha)| \leq M\|f\|_p \cdot \|g\|_q \cdot \|\varphi\|_n \) for \( -n \leq \Re \alpha \leq \beta_0 \). Hence, for such \( \alpha \),

\[
\|\varphi(T_\alpha)\| \leq M \exp \pi(\gamma^2 - \beta^2)\|\varphi\|_n,
\]

and the lemma follows.

The next two theorems give explicitly the \( C^n \)-operational calculus for \( T_\alpha \) (\( |\Re \alpha| \leq n \)) acting in \( L_p(0, 1) \), with \( 1 < p < \infty \).

Theorem 8. Let \( n \) be a nonnegative integer. Then for \( 0 \leq \Re \alpha \leq n \) and \( \varphi \in C^n[0, 1] \) the range of \( J_\alpha \) (i.e., the domain of \( J^{-\alpha} \)) is invariant under \( M(\varphi) \), and the \( C^n \)-operational calculus for \( T_\alpha \) is given by

\[
T_\alpha(\varphi) = J^{-\alpha} M(\varphi) J_\alpha, \quad \varphi \in C^n[0, 1].
\]

Proof. By Lemma 1 (3),

\[
\varphi(M) J_\alpha = J_\alpha \varphi(T_\alpha)
\]

for any polynomial \( \varphi \). In particular,

\[
\varphi(M) \Re(J_\alpha) \subset \Re(J_\alpha), \quad \varphi = \text{a polynomial}.
\]
Let \( \varphi \in C^n[0,1] \), and choose polynomials \( \varphi_k \) which converge to \( \varphi \) in \( C^n[0,1] \). In particular, \( \varphi_k \to \varphi \) uniformly in \([0,1]\), and therefore

\[
D \varphi \ni \varphi_k \to J^\alpha g \to \varphi J^\alpha g
\]

in \( L_p(0,1) \), for any \( g \in L_p(0,1) \). By (9), we have:

\[
J^{-a}(\varphi_k J^\alpha g) = \varphi_k(T_a) = T_a(\varphi_k) \to T_a(\varphi)
\]

in the uniform operator topology, since \( T_a \in (C^n) \) by Theorem 6 (for \( |\text{Re } \alpha| \leq n \) and \( 1 < p < \infty \)) and \( \varphi_k \to \varphi \) in \( C^n[0,1] \). Since \( J^{-a} \) is a closed operator, it follows from (11) and (12) that \( \varphi J^\alpha g \in D \varphi \) and

\[
J^{-a}(\varphi J^\alpha g) = T_a(\varphi),
\]

Q.E.D.

We consider next the range \(-n \leq \text{Re } \alpha < 0 \) (\( n = 1, 2, \ldots \)). Note that \( \text{Re } (\alpha + n) \geq 0 \). The notation \( T_{-n} \) is that of Lemma 5.

**Theorem 9.** Let \( n \) be a nonnegative integer. Then for \(-n \leq \text{Re } \alpha < 0 \) and \( \varphi \in C^n[0,1] \), the range of \( J^{a+n} \) (i.e., \( D_{-n}(a+n) \)) is invariant under \( T_{-n}(\varphi) \), and the \( C^n \)-operational calculus for \( T_a \) is given by

\[
T_a(\varphi) = J^{-(\alpha+n)T_{-n}(\varphi)J^{a+n}, \quad \varphi \in C^n[0,1].
\]

**Proof.** By (10), \( \mathcal{R}(J^{a+n}) \) is invariant for \( M(\varphi) \) for any polynomial \( \varphi \); it is therefore invariant for the operator

\[
T_{-n}(\varphi) = \sum_{j=0}^{n} \binom{n}{j}(-1)^j J^j M(\varphi^{(j)}), \quad \varphi \text{ a polynomial}.
\]

Thus, for any polynomial \( \varphi \), the operator

\[
S_a(\varphi) = J^{-(\alpha+n)T_{-n}(\varphi)J^{a+n}}
\]

is everywhere defined. Being closed, it is continuous by the Closed Graph Theorem.

Let \( g \in D_a = \mathcal{R}(J^{-n}) \), say \( g = J^{-n} h \) with \( h \in L_p(0,1) \). By Lemma 1,

\[
S_a(\varphi) g = J^{-(\alpha+n)T_{-n}(\varphi)J^{n} h} = J^{-(\alpha+n)J^{n} \varphi(M) h}
= J^{-a} \varphi(M) h = \varphi(T_a) J^{-a} h
= \varphi(T_a) g
\]

for any polynomial \( \varphi \).

This shows that the continuous operators \( S_a(\varphi) \) and \( \varphi(T_a) = T_a(\varphi) \) coincide on the dense subset \( D_a \) of \( L_p(0,1) \). Thus, for every polynomial \( \varphi \),

\[
T_a(\varphi) = J^{-(\alpha+n)T_{-n}(\varphi)J^{a+n}}.
\]
Let \( \varphi \in C^*[0, 1] \), and let \( \varphi_k \) be polynomials converging to \( \varphi \) in \( C^*[0, 1] \). Since \( T_\alpha \) and \( T_{-\alpha} \) are of class \( C^* \) (by Theorem 6), we have (in the uniform operator topology):

\[
T_\alpha(\varphi_k) \rightarrow T_\alpha(\varphi); \quad T_{-\alpha}(\varphi_k) \rightarrow T_{-\alpha}(\varphi)
\]

for any \( k \rightarrow \infty \).

Fix \( g \in L_p(0, 1) \). Then \( T_{-\alpha}(\varphi_k)J^{a+n}g \in \mathcal{D}_{-(\alpha+n)} \) (cf. beginning of the proof) and

\[
T_{-\alpha}(\varphi_k)J^{a+n}g \rightarrow T_{-\alpha}(\varphi)J^{a+n}g
\]

for \( k \rightarrow \infty \) (by (14)). Moreover

\[
J^{-(\alpha+n)}[T_{-\alpha}(\varphi_k)J^{a+n}g] = T_\alpha(\varphi_k)g \rightarrow T_\alpha(\varphi)g
\]

by (13) and (14). Since \( J^{-(\alpha+n)} \) is closed, it follows that \( T_{-\alpha}(\varphi)J^{a+n}g \in \mathcal{D}_{-(\alpha+n)} \) and \( J^{-(\alpha+n)}T_{-\alpha}(\varphi)J^{a+n}g = T_\alpha(\varphi)g \), Q.E.D.

### 3. The local \( C^k \)-operational calculus

Note first that the results of §2 are also relevant to the case \( p=1 \), in the sense of the local \( C^k \)-operational calculus. Let

\[
L = \bigcup_{1 < p < \infty} L_p(0, 1)
\]

This is a dense linear manifold in \( L_1(0, 1) \), which is invariant under \( T_\alpha \) for all \( \alpha \in C \). Let \( n \geq 0 \) be an integer, and let \( |\Re \alpha| \leq n \). If \( f \in L_p(0, 1) \) for some \( 1 < p < \infty \), then the mapping \( \varphi \in C^*[0, 1] \rightarrow T_\alpha(\varphi)f \in L_1(0, 1) \) is continuous (\( T_\alpha(\cdot) \) is given by Theorems 8 and 9) because

\[
\|T_\alpha(f)\|_1 \leq \|T_\alpha(\varphi)\|_p \leq \|T_\alpha(\cdot)\|_p \|f\|_p \|\varphi\|_n,
\]

where \( \|T_\alpha(\cdot)\|_p \) denotes the norm of the \( C^* \)-operational calculus for \( T_\alpha \) acting in \( L_p(0, 1) \). Thus \( W_n(T_\alpha; 1) = \mathcal{D}_{a+k} \) for \( |\Re \alpha| \leq n \), and the \( C^* \)-operational calculus for \( T_\alpha \) on \( L_1(0, 1) \) is provided by Theorems 8 and 9.

In the next two theorems, we study the manifolds \( W_k(T_\alpha; p) \) for \( k < |\Re \alpha| \) (they coincide with the whole space for \( k \geq |\Re \alpha| \), at least for \( 1 < p < \infty \), by §2). It turns out that the situation is totally different in the right and left half-planes.

**Theorem 10.** For \( \alpha \in C \) with \( \Re \alpha < 0 \) and \( 1 < p < \infty \),

\[
W_k(T_\alpha; p) = \mathcal{D}_{a+k}, \quad 0 \leq k < |\Re \alpha|,
\]

and the \( C^* \)-operational calculus for \( T_\alpha \) on \( \mathcal{D}_{a+k} \) is given by

\[
T_\alpha(\varphi) = J^{-(\alpha+k)}T_{-\alpha}(\varphi)J^{a+k}, \quad \varphi \in C^*[0, 1],
\]

(where \( T_{-\alpha}(\varphi) \) is defined in Lemma 5).

**Proof.** Fix \( p, \alpha \) and \( k \) as in the theorem, and define \( T_\alpha(\cdot) \) by (15). One verifies easily that the mapping \( \varphi \rightarrow T_\alpha(\varphi) \) is an algebra homomorphism of \( C^*[0, 1] \) into \( T(\mathcal{D}_{a+k}) \) which sends the functions \( \varphi(x) \equiv 1 \) and \( \varphi(x) \equiv x \) respectively to \( I|\mathcal{D}_{a+k} \) and \( T_\alpha|\mathcal{D}_{a+k} \) (cf. Lemma 1). Moreover, for each \( g \in \mathcal{D}_{a+k} \), the mapping \( \varphi \rightarrow T_\alpha(\varphi)g \) of \( C^*[0, 1] \) into \( L_p(0, 1) \) is continuous, since

\[
T_\alpha(g) = J^{-(\alpha+k)}T_{-\alpha}(\varphi)h
\]

for \( g = J^{-(\alpha+k)}h \) with \( h \in L_p(0, 1) \). Q.E.D.
In particular, \( W_k(T_\alpha; p) \) is dense in \( L_p(0, 1) \) for \( \Re \alpha < 0 \) and \( k \geq 0 \) arbitrary. For \( \Re \alpha \geq 1 \), we get the "other" extreme.

**Theorem 11.** For \( \alpha \in \mathbb{C} \) with \( \Re \alpha \geq 1 \) and \( 1 < p < \infty \), \( W_k(T_\alpha; p) = \{0\} \) if \( k < \lfloor \Re \alpha \rfloor \). The same is true for \( p = 1 \) if \( \alpha \) is an integer.

**Proof.** If \( \alpha \) is an integer, this is a trivial consequence of Lemma 3 and Leibnitz' formula.

Suppose then that \( 1 < p < \infty \), and that \( f \in W_k(T_\alpha; p) \) for some fixed \( k < m = \lfloor \Re \alpha \rfloor \). As in the proof of Lemma 7, we apply the Phragmén-Lindelöf principle in the strip \( 0 \leq \Re \zeta \leq \Re \alpha \) to the function \( \Phi(\zeta) = \langle e^{\alpha \zeta^2} \varphi(T_\zeta), f, g \rangle \) where \( \varphi \) is a polynomial and \( g \in L_q(0, 1) \) (both fixed). We then obtain that \( f \in W_\alpha(T_\zeta; p) \) for all \( \zeta \) in the strip, hence in particular for \( \zeta = m \). Since \( k < m \), we conclude that \( f \) is the null function.

4. Similarity and spectrality.

**Lemma 12.** Let \( \alpha \in \mathbb{C} \) and \( 1 < p < \infty \). Then every \( s \in [0, 1) = \sigma(T_\alpha) \setminus \{1\} \) is an eigenvalue of \( T_\alpha \) for \( \Re \alpha \geq 1 \) (\( \Re \alpha > 1 \) or \( \alpha = 1 \) if \( p = 1 \)), while \( T_\alpha \) has no eigenvalue for \( \Re \alpha \geq 0 \) (\( \Re \alpha > 0 \) or \( \alpha = 0 \) if \( p = 1 \)).

**Proof.** Let \( C_s \) denote the characteristic function of the interval \([s, 1)\), \( 0 \leq s < 1 \). One verifies easily that \( C_s \) is an eigenvector of \( T_{-1} \) corresponding to the eigenvalue \( s \) (for \( 1 \leq p < \infty \)).

By Lemma 1, (3),

\[
T_{-\alpha} J^{a-1} C_s = J^{a-1} T_{-1} C_s = s J^{a-1} C_s,
\]

i.e. \( J^{a-1} C_s \) (which is in \( L_p(0, 1) \) for \( \alpha \) as in the first statement of the lemma) is an eigenvector of \( T_{-\alpha} \) corresponding to the eigenvalue \( s \).

Next, suppose \( T_\alpha g = \lambda g \) for \( g \in L_p(0, 1) \) and \( \lambda \in \mathbb{C} \). If \( \Re \alpha \geq 0 \) (\( \Re \alpha > 0 \) or \( \alpha = 0 \) if \( p = 1 \)), we may apply \( J^a \) on both sides of this equation; by Lemma 1, (3), we obtain

\[
M J^a g = \lambda J^a g.
\]

Since \( M \) has no eigenvector \( \neq 0 \) and \( J^a \) is one-one, it follows that \( g \) is the zero element.

Let \( \alpha, \beta \in \mathbb{C} \). By Lemma 2, \( T_\alpha \) and \( T_\beta \) are similar if \( \Re \alpha = \Re \beta \) (and \( 1 < p < \infty \)). On the other hand, since the \( C^k \)-classification and the point spectrum are similarity invariants, it follows from Lemmas 3, 5 and 12 that \( T_\alpha \) and \( T_\beta \) are not similar if \( \alpha \) and \( \beta \) are distinct integers (for \( 1 \leq p < \infty \)).

**Conjecture.** For \( 1 < p < \infty \) and \( \alpha, \beta \in \mathbb{C} \), \( T_\alpha \) and \( T_\beta \) are similar if and only if \( \Re \alpha = \Re \beta \). (By Lemma 2, it would suffice to verify that \( T_\alpha \) and \( T_\beta \) are not similar if \( \alpha \) and \( \beta \) are distinct real numbers.)

**Proposition 13.** Let \( \alpha, \beta \in \mathbb{C} \) and \( 1 < p < \infty \). Then \( T_\alpha \) and \( T_\beta \) (acting in \( L_p(0, 1) \)) are not similar if \( \lfloor \Re \alpha \rfloor \neq \lfloor \Re \beta \rfloor \).
Proof. Assume, without loss of generality, that $\text{Re } \alpha < \text{Re } \beta$. If either $0 \leq \text{Re } \alpha$ or $\text{Re } \beta \leq 0$, this follows from Theorem 6 and the similarity invariance of the $C^k$-classification. If $\text{Re } \alpha < 0 < 1 \leq \text{Re } \beta$, $W_0(T_\alpha)$ is dense in $L_p(0, 1)$ (Theorem 10) while $W_0(T_\beta) = (0)$ (Theorem 11). Thus $T_\alpha$ and $T_\beta$ are not similar.

If $\text{Re } \alpha \leq -1 < 0 < \text{Re } \beta$, every $s \in [0, 1)$ is an eigenvalue of $T_\alpha$, while $T_\beta$ has no eigenvalue (Lemma 12), and the conclusion follows from the similarity invariance of the point spectrum. Q.E.D.

We next discuss the spectrality of $T_\alpha$ in Dunford’s sense [1].

Lemma 14. Let $T$ be a bounded spectral operator with real spectrum, acting in the Banach space $X$. Let $T = S + N$ be its canonical decomposition (cf. [1]). Then:

(a) If $W_k(T)$ is dense in $X$ for some integer $k \geq 0$, then $T$ is of finite type $\leq k$ (i.e., $N^{k+1} = 0$).

(b) If $T$ is of finite type $k$, then $W_j(T) \neq (0)$ for all $j \geq 0$; in fact, $W_j(T) = \mathcal{B}(N^{k-j})$ for $j = 0, \ldots, k-1$, and trivially $W_k(T) = X$ for $j \geq k$.

Proof. Fix a compact interval $\Delta = \sigma(T)$. Let $S(\cdot)$ be the $C$-operational calculus for $S$ (defined on $C(\Delta)$), and let $\|S(\cdot)\|$ be its norm.

(a) Let $x \in W_k(T)$. The function $e^{itN}x$ ($z \in C$) is entire of order one and minimal type (since $N$ is a quasi-nilpotent operator). For $z = t \in R$, we have:

$$\|e^{itN}x\| = \|e^{-itS}e^{itT}x\| \leq \|S(\cdot)\| \|e^{itT}x\| \leq \|S(\cdot)\| \|x\|_1 \|\varphi_t\|_{\|\cdot\|,\Delta},$$

where $\varphi_t(s) = e^{its}$, $t, s \in R$.

Thus $\|e^{itN}x\| = O(|t|^k)$, and therefore $e^{itN}x$ is a polynomial of order $\leq k$ by Theorem 3.13.8 in [3]. Hence $N^{k+1}x = 0$ for each $x \in W_k(T)$, and it follows that $N^{k+1} = 0$ since $W_k(T)$ is dense in $X$.

(b) We have $N^{k+1} = 0$ and $N^k \neq 0$. The analytic operational calculus for $T$ takes the form (cf. [1]):

$$T(\varphi) = \sum_{m=0}^{k} S(\varphi^{(m)})N^m/m!$$

If $x \in \mathcal{B}(N^{k-j})$, say $x = N^{k-j}y$ with $y \in X(0 \leq j < k)$, then

$$T(\varphi)x = \sum_{m=0}^{j} S(\varphi^{(m)})N^my/m!$$

In particular, $\|p(T)x\| \leq \|S(\cdot)\| \cdot \max_{0 \leq m \leq j} \|N^my\| \|p\|_{1,\Delta}$ for any polynomial $p$, i.e. $x \in W_j(T)$. Q.E.D.

For simplicity, we state the following result for $1 < p < \infty$, although part of the conclusion remains valid for $p = 1$.

Proposition 15. Let $1 < p < \infty$. Then $T_\alpha$ is spectral for $\text{Re } \alpha = 0$, and is not spectral for $|\text{Re } \alpha| \geq 1$.

Proof. The first statement is a trivial corollary of Lemma 2.
By Theorem 6, $T_\alpha$ is of class $C^n$ if $n \geq |\text{Re } \alpha|$. Thus, if $T_\alpha$ were spectral, it should be of finite type by Lemma 14(a). In particular, its point spectrum should be at most countable by [2, Theorem 1, p. 56]. But this contradicts Lemma 12 if $\text{Re } \alpha \leq -1$. Also all $W_j(T_\alpha)$ ($j \geq 0$) should be nontrivial by Lemma 14(b), contradicting Theorem 11 if $\text{Re } \alpha \neq 1$. Thus $T_\alpha$ is not spectral for $|\text{Re } \alpha| \leq 1$.

5. Remarks. It is interesting to regard the results of this paper as statements about the operators $\alpha^{-1} T_\alpha = J + \alpha^{-1} M$ ($0 \neq \alpha \in \mathbb{C}$), which are perturbations of $J$ by a scalar operator of arbitrarily small norm. Thus, if $\alpha$ and $\beta$ are nonzero complex numbers, the following assertions can be made (for $1 < p < \infty$):

(a) If $[|\text{Re } \alpha|] \neq [| \text{Re } \beta |]$, $J + \alpha^{-1} M$ and $J + \beta^{-1} M$ belong to distinct $(C^k)$-classes, although they differ only by the scalar operator $(\alpha^{-1} - \beta^{-1}) M$, which is of arbitrarily small norm. This shows that the commutativity hypothesis in [5, Corollary 5.6] cannot be replaced by a restriction on the norm of the perturbing scalar operator.

(b) The perturbations $J - \alpha^{-1} M$ and $J + \alpha^{-1} M$ have respectively a dense and a trivial semisimplicity manifold, a "pure" point spectrum (up to the right end point of the spectrum $[0, \alpha^{-1}]$) and no point spectrum.

(c) The perturbations $J + \alpha^{-1} M$ and $J + \beta^{-1} M$ are not similar if $|\text{Re } \alpha| \neq |\text{Re } \beta|$.

References


Yale University, New Haven, Connecticut