THE $C^k$-CLASSIFICATION OF CERTAIN OPERATORS IN $L_p(1)$

BY

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Introduction. We investigate in this paper the one-parameter family of operators

$$T_a = M + \alpha J$$

acting in $L_p(0, 1)$ ($1 \leq p < \infty$), where $\alpha \in \mathbb{C}$ (the complex field), $M: f(x) \mapsto xf(x)$ and $J: f(x) \mapsto \int_0^x f(t) \, dt$.

Extending a result of Sakhrnovich [8] from the case $p=2$ to the case $1 < p < \infty$, Kalisch [4] established recently that $T_a$ is similar to $M$ if $\text{Re} \, \alpha = 0$.

The situation becomes quite different if $\text{Re} \, \alpha \neq 0$; thus, $T_a$ is not similar to $M$ if $|\text{Re} \, \alpha| \geq 1$, and more generally, $T_a$ is not similar to $T_{\beta}$ if $|\text{Re} \, \alpha| \neq |\text{Re} \, \beta|$ (Proposition 13). This result will follow from our discussion of the $C^k$-operational calculus (both "global" and "local", in the sense of [5], [6], [7]) for the operators $T_a$. We recall briefly the terminology, restricting ourselves to bounded operators $T: X \to X$ ($X$ a Banach space) with real spectrum $\sigma(T)$.

Fix a compact interval $\Delta \subseteq \mathbb{R}$ (the real line) which contains $\sigma(T)$. For $n=0, 1, 2, \ldots$, let $C^n(\Delta)$ denote the Banach algebra of all complex valued functions of class $C^n$ on $\Delta$, with the norm

$$\|\varphi\|_{n, \Delta} = \sum_{j=0}^{n} \sup_{\Delta} |\varphi^{(j)}|^{1/j}$$

(We shall write $\|\varphi\|_{n, \Delta}$ when $\Delta = [0, 1]$.) We say that $T$ is of class $C^n$ (and we write $T \in (C^n)$ if there exists a continuous representation $\varphi \mapsto T(\varphi)$ of $C^n(\Delta)$ on $X$ such that $T(\varphi) = I$ (the identity operator) for $\varphi(t) \equiv 1$ and $T(\varphi) = T$ for $\varphi(t) \equiv t$. Such a representation is unique when it exists, and is called the $C^n$-operational calculus for $T$. For example, $T_0 = M$ is of class $C^0$ (= $C^0$), and its $C$-operational calculus is $\varphi \mapsto M(\varphi)$, where

$$M(\varphi): f(x) \mapsto \varphi(x)f(x), f \in L_p(0, 1).$$

If $W \subseteq X$ is a linear manifold, we denote by $T(W)$ the algebra of all linear transformations of $X$ with domain $W$ and range contained in $W$. If $W$ is invariant for $T$, a $C^n$-operational calculus for $T$ on $W$ is an algebra homomorphism $\varphi \mapsto T(\varphi)$ of $C^n(\Delta)$ into $T(W)$ with the following properties:

(i) $T(\varphi) = I/W$ for $\varphi(t) \equiv 1$;
(ii) $T(\varphi) = T/W$ for $\varphi(t) \equiv t$;
(iii) for each $x \in W$, the mapping $\varphi \mapsto T(\varphi)x$ of $C^n(\Delta)$ into $X$ is continuous.

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For each $n=0,1,2,\ldots$, there exists a maximal invariant linear manifold $W_n(T)$ on which $T$ admits a $C^n$-operational calculus, and the latter is uniquely determined on $W_n(T)$. In fact, $W_n(T)$ is the set of all $x \in X$ for which

$$|x|_n = \sup \{\|\varphi(T)x\| : \varphi \text{ a polynomial with } \|\varphi\|_{n,A} \leq 1\}$$

is finite.

The "semisimplicity manifold" $W_0(T)$ is particularly important (cf. Theorem 2.1 in [6]); it contains trivially the eigenvectors of $T$. We shall write $W_n(T; p)$ instead of $W_n(T_a)$ when it will be necessary to specify the $L_p$ space under consideration.

We may now describe the main results of this paper (for $1 < p < \infty$):

1. $T_a$ is of class $C^n$ if $|\Re a| \leq n$ and only if $|\Re a| < n + 1$ (Theorem 6); the $C^n$-operational calculus for $T_a$ ($|\Re a| \leq n$) is given in Theorems 8 and 9.

2. The $W_k$ manifolds of $T_a$ are dense for $\Re a < 0$ and trivial for $\Re a \geq 1$ and $k < [\Re a]$ (Theorems 10 and 11).

3. $T_a$ is not spectral (in Dunford's sense) for $|\Re a| \geq 1$ (it is of course spectral for $\Re a = 0$, by the Kalisch-Sakhnovič result).

1. **Five lemmas.** Let $\{F; \Re \zeta > 0\}$ be the Riemann-Liouville holomorphic semigroup, acting in $L_p(0, 1)$ ($1 \leq p < \infty$):

$$(J^f)(x) = (1/\Gamma(\zeta)) \int_0^x (x-t)^{\zeta-1}f(t) \, dt$$

($f \in L_p(0, 1)$, $x \in [0, 1]$, $\Re \zeta > 0$). It is known (cf. [3] for $p = 2$, and [4] for $1 < p < \infty$) that if $1 < p < \infty$, the semigroup $\{F; \Re \zeta > 0\}$ admits a strongly continuous boundary group $\{F^\gamma; \gamma \in \mathbb{R}\}$ of bounded operators, and

$$\|F^\gamma\| \leq e^{\alpha|\gamma|/2} \quad (\gamma \in \mathbb{R})$$

(see also the estimates at the end of Kalisch's paper [4]). The operator $F$ ($\Re \zeta > 0$) is one-to-one in $L_p(0, 1)$ ($p \geq 1$); its inverse, with domain $D_{-1} = \mathcal{R}(J^\zeta)$ (the range of $J^\zeta$), is a closed operator, which we denote by $J^{-\zeta}$.

For $1 < p < \infty$, note that $\mathcal{R}(J^{\beta + i\gamma}) = \mathcal{R}(J^\beta)$ ($\beta > 0$, $\gamma \in \mathbb{R}$), since $J^{\beta + i\gamma} = J^\beta J^{i\gamma}$ and $J^{i\gamma}$ is nonsingular.

For $p = 1$ and $\gamma \in \mathbb{R}$, we define $J^{i\gamma}$ as follows. Its domain is $D_{i\gamma} = U(\mathcal{R}(J^\gamma); \Re \zeta > 0)$; if $f \in D_{i\gamma}$, say $f = J^\gamma h$ for some $h \in L_1(0, 1)$ and $\zeta \in C$ with $\Re \zeta > 0$, then $J^{i\gamma} f = J^{i\gamma + i\gamma} h$. One verifies easily that $J^{i\gamma}$ is well defined.

We shall use the notation $D_\zeta$ also for $\Re \zeta > 0$, in which case $D_\zeta = L_p(0, 1)$ (the domain of $J^{\zeta}$); similarly $D_{i\gamma} = L_p(0, 1)$ for $1 < p < \infty$.

**Lemma 1.** For any $\zeta \in C$, $D_\zeta$ is invariant under $M$, and the following (trivially equivalent) identities are valid on $D_\zeta$:

1. $M J^\zeta = J^{\zeta} M = \zeta J^{\zeta + 1}$,
2. $J^\zeta M = T_+ J^\zeta$,
3. $M J^\zeta = J^\zeta T_\zeta$. 

Proof. If \( \text{Re} \zeta > 0 \) (\( \text{Re} \zeta \geq 0 \) if \( 1 < p < \infty \)), the first statement is trivial, since \( \mathcal{D}_{\zeta} = L_p(0, 1) \). For \( g \in L_p(0, 1) \) and \( \text{Re} \zeta > 0 \),

\[
(M^{\zeta} - J^{\zeta} M)g(x) = \Gamma(\zeta)^{-1} \int_0^x (x-t)^{\zeta-1} (x-t) g(t) \, dt = \Gamma(\zeta)^{-1} \Gamma(\zeta + 1) J^{\zeta+1} g(x) = \zeta J^{\zeta+1} g(x).
\]

This proves the first, and hence all three identities of the lemma (for \( 1 \leq p < \infty \) and \( \text{Re} \zeta > 0 \)). By (3), \( M \mathcal{R}(J^\zeta) \subset \mathcal{R}(J^\zeta) \), i.e. \( \mathcal{D}_{-\zeta} \) is invariant under \( M \). Apply \( J^{-\zeta} \) to both sides of (3):

\[
J^{-\zeta} M J^{\zeta} = T_{\zeta},
\]

i.e., \( J^{-\zeta} M = T_{\zeta} J^{-\zeta} \) on \( \mathcal{D}_{-\zeta} \). This proves (2) (and hence the lemma) for \( \text{Re} \zeta < 0 \).

Let \( \mathcal{D} = U(\mathcal{D}_{\zeta}; \text{Re} \zeta < 0) \). \( \mathcal{D} \) is \( M \)-invariant, and dense in \( L_p(0, 1) \) (\( 1 \leq p < \infty \)). Let \( g \in \mathcal{D} \); then \( g = J^\gamma h \) for some \( \zeta \in \mathbb{C} \) with \( \text{Re} \zeta > 0 \) and some \( h \in L_p(0, 1) \). Using (3) twice (for \( \text{Re} \zeta > 0 \)), we obtain (for \( \gamma \in \mathbb{R} \)):

\[
M J^\gamma g = M J^{\zeta+\gamma} h = J^{\zeta+\gamma} T_{\zeta+\gamma} h
= J^{\gamma} (T_{\zeta} + i\gamma J) h = J^{\gamma} (M J^{\zeta} + i\gamma J^{\zeta+1}) h
= J^{\gamma} (M + i\gamma J) J^{\zeta} h = J^{\gamma} T_{\zeta} J^\gamma g.
\]

If \( p = 1 \), this finishes the proof of the lemma for \( \text{Re} \zeta = 0 \). If \( 1 < p < \infty \), this shows that (3) is valid on the dense subset \( \mathcal{D} \) of \( L_p(0, 1) \) (for \( \gamma = i\gamma \); since \( J^{i\gamma} \) is a bounded operator, (3) is valid everywhere on \( L_p(0, 1) \).

**Lemma 2.** For \( \beta, \gamma \in \mathbb{R} \) arbitrary, the operators \( T_{\beta+i\gamma} \) and \( T_{\beta} \) acting in \( L_p(0, 1) \) (\( 1 < p < \infty \)) are similar, with \( J^{i\gamma} \) implementing the similarity:

\[
J^{i\gamma} T_{\beta+i\gamma} J^{-i\gamma} = T_{\beta},
\]

(for \( \beta = 0 \) and \( p = 2 \), this is due to Sakhnovi\'c [8]).

**Proof.** By (3),

\[
J^{i\gamma} T_{\beta+i\gamma} J^{-i\gamma} = J^{i\gamma} (T_{i\gamma} + \beta J) J^{-i\gamma} = M + \beta J = T_{\beta}.
\]

**Remark.** For any \( 1 \leq p < \infty \) and any complex numbers \( \alpha \) and \( \zeta \), \( T_a \) is unboundedly similar to \( T_{\zeta} \) (in particular, \( T_a \) is unboundedly similar to \( M \)). Indeed,

\[
J^{-\zeta+a} T_a J^{\zeta-a} = T_{\zeta} \quad \text{on } \mathcal{D}_{-\zeta-a},
\]

where everything makes sense by Lemma 1. This suggests considering the map \( \varphi \to J^{-a} M(\varphi) J^a \) as a "possible" operational calculus for \( T_a \).

**Lemma 3.** For any integer \( n \geq 0 \), and \( 1 \leq p < \infty \), the operator \( T_n \) acting in \( L_p(0, 1) \) belongs to \( (C^n)-(C^{n-1}) \), and the \( C^n \)-operational calculus for \( T_n \) is given by

\[
T_n(\varphi) = J^{-n} M(\varphi) J^n, \quad \varphi \in C^n[0, 1].
\]
Remark. (1) \( \mathcal{C}^{-1} = \Phi \) by convention. (2) \( T_n(\varphi) \) is well defined, since \( \mathcal{A}(J^n) \) is a \( \mathcal{C}^n[0, 1] \)-module.

Proof. By Leibnitz’ formula,

\[
(5) \quad T_n(\varphi) = \sum_{j=0}^{n} \binom{n}{j} M(\varphi^{(j)} J^j), \quad \varphi \in \mathcal{C}^n[0, 1].
\]

Thus \( T_n(\varphi) \) is a bounded operator on \( L_p(0, 1) \). In fact, since \( \|J^j\| \leq 1/j! \) (cf. [3, p. 664]), we have

\[
(6) \quad \|T_n(\varphi)\| \leq \binom{n}{[n/2]} \|\varphi\|_n, \quad \varphi \in \mathcal{C}^n[0, 1].
\]

The map \( \varphi \mapsto T_n(\varphi) \) is clearly an algebra homomorphism of \( \mathcal{C}^n[0, 1] \) into the bounded operators on \( L_p(0, 1) \), which is continuous by (6). If \( \varphi(x) \equiv 1 \), \( T_n(\varphi) \) is trivially the identity operator. If \( \varphi(x) \equiv x \), \( T_n(\varphi) = T_n \) by (3). Thus, \( T_n \in \mathcal{C}^n \) and its \( \mathcal{C}^n \)-operational calculus is given by (4). Finally, apply (5) to the functions

\[
\varphi_t(x) = e^{itx} \quad (x, t \in \mathbb{R}).
\]

Thus

\[
(\mathbb{e}^{itT_n g})(x) = \sum_{j=0}^{n} \binom{n}{j} (it)^j e^{itx}(J^j g)(x),
\]

and consequently \( \|e^{itT_n}\| \neq O(|t|^{-n-1}) \) (\( n = 1, 2, \ldots \)). Therefore \( T_n \notin \mathcal{C}^{n-1} \) (\( n \geq 1 \)) by Lemma 2.11 in [5].

Lemma 4. Let \( n \geq 0 \) be an integer. Then, for \( \varphi \) of class \( \mathcal{C}^n \) and \( g \in \mathcal{A}(J^n) \) (or vice versa, or for both \( \varphi \) and \( g \) in \( \mathcal{A}(J^n) \)), the following identity is valid:

\[
J^n(\varphi J^{-n} g) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j J^j (g J^{-j} \varphi).
\]

Remark. Here, as usual, \( \mathcal{A}(J^n) \) denotes the range of \( J^n \) in \( L_p(0, 1) \), with \( p \) arbitrary (1 \( \leq p < \infty \)). Note the analogy with Leibnitz’ formula (in fact, the latter may be used to prove the lemma).

Proof. We use induction on \( n \). The lemma is trivial for \( n = 0 \). Suppose it is true for \( n = k \). Let \( \varphi \) and \( g \) be as required for \( n = k + 1 \). Write \( g' = J^{-1} g \). Then \( \varphi \) and \( g' \) satisfy the hypothesis for \( n = k \). Therefore

\[
J^{k+1}(\varphi J^{-(k+1)} g) = J J^k(\varphi J^{-k} g')
\]

\[
= \sum_{j=0}^{k} (-1)^j \binom{k}{j} J^j (g' J^{-j} \varphi).
\]

An integration by parts shows that

\[
J(g' J^{-i} \varphi) = g J^{-i} \varphi - J(g J^{-i+1} \varphi).
\]
Thus
\[ J^{k+1}(gJ^{-(k+1)}g) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left\{ J^j(gJ^{-j}g) - J^{j+1}(gJ^{-(j+1)}g) \right\} \]
\[ = g\varphi + \sum_{j=1}^{k} (-1)^j \binom{k}{j} \left[ \binom{k}{j-1} \right] J^j(gJ^{-j}g) \]
\[ + (-1)^{k+1}J^{k+1}(gJ^{-(k+1)}g). \]

Since
\[ \binom{k}{j} + \binom{k}{j-1} = \binom{k+1}{j}, \]
we obtain the correct identity for \( n = k + 1 \), Q.E.D.

**Lemma 5.** For any integer \( n \geq 0 \), and for \( 1 \leq p < \infty \), the operator \( T_{-n} \) acting in \( L_p(0, 1) \) belongs to \( (C^n) - (C^{n-1}) \), and the \( C^n \)-operational calculus for \( T_{-n} \) is given by
\[ (7) \quad T_{-n}(\varphi) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j M(\varphi^{(j)}), \quad \varphi \in C^n[0, 1]. \]

**Proof.** The map \( \varphi \mapsto T_{-n}(\varphi) \) of \( C^n[0, 1] \) into the bounded operators on \( L_p(0, 1) \) is clearly linear and continuous, in fact,
\[ \|T_{-n}(\varphi)\| \leq \left( \frac{n}{[n/2]} \right) \|\varphi\|. \]

For \( g \in \mathcal{R}(J^n) \), we have by Lemma 4:
\[ (8) \quad T_{-n}(\varphi)g = J^n(\varphi J^{-n}g). \]

Therefore \( \mathcal{R}(J^n) \) is invariant under \( T_{-n}(\varphi) \) (for all \( \varphi \in C^n[0, 1] \)) and \( T_{-n}(\varphi\psi) = T_{-n}(\varphi)T_{-n}(\psi) \) on \( \mathcal{R}(J^n) \) (for all \( \varphi, \psi \in C^n[0, 1] \)). Since \( \mathcal{R}(J^n) \) is dense in \( L_p(0, 1) \) and \( T_{-n}(\varphi) \) is continuous, it follows that \( T_{-n}(\cdot) \) is multiplicative on \( C^n[0, 1] \). The relations \( T_{-n}(\varphi) = I(T_{-n}) \) for \( \varphi(x) = 1 \) (\( \equiv x \)) are trivial on \( \mathcal{R}(J^n) \) by (8) and Lemma 1; by density, they are true throughout \( L_p(0, 1) \).

Finally, one verifies that \( T_{-n} \notin (C^{n-1}) \) just as in Lemma 3.

2. **Global classification** \( (1 < p < \infty) \).

**Theorem 6.** The operator \( T_\alpha \) acting in \( L_p(0, 1) \) \( (1 < p < \infty) \) is of class \( C^n \) \( (n = 0, 1, 2, \ldots) \) if \( |\text{Re } \alpha| \leq n \) and only if \( |\text{Re } \alpha| < n + 1 \).

In other words, \( T_\alpha \) is of class \( C^n \) in the strip \( |\text{Re } \alpha| \leq n \) and is not of class \( C^n \) outside the strip \( |\text{Re } \alpha| < n + 1 \).

The theorem is an immediate corollary of Lemmas 3 and 5, together with the following

**Lemma 7.** Suppose that, for some integer \( n \geq 0 \) and some \( \alpha_0 \in \mathbb{C} \), the operator \( T_{\alpha_0} \) is of class \( C^n \) (when acting in \( L_p(0, 1) \), \( 1 < p < \infty \)). Then \( T_\alpha \) is of class \( C^n \) for all \( \alpha \) in the strip \( -n \leq \text{Re } \alpha \leq n \) if \( \text{Re } \alpha_0 \geq 0 \) \( (\text{Re } \alpha_0 \geq \text{Re } \alpha \leq n \) if \( \text{Re } \alpha_0 \leq 0 \).
Proof. To fix the ideas, suppose $\Re \alpha_0 = \beta_0 \geq 0$. Write $\alpha = \beta + i\gamma$ ($\beta, \gamma \in \mathbb{R}$). Fixing a polynomial $\varphi$ and elements $f \in L_p(0, 1)$, $g \in L_q(0, 1)$ ($p^{-1} + q^{-1} = 1$), we define

$$\Phi(\alpha) = \langle e^{\alpha a} \varphi(T_a) f, g \rangle, \quad \alpha \in \mathbb{C}.$$  

Since $|e^{\alpha a}| \leq e^{\alpha a^2}$, and since $\varphi(T_a)$ is a polynomial in $\alpha$ (with operator coefficients), we have $|\Phi(\alpha)| = O(e^{\epsilon |\alpha|^2})$ (for $|\gamma| \to \infty$) in the strip $-n \leq \Re \alpha \leq \beta_0$, for any $\epsilon > 0$.

By Lemma 2 and the estimate $\|f^{1/2}\| \leq e^{a^2/2}$, we have:

$$|\Phi(\beta + i\gamma)| \leq \exp \pi \beta^2 - \gamma^2 + |\gamma| \cdot \|f\|_p \cdot \|g\|_q \cdot \|\varphi(T_\beta)\|$$

$$\leq \exp \pi (\beta^2 + 1/4) \cdot \|f\|_p \cdot \|g\|_q \cdot \|\varphi(T_\beta)\|$$

for all $\beta, \gamma \in \mathbb{R}$.

Since $T_{-n}$ and $T_{\beta_0}$ are of class $C^n$ (by Lemma 5, the hypothesis and Lemma 2), there exists a constant $K$ (depending only on $n$, $\beta_0$ and $p$) such that

$$\|\varphi(T_{-n})\| \leq K \|\varphi\|_n \quad \text{and} \quad \|\varphi(T_{\beta_0})\| \leq K \|\varphi\|_n.$$

Hence

$$|\Phi(-n + i\gamma)| \leq M \|f\|_p \cdot \|g\|_q \cdot \|\varphi\|_n$$

and

$$|\Phi(\beta_0 + i\gamma)| \leq M \|f\|_p \cdot \|g\|_q \cdot \|\varphi\|_n$$

for all real $\gamma$, where $M = K \exp \pi (\delta^2 + 1/4)$ and $\delta = \max (\eta, \beta_0)$. By the Phragmèn-Lindelöf principle (cf. [9, p. 180]), it follows that $|\Phi(\alpha)| \leq M \|f\|_p \cdot \|g\|_q \cdot \|\varphi\|_n$ for $-n \leq \Re \alpha \leq \beta_0$. Hence, for such $\alpha$,

$$\|\varphi(T_{-n})\| \leq M \exp \pi (\gamma^2 - \beta^2) \|\varphi\|_n,$$

and the lemma follows.

The next two theorems give explicitly the $C^n$-operational calculus for $T_\alpha$ ($|\Re \alpha| \leq n$) acting in $L_p(0, 1)$, with $1 < p < \infty$.

Theorem 8. Let $n$ be a nonnegative integer. Then for $0 \leq \Re \alpha \leq n$ and $\varphi \in C^n[0, 1]$ the range of $J^\alpha$ (i.e., the domain of $J^{-\alpha}$) is invariant under $M(\varphi)$, and the $C^n$-operational calculus for $T_\alpha$ is given by

$$T_\alpha(\varphi) = J^{-\alpha} M(\varphi) J^\alpha, \quad \varphi \in C^n[0, 1].$$

Proof. By Lemma 1 (3),

$$\varphi(M) J^\alpha = J^\alpha \varphi(T_\alpha)$$

for any polynomial $\varphi$. In particular,

$$\varphi(M) \mathcal{R}(J^\alpha) \subset \mathcal{R}(J^\alpha), \quad \varphi = \text{a polynomial}.$$
Let \( \varphi \in C^n[0, 1] \), and choose polynomials \( \varphi_k \) which converge to \( \varphi \) in \( C^n[0, 1] \). In particular, \( \varphi_k \to \varphi \) uniformly in \( [0, 1] \), and therefore

\[
\mathcal{D}_{-a} \ni \varphi_k J^a g \to \varphi J^a g
\]

in \( L_p(0, 1) \), for any \( g \in L_p(0, 1) \). By (9), we have:

\[
J^{-\alpha}(\varphi_k J^a g) = \varphi_k(T_a) = T_a(\varphi_k) \to T_a(\varphi)
\]

in the uniform operator topology, since \( T_a \in (C^n) \) by Theorem 6 (for \( |\text{Re } \alpha| \leq n \) and \( 1 < p < \infty \)) and \( \varphi_k \to \varphi \) in \( C^n[0, 1] \). Since \( J^{-\alpha} \) is a closed operator, it follows from (11) and (12) that \( \varphi J^a g \in \mathcal{D}_{-a} \) and

\[
J^{-\alpha}(\varphi J^a g) = T_a(\varphi),
\]

Q.E.D.

We consider next the range \( -n \leq \text{Re } \alpha < 0 \) (\( n = 1, 2, \ldots \)). Note that \( \text{Re } (\alpha + n) \geq 0 \). The notation \( T_{-n}(\varphi) \) is that of Lemma 5.

**Theorem 9.** Let \( n \) be a nonnegative integer. Then for \( -n \leq \text{Re } \alpha < 0 \) and \( \varphi \in C^n[0, 1] \), the range of \( J^{\alpha+n} \) (i.e., \( \mathcal{D}_{-(\alpha+n)} \)) is invariant under \( T_{-n}(\varphi) \), and the \( C^n \)-operational calculus for \( T_a \) is given by

\[
T_a(\varphi) = J^{-\alpha+n}T_{-n}(\varphi)J^{\alpha+n}, \quad \varphi \in C^n[0, 1].
\]

**Proof.** By (10), \( \mathcal{R}(J^{\alpha+n}) \) is invariant for \( M(\varphi) \) for any polynomial \( \varphi \); it is therefore invariant for the operator

\[
T_{-n}(\varphi) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j J^j M(\varphi^{(j)}), \quad \varphi \text{ is a polynomial}.
\]

Thus, for any polynomial \( \varphi \), the operator

\[
S_a(\varphi) = J^{-\alpha+n}T_{-n}(\varphi)J^{\alpha+n}
\]

is everywhere defined. Being closed, it is continuous by the Closed Graph Theorem.

Let \( g \in \mathcal{D}_a = \mathcal{R}(J^{-\alpha}) \), say \( g = J^{-\alpha}h \) with \( h \in L_p(0, 1) \). By Lemma 1,

\[
S_a(\varphi)g = J^{-\alpha+n}T_{-n}(\varphi)J^\alpha h = J^{-\alpha+n}J^\alpha \varphi(M)h
= J^{-\alpha} \varphi(M)h = \varphi(T_a)J^{-\alpha}h
= \varphi(T_a)g
\]

for any polynomial \( \varphi \).

This shows that the continuous operators \( S_a(\varphi) \) and \( \varphi(T_a) = T_a(\varphi) \) coincide on the dense subset \( \mathcal{D}_a \) of \( L_p(0, 1) \). Thus, for every polynomial \( \varphi \),

\[
T_a(\varphi) = J^{-\alpha+n}T_{-n}(\varphi)J^{\alpha+n}.
\]
Let \( \varphi \in C^a[0, 1] \), and let \( \varphi_k \) be polynomials converging to \( \varphi \) in \( C^a[0, 1] \). Since \( T_a \) and \( T_{-a} \) are of class \( C^a \) (by Theorem 6), we have (in the uniform operator topology):

\[
T_a(\varphi_k) \to T_a(\varphi); \quad T_{-n}(\varphi_k) \to T_{-n}(\varphi)
\]

for any \( k \to \infty \).

Fix \( g \in L_p(0, 1) \). Then \( T_{-n}(\varphi_k)J^{a+n}g \in \mathcal{D}_{-(\alpha+n)} \) (cf. beginning of the proof) and

\[
T_{-n}(\varphi_k)J^{a+n}g \to T_{-n}(\varphi)J^{a+n}g
\]

for \( k \to \infty \) (by (14)). Moreover

\[
J^{-(\alpha+n)}[T_{-n}(\varphi_k)]J^{a+n}g = T_a(\varphi_k)g \to T_a(\varphi)g
\]

by (13) and (14). Since \( J^{-(\alpha+n)} \) is closed, it follows that \( T_{-n}(\varphi)J^{a+n}g \in \mathcal{D}_{-(\alpha+n)} \) and \( J^{-(\alpha+n)}T_{-n}(\varphi)J^{a+n}g = T_a(\varphi)g \), Q.E.D.

3. The local \( C^k \)-operational calculus. Note first that the results of \S 2 are also relevant to the case \( p=1 \), in the sense of the local \( C^k \)-operational calculus. Let \( L = \bigcup_{1 \leq p < \infty} L_p(0, 1) \). This is a dense linear manifold in \( L_a(0, 1) \), which is invariant under \( T_a \) for all \( \alpha \in C \). Let \( n \geq 0 \) be an integer, and let \( |\text{Re} \alpha| \leq n \). If \( f \in L \), say \( f \in L_p(0, 1) \) for some \( 1 < p < \infty \), then the mapping \( \varphi \in C^a[0, 1] \to T_a(\varphi)f \in L_1(0, 1) \) is continuous (\( T_a(\cdot) \) is given by Theorems 8 and 9) because

\[
\|T_a(\varphi)f\|_1 \leq \|T_a(\varphi)f\|_p \leq \|T_a(\cdot)\|_p f_1\|_p \|\varphi\|_n,
\]

where \( \|T_a(\cdot)\|_p \) denotes the norm of the \( C^a \)-operational calculus for \( T_a \) acting in \( L_1(0, 1) \). Thus \( W_n(T_a; 1) := L \) for \( |\text{Re} \alpha| \leq n \), and the \( C^a \)-operational calculus for \( T_a \) on \( L \) is provided by Theorems 8 and 9.

In the next two theorems, we study the manifolds \( W_k(T_a; p) \) for \( k < |\text{Re} \alpha| \) (they coincide with the whole space for \( k \geq |\text{Re} \alpha| \), at least for \( 1 < p < \infty \), by \S 2). It turns out that the situation is totally different in the right and left half-planes.

**Theorem 10.** For \( \alpha \in C \) with \( \text{Re} \alpha < 0 \) and \( 1 < p < \infty \),

\[
W_k(T_a; p) = \mathcal{D}_{a+k}, \quad 0 \leq k < |\text{Re} \alpha|,
\]

and the \( C^a \)-operational calculus for \( T_a \) on \( \mathcal{D}_{a+k} \) is given by

\[
T_a(\varphi) = J^{-(a+k)}T_{-a}(\varphi)J^{a+k}, \quad \varphi \in C^k[0, 1],
\]

(where \( T_{-a}(\varphi) \) is defined in Lemma 5).

**Proof.** Fix \( p, \alpha \) and \( k \) as in the theorem, and define \( T_a(\cdot) \) by (15). One verifies easily that the mapping \( \varphi \to T_a(\varphi) \) is an algebra homomorphism of \( C^k[0, 1] \) into \( T(\mathcal{D}_{a+k}) \) which sends the functions \( \varphi(x) \equiv l \) and \( \varphi(x) \equiv x \) respectively to \( lI|\mathcal{D}_{a+k} \) and \( T_a|\mathcal{D}_{a+k} \) (cf. Lemma 1). Moreover, for each \( g \in \mathcal{D}_{a+k} \), the mapping \( \varphi \to T_a(\varphi)g \) of \( C^k[0, 1] \) into \( L_p(0, 1) \) is continuous, since

\[
T_a(\varphi)f = J^{-(a+k)}T_{-a}(\varphi)h
\]

for \( g = J^{-a-k}h \) with \( h \in L_p(0, 1) \). Q.E.D.
In particular, $W_k(T_a; p)$ is dense in $L_p(0, 1)$ for $Re \alpha < 0$ and $k \geq 0$ arbitrary. For $Re \alpha \geq 1$, we get the "other" extreme.

**Theorem 11.** For $\alpha \in \mathbb{C}$ with $Re \alpha \geq 1$ and $1 < p < \infty$, $W_k(T_a; p) = (0)$ if $k < [Re \alpha]$. The same is true for $p = 1$ if $\alpha$ is an integer.

**Proof.** If $\alpha$ is an integer, this is a trivial consequence of Lemma 3 and Leibnitz' formula.

Suppose then that $1 < p < \infty$, and that $f \in W_k(T_a; p)$ for some fixed $k < m = [Re \alpha]$. As in the proof of Lemma 7, we apply the Phragmén-Lindelöf principle in the strip $0 \leq Re \zeta \leq Re \alpha$ to the function $\Phi(\zeta) = \langle e^{\zeta^2 \varphi(T_a)} f, g \rangle$ where $\varphi$ is a polynomial and $g \in L_q(0, 1)$ (both fixed). We then obtain that $f \in W_k(T_a; p)$ for all $\zeta$ in the strip, hence in particular for $\zeta = m$. Since $k < m$, we conclude that $f$ is the null function.

4. Similarity and spectrality.

**Lemma 12.** Let $\alpha \in \mathbb{C}$ and $1 < p < \infty$. Then every $s \in [0, 1) = \sigma(T_a) \backslash \{1\}$ is an eigenvalue of $T_a$ for $Re \alpha \geq 1$ (Re $\alpha > 1$ or $\alpha = 1$ if $p = 1$), while $T_a$ has no eigenvalue for $Re \alpha \geq 0$ (Re $\alpha > 0$ or $\alpha = 0$ if $p = 1$).

**Proof.** Let $C_s$ denote the characteristic function of the interval $[s, 1)$, $0 \leq s < 1$. One verifies easily that $C_s$ is an eigenvector of $T^{-1}$ corresponding to the eigenvalue $s$ (for $1 \leq p < \infty$).

By Lemma 1, (3),

$$T^{-1}J^a^{-1}C_s = J^a^{-1}T^{-1}C_s = sJ^a^{-1}C_s,$$

i.e. $J^a^{-1}C_s$ (which is in $L_p(0, 1)$ for $\alpha$ as in the first statement of the lemma) is an eigenvector of $T_a$ corresponding to the eigenvalue $s$.

Next, suppose $T_a g = \lambda g$ for $g \in L_p(0, 1)$ and $\lambda \in \mathbb{C}$. If $Re \alpha \geq 0$ (Re $\alpha > 0$ or $\alpha = 0$ if $p = 1$), we may apply $J^a$ on both sides of this equation; by Lemma 1, (3), we obtain

$$MJ^a g = \lambda J^a g.$$

Since $M$ has no eigenvector $\neq 0$ and $J^a$ is one-one, it follows that $g$ is the zero element.

Let $\alpha, \beta \in \mathbb{C}$. By Lemma 2, $T_a$ and $T_\beta$ are similar if $Re \alpha = Re \beta$ (and $1 < p < \infty$). On the other hand, since the $C^k$-classification and the point spectrum are similarity invariants, it follows from Lemmas 3, 5 and 12 that $T_a$ and $T_\beta$ are not similar if $\alpha$ and $\beta$ are distinct integers (for $1 \leq p < \infty$).

**Conjecture.** For $1 < p < \infty$ and $\alpha, \beta \in \mathbb{C}$, $T_a$ and $T_\beta$ are similar if and only if $Re \alpha = Re \beta$. (By Lemma 2, it would suffice to verify that $T_a$ and $T_\beta$ are not similar if $\alpha$ and $\beta$ are distinct real numbers.)

**Proposition 13.** Let $\alpha, \beta \in \mathbb{C}$ and $1 < p < \infty$. Then $T_a$ and $T_\beta$ (acting in $L_p(0, 1)$) are not similar if $[Re \alpha] \neq [Re \beta]$. 

Proof. Assume, without loss of generality, that $\text{Re } \alpha < \text{Re } \beta$. If either $0 \leq \text{Re } \alpha$ or $\text{Re } \beta \leq 0$, this follows from Theorem 6 and the similarity invariance of the $C^k$-classification. If $\text{Re } \alpha < 0 < 1 \leq \text{Re } \beta$, $W_0(T_\alpha)$ is dense in $L_p(0, 1)$ (Theorem 10) while $W_0(T_\beta) = (0)$ (Theorem 11). Thus $T_\alpha$ and $T_\beta$ are not similar.

If $\text{Re } \alpha \leq -1 < \text{Re } \beta$, every $s \in [0, 1)$ is an eigenvalue of $T_\alpha$, while $T_\beta$ has no eigenvalue (Lemma 12), and the conclusion follows from the similarity invariance of the point spectrum. Q.E.D.

We next discuss the spectrality of $T_\alpha$ in Dunford’s sense [1].

**Lemma 14.** Let $T$ be a bounded spectral operator with real spectrum, acting in the Banach space $X$. Let $T = S + N$ be its canonical decomposition (cf. [1]). Then:

(a) If $W_k(T)$ is dense in $X$ for some integer $k \geq 0$, then $T$ is of finite type $\leq k$ (i.e., $N^{k+1} = 0$).

(b) If $T$ is of finite type $k$, then $W_j(T) \neq (0)$ for all $j \geq 0$; in fact, $W_j(T) \supseteq \mathcal{R}(N^{k-j})$ for $j = 0, \ldots, k-1$, and trivially $W_k(T) = X$ for $j \geq k$.

Proof. Fix a compact interval $\Delta \supseteq \sigma(T)$. Let $S(\cdot)$ be the $C$-operational calculus for $S$ (defined on $C(\Delta)$), and let $\|S(\cdot)\|$ be its norm.

(a) Let $x \in W_k(T)$. The function $e^{itN}x \ (z \in \mathbb{C})$ is entire of order one and minimal type (since $N$ is a quasi-nilpotent operator). For $z = t \in \mathbb{R}$, we have:

$$\|e^{itN}x\| \leq \|S(\cdot)\| \|e^{itT}x\| \leq \|S(\cdot)\| \|x\|_{k} \|\varphi_{1,k,\Delta}\|,$$

where $\varphi_{k}(s) = e^{its}$, $t, s \in \mathbb{R}$.

Thus $\|e^{itN}x\| = O(|t|^k)$, and therefore $e^{itN}x$ is a polynomial of order $\leq k$ by Theorem 3.13.8 in [3]. Hence $N^{k+1}x = 0$ for each $x \in W_k(T)$, and it follows that $N^{k+1} = 0$ since $W_k(T)$ is dense in $X$.

(b) We have $N^{k+1} = 0$ and $N^k \neq 0$. The analytic operational calculus for $T$ takes the form (cf. [1]):

$$T(\varphi) = \sum_{m=0}^{k} S(\varphi^{(m)})N^m/m!$$

If $x \in \mathcal{R}(N^{k-j})$, say $x = N^{k-j}y$ with $y \in X(0 \leq j < k)$, then

$$T(\varphi)x = \sum_{m=0}^{j} S(\varphi^{(m)})N^my/m!$$

In particular, $\|p(T)x\| \leq \|S(\cdot)\| \max_{0 \leq n \leq j} \|N^ny\| \|p\|_{1,\Delta}$ for any polynomial $p$, i.e. $x \in W_j(T)$. Q.E.D.

For simplicity, we state the following result for $1 < p < \infty$, although part of the conclusion remains valid for $p = 1$.

**Proposition 15.** Let $1 < p < \infty$. Then $T_\alpha$ is spectral for $\text{Re } \alpha = 0$, and is not spectral for $|\text{Re } \alpha| \geq 1$.

Proof. The first statement is a trivial corollary of Lemma 2.
By Theorem 6, $T_a$ is of class $C^n$ if $n \geq |\text{Re } a|$. Thus, if $T_a$ were spectral, it should be of finite type by Lemma 14(a). In particular, its point spectrum should be at most countable by [2, Theorem 1, p. 56]. But this contradicts Lemma 12 if $\text{Re } a \leq -1$. Also all $W_j(T_a)$ ($j \geq 0$) should be nontrivial by Lemma 14(b), contradicting Theorem 11 if $\text{Re } a \geq 1$. Thus $T_a$ is not spectral for $|\text{Re } a| \leq 1$.

5. Remarks. It is interesting to regard the results of this paper as statements about the operators $\alpha^{-1}T_a = J + \alpha^{-1}M$ ($0 \neq \alpha \in C$), which are perturbations of $J$ by a scalar operator of arbitrarily small norm. Thus, if $\alpha$ and $\beta$ are nonzero complex numbers, the following assertions can be made (for $1 < p < \infty$):

(a) If $\|\text{Re } \alpha\| \neq \|\text{Re } \beta\|$, $J + \alpha^{-1}M$ and $J + \beta^{-1}M$ belong to distinct $(C^k)$-classes, although they differ only by the scalar operator $(\alpha^{-1} - \beta^{-1})M$, which is of arbitrarily small norm. This shows that the commutativity hypothesis in [5, Corollary 5.6] cannot be replaced by a restriction on the norm of the perturbing scalar operator.

(b) The perturbations $J - \alpha^{-1}M$ and $J + \alpha^{-1}M$ have respectively a dense and a trivial semisimplicity manifold, a "pure" point spectrum (up to the right end point of the spectrum $[0, \alpha^{-1}]$) and no point spectrum.

(c) The perturbations $J + \alpha^{-1}M$ and $J + \beta^{-1}M$ are not similar if $\|\text{Re } \alpha\| \neq \|\text{Re } \beta\|$.

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