ON UNIFORM SPACES WITH QUASI-NESTED BASE

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Introduction. Uniform spaces with a *nested* base (briefly "*nested spaces*") were largely studied under the "disguise" of Fréchet's "espaces à écart" [6], [7], [1], [3], [5]. That the two kinds of spaces are actually the same was proved by Colmez [3]. Interesting cases of nonmetrizable nested spaces were studied by L. W. Cohen and C. Goffman [2] who extended a generalized version of Baire's theory of category to nested spaces with an additional property, stated in their Axiom 4, which we call *pseudocompleteness*. Scattered statements on nested spaces are also found in [10, p. 204, Problems C, D], [11], [13] etc.

In the present note we consider a more general class of spaces which we call *quasi-nested* (cf. §1) and which include, as a special case, also those spaces which *admit an infinite cardinal* (cf. Isbell [9, p. 133]). We study the structure of such spaces by introducing what we call the *upper* and *lower types* of a uniform space. Our results include a strengthening of Isbell's Propositions 26 and 27 of [9, p. 133] and of a theorem due to Doss [5], by extending them, in a certain sense, to quasi-nested spaces. It is also our aim to investigate those topological conditions which imply the uniform metrizability of a quasi-nested space. Such conditions are, e.g., separability, the strong (i.e. hereditary) Lindelöf property, total boundedness, and some generalized versions of these properties. Stronger results will be obtained for pseudocomplete spaces which have "few" isolated points (see §1, VIII below); we regard these as the main object of our study. Some examples will be given in §4 to show that the assumptions of pseudocompleteness and absence of "too many" isolated points are essential. Finally, in §5, we give some remarks on Baire's theory of category, for quasi-nested spaces. We are indebted to the referee for many valuable suggestions, incorporated in our theorems.

1. **Preliminaries. Terminology and notation.** We shall use the terminology and notation of [10] with the following changes and additions:

I. (X, U) denotes a uniform space X, with U a given base for the uniformity $\bar{U}$ (i.e., filter of entourages) in X. For brevity, U is also called a base *of* X. Unless otherwise stated, X is separated, i.e. a Hausdorff uniform space.

II. By a *string* we mean a (in general transfinite) sequence of entourages

$$(1.1) \qquad U_0 \supseteq U_1 \supseteq \cdots \supseteq U_\alpha \supseteq \cdots \qquad (\alpha < \omega_\xi,\ U_\alpha \in \bar{U})$$

whose order type ω_ξ is a regular initial ordinal (so that it has no cofinal

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subsequences of type $<\omega_\xi$), and which satisfies:

(1.2) $$U_{\alpha+1} \circ U_{\alpha+1}^{-1} \subseteq U_\alpha \quad \text{for all } \alpha < \omega_\xi \quad (\text{“}\textit{base property}\text{”}).$$

A string (1.1) is said to be *proper* if $\bigcap_{\alpha<\omega_\xi} U_\alpha$ is not an entourage. We denote by J the supremum of all order types ω_ξ of proper strings in X, and call J the (*upper*) *type of* X. Its cardinal is denoted by $\bar{J}$. If X is uniformly discrete (i.e., if the diagonal Δ of $X \times X$ is an entourage), we adopt the convention that $\bar{J}$ is the least cardinal exceeding that of $\tilde{\boldsymbol{U}}$, and J is the corresponding initial ordinal. We say that X is of *proper type* (or that J is *proper*) if $J=\omega_\mu$ for some *nonlimit* index μ(1). $(X, \boldsymbol{U}, J)$ denotes a space $(X, \boldsymbol{U})$ of type J.

III. $(X, \boldsymbol{U})$ is said to be *nested* if some string is a base for $\tilde{\boldsymbol{U}}$(2).

IV. $(X, \boldsymbol{U})$ is *quasi-nested* if it is nested or of type $J>\omega$.

V. We say that X *admits* a cardinal $\boldsymbol{m}$ if $\tilde{\boldsymbol{U}}$ is of power $\geqq \boldsymbol{m}$ and if the intersection of every $\boldsymbol{m}$ entourages is itself an entourage. The least cardinal *not* admitted by X is denoted by $\bar{I}$, and the corresponding initial ordinal I is called the *lower* type of X (it is always a *regular* initial ordinal). Clearly, X admits $\aleph_\xi$ iff $\omega_\xi < I$. Hence, if X is not uniformly discrete, one can always inductively construct a proper string of type I. It follows that $I \leqq J$. In a nested space necessarily $I=J$, but otherwise we may well have $\omega = I < J$ (e.g., take X=product of a metric space by a nested space of type $J>\omega$, to obtain this result). *It follows that every space which admits infinite cardinals is quasi-nested, with* $J>\omega$, but the converse is not true.

VI. $(X, \boldsymbol{U})$ is said to be *uniformly* (*topologically*) metrizable if its uniformity (resp., its topology) is compatible with some metric. Except for the uniformly discrete case, a quasi-nested space is uniformly metrizable iff $J=\omega$ (for a *nested* X, it suffices that $I=\omega$); cf. [10, p. 186]. Thus in all our metrization theorems the problem is to find those conditions which exclude the case $J>\omega$. It should be well noted that, even when using the notation $(X, \boldsymbol{U}, J)$, we do not assume the upper or lower type of X as *given* in advance. It is our aim in those theorems to characterize metrizability in terms of other topological conditions, not in terms of J or I (where there is no problem).

VII. Given a finite or infinite cardinal $\boldsymbol{m}$, we say that X has the *strong* (resp., *plain*) ($<\boldsymbol{m}$)-*covering property* if every open covering of any set $A \subseteq X$ (resp., of X itself) can be reduced to a subcovering of power *less* than $\boldsymbol{m}$. For $\boldsymbol{m}=\aleph_1$, this is the hereditary (resp., plain) Lindelöf property. The ($\leqq \boldsymbol{m}$)-covering property (strong or plain) is defined accordingly. The plain ($<\aleph_0$)-covering property is ordinary compactness. X is said to be *totally* [resp., σ-*totally*, ($<\boldsymbol{m}$)-*totally* or ($\leqq \boldsymbol{m}$)-*totally*] *bounded* if it has a base $\boldsymbol{U}$ such that, for every entourage $U \in \boldsymbol{U}$, X can be covered by finitely many (resp., countably many, less than $\boldsymbol{m}$, or $\leqq \boldsymbol{m}$) neighborhoods of the form $U[x]=\{y \mid (x, y) \in U\}$. If, instead, every neighborhood $V[x]$ ($V \in \boldsymbol{U}$) can be so covered, we replace the term "totally" by "locally" in these definitions. X is

(1) Note that if J is proper, there must be a *proper string of type* J.

(2) It suffices that some base of $\tilde{\boldsymbol{U}}$ is *linearly* ordered by $\supset$ (cf. [8, p. 142]).

said to be ***m*-*compact*** if every subset of power $\geq m$ has an accumulation point in X. We say that X is ***m*-*separable*** if it has a dense subset of power $\leq m$; for $m=\aleph_0$, this is ordinary separability.

VIII. X is said to have *few* isolated points if at least one neighborhood $U[x]$ is free of such points.

IX. We often say "clopen" instead of "both closed and open". "$\mathfrak{U}$-open" means "open under the topology generated by the uniformity $\mathfrak{U}$".

X (suggested by the referee). Given a uniform space (X, U, J), we denote by $\mathfrak{U}^0$ the set of all entourages $U \in \mathfrak{U}$ such that $U = U_0$ for some string $\{U_\alpha\}$ of type $>\omega$. By the definition of J, such strings exist if $J > \omega$, and then, as is easily seen, $\mathfrak{U}^0$ is a (possibly nonseparated) uniformity on X, coarser (smaller) than $\mathfrak{U}$. In particular, $\mathfrak{U}^0$ is closed under finite intersections; for if $\{U_\alpha\}_{\alpha<\omega_\xi}$ and $\{V_\alpha\}_{\alpha<\omega_\eta}$ are strings of type $>\omega$, so is $\{U_\alpha \cap V_\alpha\}_{\alpha<\omega_\xi}$ if $\xi \leq \eta$. We call $\mathfrak{U}^0$ the *reduced* uniformity (relative to $\mathfrak{U}$). If $I > \omega$, then $\mathfrak{U}^0 = \mathfrak{U}$; for, in this case, *every* entourage $U \in \mathfrak{U}$ initiates an (inductively constructed) string of type $>\omega$, and thus $\mathfrak{U} \subseteq \mathfrak{U}^0 \subseteq \mathfrak{U}$. In nested spaces and, more generally, in those with $I = J$, we always have $\mathfrak{U}^0 = \mathfrak{U}$. Note that $\mathfrak{U}^0$ contains all strings of type $>\omega$, contained in $\mathfrak{U}$; and if such a string is proper in $(X, \mathfrak{U})$, it is so in $(X, \mathfrak{U}^0)$ as well. Thus, if $\omega < J$ and $\Delta \notin \mathfrak{U}$, there is a proper string $\{U_\alpha\} \subseteq \mathfrak{U}^0$, of type $>\omega$.

Most of our theorems on quasi-nested spaces could be proved by a reduction to the nested case (referee's remark). However, to the best of our knowledge, our results are not found, in full generality, even in the literature on nested spaces. Since the proof in the general case is not much longer than in the nested case, we prefer to prove them for quasi-nested spaces right from the start.

2. **Some theorems on quasi-nested spaces in general.** We begin by generalizing a theorem due to Doss [5, pp. 111–117](3). This will also strengthen Isbell's Propositions 26 and 27 [9, p. 133]. A further strengthening will result in 2.2.

2.1. *If* (X, U, J) *is not uniformly discrete and if* $J > \omega$ (resp., $I > \omega$), *then the reduced uniformity* $\mathfrak{U}^0$ (resp., $\mathfrak{U}$ *itself*) *has a base* V *such that*:

(a) *For each entourage* $V \in V$, *the neighborhoods* $V[x]$, $x \in X$, *are both* $\mathfrak{U}^0$-*open and* $\mathfrak{U}^0$-*closed* (*hence also* "$\mathfrak{U}$-*clopen*"), *with* $V[x] = V[y]$ *iff* $V[x] \cap V[y] \neq \varnothing$, *thus forming a partition of* X *into disjoint sets* $V[x]$ (*the* "V-*induced partition*").

(b) *For each cardinal* $m < \bar{J}$, *there is an entourage* $V \in V$ *which induces a partition of* X *into* m *or more disjoint sets* $V[x]$, *and all such entourages* (*for a fixed* m) *constitute another base, equivalent to* V *and satisfying* (a).

(c) *For each ordinal* $\mu < J$, *there is a proper string* $\{V_\nu\} \subseteq V$, *of type* $>\mu$. *If* J *is proper*, V *also contains a proper string of type* J(4).

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(3) Doss considers only Fréchet's "espaces à écart", and constructs the "clopen" base V of our Theorem 2.1 only for the *topology* of X, without obtaining its *uniform* properties and the cardinalities of the partitions described in 2.1. Our proof is also considerably shorter.

(4) By a "proper string" we mean one that is proper in both (X, U) and $(X, \mathfrak{U}^0)$.

(d) *If X is nested, V can be made a wellordered base for all of $\tilde{U}$ (under $\supset$), and then the partitions induced by all $V \in \boldsymbol{V}$ are wellordered by refinement.*

Proof. As $J > \omega$, there is a strictly decreasing proper string $\{U_\alpha\} \subseteq \tilde{U}^0$, of some type $\omega_\xi > \omega$. By (1.2), $\{U_\alpha\}$ is a (wellordered) base for some coarser uniformity $\tilde{V} \subseteq \tilde{U}^0 \subseteq \tilde{U}$ for X. As a first step, we shall construct another base *for* $\tilde{V}$, satisfying (a). Let J' be the set of all *limit* ordinals $> \omega_\xi$. Then for each (fixed) $\gamma \in J'$, the entourages U_α $(\nu \leqq \alpha < \nu + \omega)$ form a base for a still coarser nonseparated uniformity $\tilde{U}^\nu$ for X [indeed, they inherit the base property (1.2) from the string $\{U_\alpha\}$]. Under the $\tilde{U}^\nu$-topology, the closure $\bar{x}^\nu$ of a singleton $\{x\}$ is given by the well known formula (cf. [12, p. 9]):

$$\bar{x}^\nu = \bigcap_\alpha \{U_\alpha[x] \mid \nu \leqq \alpha < \nu + \omega\}. \tag{2.1.1}$$

As $\bar{x}^\nu$ is closed under the $\tilde{U}^\nu$-topology, it is certainly so under the finer $\tilde{U}^0$ and $\tilde{U}$-topologies. Moreover, onepoint closures are either identical or disjoint under any uniform topology. Thus, for all $y \in \bar{x}^\nu$, $\bar{x}^\nu = \bar{y}^\nu = \bigcap_\alpha \{U_\alpha[y] \mid \nu \leqq \alpha < \nu + \omega\} \supseteq U_{\nu+\omega}[y]$. Hence $\bar{x}^\nu$ is also $\tilde{U}^0$-open (each $y \in \bar{x}^\nu$ being an *interior* point of $\bar{x}^\nu$). Setting $V_\nu = \{(x, y) \mid y \in \bar{x}^\nu\}$, $\nu \in J'$, we have $V_\nu \supseteq U_{\nu+\omega}$ and $V_\nu \subseteq U_\alpha$ for $\nu \leqq \alpha < \nu + \omega$, as easily follows from (2.1.1). Hence the entourages V_ν $(\nu \in J')$ form a new base $\boldsymbol{V}'$ for the uniformity $\tilde{V} \subseteq \tilde{U}^0$ and, by construction, the clopen disjoint sets $\bar{x}^\nu = V_\nu[x]$ form a partition of X for each (fixed) $\nu \in J'$, as desired. Moreover, as the string $\{U_\alpha\}$ is proper and strictly decreasing, so is $\{V_\nu\}$; the order type of both is ω_ξ; and each U_α contains some V_ν. As the V_ν decrease, the V_ν-induced partitions grow strictly finer, i.e., each consecutive partition increases the number of the sets $V_\nu[x]$ by one at least. Since there are $\aleph_\xi$ entourages V_ν, the number of these sets must eventually reach any cardinal $\boldsymbol{m} < \aleph_\xi$ for some V_ν $(\nu \geqq \nu_0)$; all such V_ν together form another base for $\tilde{V}$.

It is clear that this process works also if the string $\{U_\alpha\}$ is not proper (then also the resulting string $\{V_\nu\}$ is not). Thus, from each string $\{U_\alpha\}$ of type $> \omega$, we obtain a new string $\{V_\nu\}$, of the same order type, satisfying (a). The members of all strings $\{V_\nu\}$ so obtained, when combined, constitute a base $\boldsymbol{V}$ for the uniformity $\tilde{U}^0$; indeed, by the definition of $\tilde{U}^0$ (§1, IX), each entourage $U \in \tilde{U}^0$ is in some string $\{U_\alpha\}$ of type $> \omega$, and hence contains some $V_\nu \in \boldsymbol{V}$. By construction, each $V \in \boldsymbol{V}$ satisfies (a). Property (c) easily follows from the definition of J (§1, II) and footnote 1 to it. Property (b) is likewise satisfied. Indeed, if J is proper, there is a proper string $\{V_\nu\} \subseteq \boldsymbol{V}$ of type J. If J is not proper, each cardinal $\boldsymbol{m} < \bar{J}$ is exceeded by some cardinal of the form $\aleph_{\mu+1} < \bar{J}$. In both cases there are $V \in \boldsymbol{V}$ satisfying (b), as was explained above. Furthermore, if X is nested then, as is easily seen, every proper string is a base for $\tilde{U}^0 = \tilde{U}$; thus any one of the (proper) strings $\{V_\nu\}$ can serve as the required base $\boldsymbol{V}$, satisfying (d) as well. Finally, if $I > \omega$ then, as was noted in §1, IX, $\tilde{U}^0 = \tilde{U}$, and thus $\boldsymbol{V}$ is a base for all of $\tilde{U}$. This completes the proof.

We note that, even if $\tilde{U}^0 \neq \tilde{U}$, the assumptions of 2.1 suffice for the construction of a proper strictly decreasing string $\{V_\nu\}$ satisfying 2.1(a), and that such a $\{V_\nu\}$ can be chosen to be of order type greater than any $\mu < J$. We use this fact in the proof of 2.2 below.

Henceforth V will always denote a base (for $\bar{U}^0$ or $\bar{U}$, as the case may be) satisfying 2.1(a) (if $J > \omega$), or a countable base (if $J = \omega$). It will be called a *standard base*.

2.2. *If (X, U, J) is not uniformly discrete and if $\bar{J} > \aleph_\mu$, then X contains $\aleph_{\mu+1}$ or more clopen disjoint neighborhoods. Hence it has a discrete subspace of power $\aleph_{\mu+1}$ at least*(5).

Proof. Using the base V of 2.1, construct a proper strictly decreasing string $\{V_\nu\} \subseteq V$ of type $\omega_\xi > \omega_\mu$, satisfying 2.1(a). There are two cases:

(i) If some point $x_0 \in X$ has no *smallest* neighborhood of the form $V_\nu[x_0]$ (with V_ν from that string), we may assume that the sequence of neighborhoods $V_\nu[x_0]$, $\nu < \omega_\xi$, is *strictly* decreasing [indeed, as ω_ξ is a *regular* initial ordinal, there are no cofinal subsequences of type $< \omega_\xi$; thus, dropping repeating terms $V_\nu[x_0]$ (if any), we get a strictly decreasing subsequence of the same type ω_ξ]. Then the sets $N_\nu = V_\nu[x_0] - V_{\nu+1}[x_0]$, $\nu < \omega_\xi$, are nonempty, disjoint and clopen. Their number is $\aleph_\xi \geqq \aleph_{\mu+1}$. Thus they are the desired neighborhoods.

(ii) If, on the other hand, each $x \in X$ has a smallest neighborhood of the form $V_\lambda[x]$ $(\lambda < \omega_\xi)$, then X is covered by such neighborhoods. Moreover, their minimality along with 2.1(a) easily implies that any two such neighborhoods are either identical or disjoint. The subsequence $\{V_\lambda\}$ of those entourages that were used to form the minimal neighborhoods $V_\lambda[x]$, cannot be of type $< \omega_\xi$; for, otherwise, the wellordering of $\{V_\nu\}$ would yield some V_ν smaller than *all* V_λ; but then the $V_{\nu+1}$-induced partition of X would be strictly finer than the one formed by the *minimal* sets $V_\lambda[x]$, which is a contradiction. Thus the number of the V_λ is $\aleph_\xi$, i.e. at least $\aleph_{\mu+1}$. A fortiori, so is the number of the disjoint minimal sets $V_\lambda[x]$. This completes the proof.

NOTE 1. In case (ii) we have even obtained a *covering* of X by more than $\aleph_\mu$ disjoint clopen neighborhoods. However, this fails in the general case (a counterexample will be given in §4).

2.3. *The lower type I of (X, U, J) never exceeds the lower type I' of any nondiscrete subspace X' of X. Hence, if X is quasi-nested and $I = J$, then the uniform metrizability of X' is equivalent to that of all of X.*

Proof. Every cardinal $\aleph_\mu$ admitted by X is also admitted by X'. Indeed, if the intersection of $\aleph_\mu$ entourages is an entourage in X, it becomes one in X' when all is relativized to X'. We must only verify that X' does have at least $\aleph_\mu$ distinct entourages. This is achieved by constructing (inductively) an $\omega_{\mu+1}$-type sequence of distinct neighborhoods $U_\alpha[x_0]$ of some nonisolated point $x_0 \in X'$ (which is possible

(5) This strengthens Isbell's Proposition 27 and the second clause in Proposition 31 in [9, p. 133 ff]. Isbell proves only the existence of a discrete subspace of power $\aleph_\mu$, under the stronger assumption that $\bar{I} > \aleph_\mu$.

since x_0 has no *smallest* neighborhood in X'). Then, a fortiori, the relativized entourages U_α are distinct, hence $\aleph_{\mu+1}$ in number. Thus, indeed, $\boldsymbol{m}<\bar{I}$ implies $\boldsymbol{m}<\bar{I}'$, whence $I \leqq I'$. If, further, X' is metrizable, then $\omega = J' \geqq I' \geqq I$, and the second assertion of 2.3 follows.

NOTE 2. For *nested* spaces, 2.3 holds also under the weaker assumption that X' is not *uniformly* discrete (and then necessarily $I'=I=J=J'$). Indeed, by nestedness, $I=J$ and $I'=J'$; and some proper I-type string $\{U_\alpha\}$ is a base for X. When relativized, it is a base for X', *with no smallest term in it*. Thus, deleting repeated terms (if any), we still are left with a cofinal string of type $I'=I$ in X', as required.

The structural propositions proved above yield several immediate metrization corollaries which, despite their simplicity, seem not yet to be known. We combine some of them in one theorem, sketching the proof briefly:

2.4. *Each of the following conditions entails the uniform metrizability of a quasi-nested space* (X, U, J):

(a) *X has the strong Lindelöf property, or is separable, or totally bounded* (*e.g.*, *compact*); *or locally bounded and not discrete, with* $J=I$.

(b) *For some cardinal* $\boldsymbol{m}<\bar{J}$, *X is* $\boldsymbol{m}$*-separable or* $(<\boldsymbol{m})$*-totally bounded, or has the plain* $(<\boldsymbol{m})$*-covering property or its strong* $(\leqq \boldsymbol{m})$*-variety.*

(c) *X has the strong* $(<\bar{I})$*-covering property, or the strong* $(<\bar{J})$*-covering property* [*provided that* $\bar{J}=\aleph_\mu$ *for a nonlimit index* μ, *i.e.*, *J is proper*].

Moreover, in cases (b) *and* (c), *X is necessarily uniformly discrete.*

Proof. Suppose that X is not uniformly metrizable, hence certainly not uniformly discrete. Then $\bar{J}>\aleph_0$, and 2.2 yields a family $\{N_\nu\}$ of $\aleph_1$ disjoint open-closed neighborhoods. This, however, violates separability and the strong Lindelöf property, since $\bigcup N_\nu$ is a set with a disjoint (hence irreducible) open covering of power $\aleph_1$(6). Similarly 2.1(b) shows that X cannot be totally bounded. Properties (b) and (c) in 2.4 are excluded by an analogous argument, and here, as is easily seen, the desired contradiction already arises if one assumes that X is *not uniformly discrete* (instead of "not metrizable"); this proves the *last* assertion in 2.4. Finally, if X is locally bounded and not discrete, with $J=I$, some neighborhood $V[x_0]$ constitutes a nondiscrete totally bounded subspace. By (a), its type (and certainly its lower type I') cannot exceed ω. Thus 2.3 yields $\omega=I'\geqq I=J$, and X is uniformly metrizable. Q.E.D.

Simultaneously, the last part of the proof also yields:

2.5. (a) *If a uniform space* (X, U, J) *has a nondiscrete*(7) *subspace with one of the properties* 2.4(a), *then the lower type I of X is* ω; *i.e.*, *X cannot admit infinite cardinals.* (b) *If further X is quasi-nested and* $J=I$, *then X is uniformly metrizable. This is the case, in particular, if X is quasi-nested with* $J=I$ *and has a countable neighborhood base at some nonisolated point* x_0.

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(6) In the separable case, there also is a simple proof which does not use 2.1 and 2.2 at all; cf. [14, Footnote 13].

(7) Resp., not *uniformly* discrete (if X is nested).

[In fact, in the latter case there is an ω-type sequence $x_n \to x_0$ $(x_n \neq x_0)$; and $\{x_0, x_1, \ldots\}$ is a nondiscrete compact subspace; so 2.4(a) applies.] Hence:

2.6. *Topological and uniform metrizability are equivalent for nondiscrete quasi-nested spaces with $J = I$.* (This is immediate from 2.5(b), second clause.)

It is also easily seen that no uniform space can have $\boldsymbol{m}$-compact subspaces of power $\geqq \boldsymbol{m}$ for any infinite cardinal $\boldsymbol{m} < \bar{I}$ (note that $\bar{I} > \boldsymbol{m} \geqq \aleph_0$ *implies* that X is quasi-nested). We omit the obvious proof.

A further immediate corollary of 2.1 is this:

2.7. *If (X, U, J) is a quasi-nested space with $J > \omega$, then its reduced uniformity $\tilde{U}^0$*(8) *is generated by a family of pseudometrics d_ν, each taking only the values* 0 *and* 1, *so that each pseudometric space (X, d_ν) has a discrete separated quotient space. If further X is nested, these d_ν form a nondecreasing J-type sequence* [*i.e.*, $\nu \leqq \lambda < J$ *implies* $d_\nu(x, y) \leqq d_\lambda(x, y)$ *for* $x, y \in X$], *and generate all of* $\tilde{U}$.

Proof. If X is nested, some string of type J is its base V, satisfying 2.1. We then define d_ν by $d_\nu(x, y) = 0$ if $y \in \bar{x}^\nu = V_\nu[x]$, and $d_\nu(x, y) = 1$ if $y \notin \bar{x}^\nu$ (we use the notation of the proof of 2.1). Then, as is readily seen, the pseudometrics d_ν have all the required properties. If X is only quasi-nested, the proof of 2.1 shows that $\tilde{U}^0$ is the union (i.e., the supremum under inclusion order) of coarser uniformities $\tilde{V}$, all with wellordered bases of the form $\{V_\nu\}$. Each of these, in turn, is generated by a sequence of pseudometrics d_ν, as described above. The union of all these sequences then is a family of pseudometrics that generates all of $\tilde{U}^0$ (or all of $\tilde{U}$, if $I > \omega$). Q.E.D.

Note 3. All this shows that quasi-nested spaces have a structure similar to that of nested spaces. This justifies the name "quasi-nested space".

The cardinality assertions in 2.1(b) can be strengthened if X is pseudocomplete (cf. §3) and has "few" isolated points. In the next section we proceed to obtain these strengthened properties of X. This, in turn, will lead to a certain relaxation of the metrization conditions (a)–(c) in 2.4. The investigation of pseudocomplete spaces will be based on the *lower* type I, rather than the upper type J, of X. In this connection, we shall use the notation (X, V, I) for a space (X, U, J) of lower type I, with a standard base V for $\tilde{U}^0$. We use the notation (X, V, J) wherever the upper type J is involved, too. We recall again that V is a base for *all* of $\tilde{U}$ if $I > \omega$. It is this case that we shall mainly deal with, and so the notation (X, V, I) is convenient.

3. **Pseudocomplete spaces.** A quasi-nested space (X, V, I) is said to be *pseudo-complete* (relative to the given standard base V) if we have $\bigcap_{\nu < \eta} V_\nu[x_\nu] \neq \varnothing$ for every decreasing sequence of neighborhoods $V_\nu[x_\nu]$ $(V_\nu \in V)$, of order type $\eta < I$. If this holds also for $\eta = I$, we say that X is *I-complete*(9). Finally, if this holds with

(8) Resp., its original uniformity $\tilde{U}$ (if $I > \omega$).

(9) Here, in general, V is only a base for the *reduced* uniformity $\tilde{U}^0$ on X (but we recall that $\tilde{U}^0 = \tilde{U}$ if $I > \omega$). Note that, for $I > \omega$, our "I-completeness" is weaker than De Groot's "subcompactness", relative to V (cf. [4]). For $I = \omega$, pseudocompleteness is trivial.

I replaced by some other (fixed) ordinal α, we say that X is *α-pseudocomplete* (resp., *α-complete*). Below, $\boldsymbol{c}$ denotes the power of the continuum ($2^{\aleph_0}$), and $|V|$ is the total number of the disjoint sets $V[x]$ in the V-induced partition of X (cf. 2.1(a)), for a given entourage $V \in \boldsymbol{V}$.

3.1. *If a quasi-nested space $(X, \boldsymbol{V}, I)$ is pseudocomplete, has few isolated points* (cf. §1, VIII) *and admits an infinite cardinal $\aleph_\mu$ ($\bar{I} > \aleph_\mu$), then, for some entourage $V \in \boldsymbol{V}$, the number $|V|$ of the disjoint open-closed sets $V[x]$ ($x \in X$) is not less than any of the cardinals $\boldsymbol{c}$, $\aleph_{\mu+1}$ and $2^{\boldsymbol{m}}$, where $\boldsymbol{m}$ is the least cardinal such that $\boldsymbol{m} < \bar{I} \leqq 2^{\boldsymbol{m}}$ (if such a cardinal exists). Moreover, all entourages V with that property form an equivalent base for the uniformity $\boldsymbol{\bar{U}}$ of X.*

Proof. By assumption, there is a neighborhood $V_0[x_0]$ free of isolated points. By the Hausdorff property, x_0 has no *smallest* neighborhood. Thus, for some entourage $V_1 \subset V_0$ ($V_1 \in \boldsymbol{V}$), $V_0[x_0]$ must split into smaller disjoint sets of the form $V_1[x]$. We fix exactly two such sets and call them the "*fixed sets of grade* 1". Inductively, we can define a strictly decreasing ω-type sequence of entourages $V_\nu \in \boldsymbol{V}$ and select for each $\nu < \omega$ exactly 2^ν disjoint "*fixed sets of grade* ν" in such a manner that they are of the form $V_\nu[x]$ and

(3.1.1) *each fixed set of grade ν contains exactly two fixed sets of grade $\nu+1$.*

Now consider all decreasing sequences of "fixed sets" of the form

$$V_0[x_0] \supset V_1[x_1] \supset \cdots \supset V_\nu[x_\nu] \supset \cdots \qquad (\nu < \omega) \tag{3.1.2}$$

where $V_\nu[x_\nu]$ is a fixed set of grade ν. By (3.1.1), there are precisely $2^{\aleph_0}$ such sequences. The intersections $\bigcap_\nu V_\nu[x_\nu]$ are disjoint for distinct sequences (3.1.2), and none is empty, by pseudocompleteness. As X admits $\aleph_0$, we can fix an entourage $V_\omega \subset \bigcap_{\nu<\omega} V_\nu$, and then choose a point $x \in \bigcap_\nu V_\nu[x_\nu]$ for each sequence (3.1.2), so that $V_\omega[x] \subseteq \bigcap_\nu V_\nu[x] = \bigcap_\nu V_\nu[x_\nu]$, by 2.1(a). The $2^{\aleph_0} = \boldsymbol{c}$ disjoint sets $V_\omega[x]$ so obtained will be called the "*fixed sets of grade* ω". This already proves our theorem for the case $\bar{I} \leqq \boldsymbol{c}$ (with $V = V_\omega$ and $\boldsymbol{m} = \aleph_0$)[10].

If $\bar{I} > \boldsymbol{c}$, we continue (transfinitely) the inductive definition of "fixed sets" as follows. For each of the $\boldsymbol{c}$ fixed sets of grade ω, we find an entourage $V \in \boldsymbol{V}$ for which that particular fixed set splits into at least two sets of the form $V[x]$. As X admits $\boldsymbol{c}$, the intersection of the $\boldsymbol{c}$ entourages so chosen contains an entourage $V_{\omega+1} \in \boldsymbol{V}$. Clearly, the $V_{\omega+1}$-induced partition is finer than all the preceding ones. Thus *each* fixed set of grade ω contains at least two sets of the form $V_{\omega+1}[x]$. We select exactly two such sets for each fixed set of grade ω and call the sets $V_{\omega+1}[x]$ so chosen the *fixed sets of grade* $\omega+1$. This process can be inductively continued and *cannot stop before an ordinal μ is reached such that* $\bar{I} \leqq 2^\mu$ (=number of the fixed sets of grade μ)[11]. Indeed, if $\mu = \lambda + 1$ and $2^\lambda < \bar{I}$, we can define the fixed sets

[10] Note that this case requires only ω-completeness.

[11] Here μ denotes also the cardinal of the ordinal number μ. If there is no $\mu < I$ with $\bar{I} \leqq 2^\mu$, induction reaches *all* $\mu < I$. To prove it, we now make the inductive assumption that "fixed sets" of all grades less than some μ have been defined, and show how to continue induction.

of grade $\lambda+1$ by the process described above, with $\omega+1$ replaced by $\lambda+1$. If however μ is a limit number, we define the fixed sets of grade μ as it was done for $\mu=\omega$, by considering sequences (3.1.2) *of type* μ (instead of ω). Thus, if there is an initial ordinal ω_ξ with $\aleph_\xi<\bar{I}\leqq 2^{\aleph_\xi}$, it will eventually be *reached*, and the number of the fixed sets of grade ω_ξ will reach $2^{\aleph_\xi}\geqq\bar{I}\geqq\aleph_{\mu+1}$. If however no such ω_ξ exists, then $2^m<\bar{I}$ for each $m<\bar{I}$ and, by 2.1(b), there is an entourage $V\in \mathbf{V}$ with $|V|\geqq 2^m$; in particular, $|V|\geqq 2^{\aleph_\mu}\geqq\aleph_{\mu+1}$. In either case our theorem is proved (the last assertion in 3.1 easily follows from the first). Q.E.D.

3.2. *If* $(X, \mathbf{V}, J)$ *is pseudocomplete and has few isolated points, with* $J=I$, *then each of the following conditions implies the uniform metrizability of* X:

(a′) X *is* $(<\mathbf{c})$*-totally or* $(<\mathbf{c})$*-locally bounded* (*e.g.*, σ*-locally bounded*);

(b′) X *has the plain* $(<\mathbf{c})$*-covering property* (*e.g.*, X *is a Lindelöf space*);

(c′) X *is* $\mathbf{c}$*-compact.*

Proof. If $\bar{J}=\bar{I}>\aleph_0$ then, by 3.1, X splits into $|V|\geqq\mathbf{c}$ disjoint sets $V[x]$, for some $V\in\mathbf{V}$, and so both (a′) and (b′) are violated. In particular, also the $(<\mathbf{c})$-local boundedness is violated. For, as was shown in the proof of 3.1, the partition into $|V|\geqq\mathbf{c}$ sets applies not only to X itself but also to any neighborhood $V_0[x_0]$, free of isolated points. Hence, for suitable $x_0\in X$ and $V_1\in\bar{\mathbf{U}}$, and for each $V_0\in\bar{\mathbf{U}}$ with $V_0\subseteq V_1$, there is such a V. Finally, (c′) is likewise violated; for, by choosing a point from each of $\mathbf{c}$ disjoint sets $V[x]$, we obtain a set of power $\mathbf{c}$, without accumulation points in X. Q.E.D.

NOTES. (1) If $J\neq I$, conditions (a′)–(c′) only imply that $I=\omega$. (2) By 2.3, it suffices that one of (a′)–(c′) hold for some pseudocomplete or open *subspace* of X, with few isolated points in it.

3.3. *If* $(X, \mathbf{V}, J)$ *is pseudocomplete, with few isolated points, and if I or J has the form* ω_μ, *with* μ *a nonlimit ordinal, then each of the following conditions implies the uniform metrizability of* X:

(a″) X *is* $(<\bar{I})$*-totally bounded* [*or* $(<\bar{I})$*-locally bounded, with* $I=J$];

(b″) X *is* $\bar{I}$*-compact or has the plain* $(<\bar{I})$*-covering property.*

Moreover, in case (a″) X *is necessarily totally* (*resp., locally*) *bounded, and in case* (b″), X *is necessarily compact.*

Proof. If $I<J$ then, by 2.1(b) with $m=\bar{I}$, X cannot satisfy (a″) or (b″); thus either condition implies $I=J$. Moreover, $I=J=\omega$; for, otherwise, by the assumed properness of J or I, we would have $\bar{I}=\aleph_{\mu+1}>\aleph_\mu$ for some μ, and so, by 3.1, X would split into at least $\aleph_{\mu+1}=\bar{I}$ disjoint sets $V[x]$ for some $V\in\mathbf{V}$, contrary to both (a″) and (b″); [the local case in (a″) is excluded by a similar argument as in the proof of 3.2]. Since $\bar{I}=\bar{J}=\aleph_0$, X is uniformly metrizable, and also the last assertion in 3.3 follows. Q.E.D.

The propositions proved so far yield the rather negative result that nonmetrizable quasi-nested spaces (especially pseudocomplete spaces with few isolated points) cannot have any of the "pleasant" topological properties ($a, b, c,\ldots$) specified in our metrization corollaries. As we show in another paper [15], even less restrictive properties, such as $\bar{J}$-separability or $(\leqq\bar{J})$-total boundedness are excluded in nonmetrizable pseudocomplete nested spaces with few isolated points, unless some form of the generalized continuum hypothesis is true. Thus no effective examples of such spaces can be produced; they can however be constructed if the requirement of pseudocompleteness or of "few isolated points" is dropped, or if the continuum hypothesis is assumed([12]).

4. **Examples.** The following examples show that the assumptions of pseudocompleteness and absence of "too many" isolated points are essential in propositions 3.1–3.3. Example (1) is a slight modification of Kelley's Problem C [10, p. 204]. In (2) we use some ideas of Cohen and Goffman [2, p. 270 ff.].

(1) Let X be the set of all ordinals $\leqq \omega_1$. For each $x \in X$ and $\alpha < \omega_1$, let $U_\alpha[x] = \{x\}$ if $x < \alpha$, and $U_\alpha[x] = \{y \in X \mid y \geqq \alpha\}$ if $x \geqq \alpha$. Equivalently, let $U_\alpha = \{(x, y) \mid x = y$ or $x, y \geqq \alpha\}$ and $U = \{U_\alpha \mid \alpha < \omega_1\}$. Then, as is easily seen, (X, U) is a nested space with $I = J = \omega_1$ (hence not metrizable); moreover, the base U is standard (i.e., satisfies 2.1), and the space is pseudocomplete relative to U; it is even J-complete and complete in the ordinary sense. Furthermore, X is $\aleph_1$-compact, σ-totally bounded and has the plain (but not the strong) Lindelöf property. The reason for the apparent failure of 3.1 and 3.2 is that X has only one nonisolated point ω_1, and there is no nonvoid subspace free of isolated points (in its relative topology). This example also shows that the strong covering property in 2.4(a,c) cannot be replaced by its plain variety. Finally, note that X contains $\aleph_1$ disjoint one-point neighborhoods but there is no *covering* of X by $\aleph_1$ disjoint neighborhoods (cf. Note 1, §2).

(2) Let S be the set of all nondecreasing ω-type sequences $\mathbf{x} = (x_1, x_2, \ldots)$ of ordinals $x_k \leqq \omega_1$, terminating in ω_1 i.e., with all but finitely many terms equal to ω_1. For every ordinal $\alpha < \omega_1$, let the entourage V_α consist of those pairs of sequences $(\mathbf{x}, \mathbf{y}) \in S \times S$ in which $\mathbf{x}$ and $\mathbf{y}$ have exactly the same terms *less* then α (while the remaining terms are arbitrary but $\geqq \alpha$). Thus if $\mathbf{x}$ contains exactly k terms less than α, $x_1, x_2, \ldots, x_k < \alpha$, then the neighborhood $V_\alpha[\mathbf{x}]$ consists of those $\mathbf{y} \in S$ which start with *the same* terms $x_1, \ldots, x_k$ and in which the remaining terms are $\geqq \alpha$; in particular, $V_\alpha[\mathbf{x}]$ contains the sequence $\mathbf{w} = (x_1, x_2, \ldots, x_k, \omega_1, \omega_1, \ldots)$ which we shall call the *main point* of $V_\alpha[\mathbf{x}]$. For a fixed $\alpha < \omega_1$, there clearly exist at most $\aleph_0$ distinct finite sequences $x_1 \leqq x_2 \leqq \cdots \leqq x_k < \alpha$; hence there are at most $\aleph_0$ distinct neighborhoods $V_\alpha[\mathbf{x}]$ (all disjoint, and each of them uniquely determined by its main point $\mathbf{w}$). Thus, setting $V = \{V_\alpha \mid \alpha < \omega_1\}$ and $J = I = \omega_1$, we obtain a nested space (S, V, J) of type $J = \omega_1$, with V a standard base. As is easily seen, S has no isolated points and is not metrizable, even though it is σ-totally bounded, as was shown above. The apparent failure of 3.1–3.3 is this time due to the fact that (S, V, J) is *not pseudocomplete* (even though it is complete in the ordinary sense). Indeed, setting $\mathbf{w}^k = (1, 2, \ldots, k, \omega_1, \omega_1, \ldots)$ it is immediate that $\bigcap_{k<\omega} V_k[\mathbf{w}^k] = \varnothing$.

([12]) Such examples are given in [15].

It is again of some interest that S has the plain (but not the strong) Lindelöf property. We briefly sketch a proof.

Suppose that S has an open covering $\{G_i\}$ which does not reduce to a countable one, and let S_n be the subspace of those $x \in S$ which have *at most n* terms other than ω_1. As $S = \bigcup_{n=1}^{\infty} S_n$, some S_{n_0} cannot be covered by $\aleph_0$ sets G_i. Let $w^0 \in S_{n_0}$ be the sequence with *all* terms equal to ω_1. As w^0 is in one of the G_i (call it G_0), there is $\alpha_0 < \omega_1$ such that $V_{\alpha_0}[w^0] \subseteq G_0$. Now, as was shown above, S_{n_0} splits into $\leqq \aleph_0$ disjoint subsets of the form $V'_{\alpha_0}[x] = S_{n_0} \cap V_{\alpha_0}[x]$. Hence at least one of $V'_{\alpha_0}[x]$ cannot be covered by $\aleph_0$ sets G_i. Let it be $V'_{\alpha_0}[w^1]$, with main point $w^1 = (x_1, x_2, \ldots, x_{k_1}, \omega_1, \omega_1, \ldots)$, $x_{k_1} < \alpha_0$, $w^1 \neq w^0$. Again, w^1 is in one of the sets G_i (call it G_1) and $V_{\alpha_1}[w^1] \subseteq G_1$ for some $\alpha_1 > \alpha_0$, so that the process can be repeated. Inductively, one obtains a sequence of "main points" $w^m \in S_{n_0}$ and a decreasing sequence of neighborhoods $V'_{\alpha_m}[w^{m+1}]$ $(m = 1, 2, \ldots)$ such that no $V'_{\alpha_m}[w^{m+1}]$ can be covered by $\aleph_0$ sets G_i, and w^m has *fewer* terms $\neq \omega_1$ than has w^{m+1}. This inductive process continues until the number of such terms $(\neq \omega_1)$ reaches n_0. But then the corresponding "main point" $w^{m+1} = (x_1, x_2, \ldots, x_{n_0}, \omega_1, \omega_1, \ldots)$ is the *unique* element of $V'_{\alpha_m}[w^{m+1}]$ as follows from the definition of S_{n_0}. Thus $V'_{\alpha_m}[w^{m+1}]$ must be covered by a single set G_i (contradiction!). This shows that S is, indeed, a Lindelöf space. Q.E.D.

These examples demonstrate that nonmetrizable quasi-nested spaces may still possess some "nice" topological properties, even if they have no isolated points. That they are also of interest in some other respects was sufficiently demonstrated by Cohen and Goffman in [2] (for *nested* spaces).

5. **Baire's categories.** We conclude with a few remarks on the Baire category theorem. De Groot [4] proved that a subcompact regular topological space X is an "$\boldsymbol{m}$-Baire space" for every cardinal $\boldsymbol{m}$. This very result shows, however, that De Groot's "subcompactness" is a very strong property which implies much *more* than the ordinary Baire theorem, and may be rather rare. For quasi-nested spaces, a variant of Baire's theorem can easily be obtained by using the notion of α-*completeness* as defined in §3. Indeed, we have:

5.1. *In an I-complete space (X, V, I), no open set $G \neq \varnothing$ is the union of $\bar{I}$ (or less) nowhere dense sets. If X is only α-complete for some $\alpha \leqq I$, the theorem is true with $\bar{I}$ replaced by the cardinal number of α.*

The proof runs on the same lines as that of the ordinary Baire theorem, on noting that the intersection of less than $\bar{I}$ open sets is itself an open set(13). This fact makes it also superfluous to consider De Groot's "$\boldsymbol{m}$-thin" sets instead of nowhere dense sets, as far as α-completeness is concerned $(\alpha \leqq I)$.

(13) Indeed, if $x \in \bigcap_{\nu<\eta} G_\nu$ $(\eta < I)$, with all G_ν open, we can find for each ν an entourage $V_\nu \in V$ such that $V_\nu[x] \subseteq G_\nu$. As $\eta < I$, $\bigcap_{\nu<\eta} V_\nu$ must contain an entourage $V \in V$. Then clearly $V[x] \subseteq \bigcap V_\nu[x] \subseteq \bigcap G_\nu$. Thus each $x \in \bigcap G_\nu$ is an interior point. [This fact was unnecessarily assumed as a *postulate* by Cohen and Goffman in the formulation of their Axiom 4.]

Finally, we note that nothing in our previous results will change if, in the definition of α-completeness, we replace decreasing sequences $V_\nu[x_\nu]$ by those in which *every* $V_\nu[x_\nu]$ *contains the closure of each* $V_\mu[x_\mu]$ $(\mu>\nu)$. [If $I>\omega$, this makes no difference at all since the $V_\nu[x]$ are closed by 2.1(a).] Furthermore, if X is a metric space, we adopt the additional convention that the standard base V consists of all entourages of the form $V_n=\{(x,y) \mid d(x,y)<1/n\}$, $n=1,2,\ldots$. With these conventions, it can easily be shown that the I-completeness of a metric space is equivalent to ordinary completeness, and thus 5.1 is a natural generalization of the ordinary Baire theorem.

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