UNBOUNDED NORMAL OPERATORS ON BANACH SPACES(1)

BY
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Introduction. In this paper a theory of unbounded normal operators on complex Banach spaces is developed which generalizes many features of the classical theory of maximal normal operators on Hilbert spaces. In particular the operators studied are closely related to the unbounded scalar type spectral operators introduced by Bade [1] and to a generalization of these operators defined here. The definition of these normal operators is given in terms of generators of strongly continuous groups of isometries contained in a commutative V^* algebra [6]. Many of the techniques used were suggested by and generalize results in [21, 5, and 19].

In this paper all Banach spaces ($\mathfrak{X}$, $\mathfrak{Y}$ etc.) are defined over the complex numbers C, and all operators are (not necessarily bounded, closed, or densely defined) linear transformations with domain and range contained in the same Banach space.

An operator R is called self conjugate iff iR generates a strongly continuous group of isometries (cf. [19]). This definition generalizes both the notion of an unbounded self adjoint operator on Hilbert space and a concept introduced by I. Vidav [21] for elements in an arbitrary Banach algebra. A norm closed commutative subalgebra $\mathfrak{A}$ of the algebra $[\mathfrak{X}]$ of all everywhere defined bounded operators on $\mathfrak{X}$ is called a commutative V^* algebra if every element $S \in \mathfrak{A}$ can be written as $R+iJ$ with R and J self conjugate elements of $\mathfrak{A}$ [6]. In §2, it is shown that the weak closure of such an algebra is again a commutative V^* algebra. Also if $\mathfrak{X}$ is weakly complete a weakly closed commutative V^* algebra contains a self conjugate resolution of the identity for each of its elements. A commutative V^* algebra with the same property is also constructed when $\mathfrak{X}=\mathfrak{Y}^*$.

The numerical range of an operator on a Banach space is defined in a way resembling that of [17] but more easily applied to unbounded operators. Several criteria are given in §3 for a densely defined operator with real numerical range to have a self conjugate closure. Self conjugate operators R and J are defined to commute iff the groups generated by iR and iJ commute in the ordinary sense (cf. [20, p. 647]). It is shown that this definition agrees with other ideas of

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commutativity when they apply, and that the closure of a real linear combination of commuting self conjugate operators is necessarily self conjugate, hence, in particular, densely defined.

A possibly unbounded operator is called normal iff it can be written as $R+iJ$ where R and J are self conjugate operators with the groups generated by iR and iJ contained in a common commutative V^* algebra. In §4 it is shown that normal operators are necessarily closed and densely defined, and a number of concrete criteria for normality are developed.

Section 5 expounds a theory of integration for unbounded functions with respect to a spectral measure on $\mathfrak{X}$ which is merely countably additive in some weak topology defined by a total family Γ of linear functionals in $\mathfrak{X}^*$. In terms of this integral, unbounded scalar type operators on $\mathfrak{X}$ of class Γ are defined in a fashion analogous to that of [1] and [10]. It is then shown that the adjoint S^* of any normal operator S on $\mathfrak{X}$ is scalar of class $\mathfrak{X}$ on $\mathfrak{X}^*$ and that S is itself scalar of class $\mathfrak{X}^*$ on $\mathfrak{X}$ if $\mathfrak{X}$ is weakly complete, or is scalar of class $\mathfrak{Y}$ on $\mathfrak{X}$ if $\mathfrak{X}=\mathfrak{Y}^*$. Converses of the exact statements of these facts are also obtained and these together with the results of §4 lead to several characterizations of unbounded scalar type operators. These include extensions of criteria previously given for the bounded case in [5] and [16].

In general the terminology is that of [11]. Some exceptions and additions are noted above, and we mention others here. The symbol $\mathfrak{S}$ is used to denote the unit sphere $\mathfrak{S}=\{x\in\mathfrak{X} \mid \|x\|=1\}$ in $\mathfrak{X}$. The spectral radius of an element $T\in[\mathfrak{X}]$ is denoted by $\|T\|_\sigma$. Since $(^-)$ is used for complex conjugation, the closure of a set is usually denoted by a subscript c. The symbol $\boldsymbol{R}$ denotes the set of real numbers. If $\mathfrak{A}$ and $\mathfrak{B}$ are subsets of a Banach space, then $\mathfrak{A}+\mathfrak{B}$ represents the set of sums with one term from $\mathfrak{A}$ and one from $\mathfrak{B}$. Similar notation is used freely. Finally $(f\mid\mathfrak{G})$ represents the restriction of a function f to some subset $\mathfrak{G}$ of its domain.

1. **Bounded symmetric and self conjugate operators.** In this section we give basic definitions and investigate their significance for bounded operators. In some instances these results are known or are closely related to known results from [17] and [21], but they will all be needed later in developing the theory for unbounded operators and are given here for convenience.

Definition 1.1. An operator R will be called self conjugate iff iR generates a strongly continuous group $\{U(t, R)\}$ of isometries.

Explicitly we require that there exist a family $\{U(t, R) \mid t\in\boldsymbol{R}\}$ of operators satisfying the following conditions:

(1) $U(t, R)U(s, R)=U(t+s, R)$, $\forall t, s\in\boldsymbol{R}$ and $U(0, R)=I$,

(2) $\lim_{t\to s}\|U(t, R)x-U(s, R)x\|=0$, $\forall x\in\mathfrak{X}$, $\forall s\in\boldsymbol{R}$,

(3) $\|U(t, R)\|=1$, $\forall t\in\boldsymbol{R}$,

(4) $$\mathfrak{D}(R)=\left\{x \,\middle|\, \lim_{t\to 0}\frac{U(t, R)x-x}{t}\text{ exists}\right\}$$

and if $x \in \mathfrak{D}(R)$, then

$$iRx = \lim_{t \to 0} \frac{U(t, R)x - x}{t}.$$

We assume the elementary properties of such groups of operators to be known. See, for instance, [11, Chapter VIII]. In particular we will need to know that if R is self conjugate, then $\sigma(R)$ is real.

Next we introduce a class of operators that play the role of symmetric operators in Hilbert space. This necessitates the following preliminary definitions.

DEFINITION 1.2. For each $x \in \mathfrak{X}$, the set

$$C(x) = \{x^* \in \mathfrak{X}^* \mid \|x^*\| = \|x\| \text{ and } x^*x = \|x\|^2\}$$

will be called the conjugate set of x.

The conjugate set of x consists of all suitably normalized functionals which take their maximum on $\|x\|\mathfrak{S}$ at x. By the Hahn-Banach theorem, $C(x)$ is never empty, and it is obviously also convex and compact in the $\mathfrak{X}$ topology on $\mathfrak{X}^*$. By definition the unit ball in $\mathfrak{X}$ is smooth iff $C(x)$ is a singleton for each $x \in \mathfrak{X}$; and $\mathfrak{X}$ is strictly convex iff the conjugate sets are disjoint. The space $\mathfrak{X}$ is reflexive iff $\mathfrak{C} = \bigcup \{C(x) \mid x \in \mathfrak{X}\}$ is all of $\mathfrak{X}^*$ [15], but it is more important for our purposes to know that $\mathfrak{C}$ is always dense in $\mathfrak{X}^*$ with respect to the norm [8].

For an operator T on a Hilbert space $\mathfrak{X}$, the numerical range $W(T)$ of T is customarily defined by $W(T) = \{(Tx, x) \mid x \in \mathfrak{S} \cap \mathfrak{D}(T)\}$. However if x is an element of a Hilbert space, then the functional $(\cdot, x)$ is clearly the sole member of $C(x)$. Thus the following definition of the numerical range for an operator on an arbitrary Banach space generalizes the definition for Hilbert space.

DEFINITION 1.3. For an operator T on $\mathfrak{X}$, the set of complex numbers

$$W(T) = \{x^*Tx \mid x \in \mathfrak{S} \cap \mathfrak{D}(T); x^* \in C(x)\}$$

is called the numerical range of T. If $T \in [\mathfrak{X}]$, then $\sup \{|\lambda| \mid \lambda \in W(T)\}$ is called the numerical radius of T and is denoted by $\|T\|_W$.

This definition of the numerical range for an operator on a Banach space does not agree with, although it is closely related to, that given previously by Lumer [17]. His definition was given in terms of a semi-inner-product space obtained by an arbitrary choice of one member from each $C(x)$. The present definition is more convenient for dealing with unbounded operators.

Most of the properties of the numerical range proved in [17] remain true with this new definition. In particular it is trivial to show that the numerical range satisfies $W(T_1 + T_2) \subseteq W(T_1) + W(T_2)$ and $W(\alpha T) = \alpha W(T)$, and that its closure includes the approximate point spectrum which in turn includes the boundary of the spectrum and all of the point spectrum and the continuous spectrum of a closed operator. The usefulness of the numerical range depends largely on these results and on its stability under limits (see Lemma 1.5, (6) and (9) below). It also remains true that for $T \in [\mathfrak{X}]$

$$\sup\{\operatorname{Re}(\lambda) \mid \lambda \in W(T)\} = \lim_{t\to 0^+} \frac{\|I+tT\|-1}{t}.$$

From this and the subadditivity of $f(t) = \log \|\exp(\lambda t T)\|$ for $\lambda \in \boldsymbol{C}$, it follows easily that

$$\|T\|_W = \sup\{\log \|\exp(\lambda T)\|/|\lambda| \mid \lambda \in \boldsymbol{C},\ \lambda \neq 0\}.$$

Thus $\|\cdot\|_W$ is the function Ψ studied in [9]. The above remarks are enough to show that $\|\cdot\|_W$ is a norm on $[\mathfrak{X}]$ satisfying (cf. [17])

$$\|T\|_\sigma \leqq \|T\|_W \leqq \|T\| \leqq e\|T\|_W.$$

The last inequality is obtained by integrating $\exp(\lambda T)/\lambda^2$ around the origin on the circle of radius $\|T\|_W^{-1}$. However since $\|\cdot\|_W$ does not in general satisfy $\|T\|_W\|S\|_W \geqq \|TS\|_W$, $[\mathfrak{X}]$ is not a normed algebra relative to $\|\cdot\|_W$.

The following definition is known [17, p. 37] to describe the same operators in $[\mathfrak{X}]$ as does Lumer's definition of Hermitian operators. However when R is not bounded, this concept differs from the most natural generalization of Lumer's definition, and thus we use the word symmetric rather than Hermitian.

Definition 1.4. An operator R will be called symmetric iff $W(R)$ is contained in the real axis.

In Hilbert space, reality of the numerical range implies $(Rx, y) = (x, Ry)$ by polarization, so that Definition 1.4 reduces to the classical definition in this case. This in turn shows that any power of a symmetric operator on a Hilbert space is symmetric. All the differences between the theory of this paper and the similar Hilbert space theory can be attributed to the failure of the polarization argument to generalize.

It is obvious from the definition that any real linear combination of symmetric operators is symmetric. Furthermore since $\|\cdot\|_W$ is equivalent to $\|\cdot\|$, any operator $R \in [\mathfrak{X}]$ for which R and iR are both symmetric must be equal to zero.

We shall need all of the following conditions.

Lemma 1.5. *If R is an operator in $[\mathfrak{X}]$, then the following conditions are equivalent:*

(1) *R is self conjugate.*

(2) *$\|\exp(itR)\| = 1$ for all $t \in \boldsymbol{R}$.*

(3) *$\|I+itR\| = 1 + o(t)$ for $t \in \boldsymbol{R}$, i.e.*

$$\lim_{t\to 0} \frac{\|I+itR\|-1}{t} = 0.$$

(4) *R is symmetric.*

(5) *R is the closure of a symmetric operator.*

(6) *R is the limit in the weak operator topology of a net of self conjugate operators in* $[\mathfrak{X}]$.

(7) *R^* is self conjugate.*

If $\mathfrak{X}=\mathfrak{Y}^*$, then the following conditions are also equivalent to those above:

(8) *$Ry^*y \in \boldsymbol{R}$ for all $y \in \mathfrak{Y}$ and all $y^* \in C(y)$.*

(9) *R is the limit in the $\mathfrak{Y}$ operator topology of a net of self conjugate operators in* $[\mathfrak{X}]$.

Proof. The equivalence of (1) through (7) is proved piecemeal in references [17] and [21] and is in any case a trivial consequence of facts already cited above and of Lemmas 3.1 and 3.2 below. The implications (1) $\Rightarrow$ (9) $\Rightarrow$ (8) are also obvious. To prove (8) $\Rightarrow$ (1) we simply note that since $\mathfrak{C}=\bigcup \{C(y) \mid y \in \mathfrak{Y}\}$ is dense in $\mathfrak{Y}^*=\mathfrak{X}$, R is the closure of its restriction to $\mathfrak{C}$. However this restriction satisfies the inequality of Lemma 3.1, (1) and hence its closure does also. Finally since R is bounded, no half plane can be a subset of $\sigma_r(R)$ so R is self conjugate by Lemma 3.2.

Lemma 1.6. *If $S \in [\mathfrak{X}]$ can be written in the form $S=R+iJ$ with R and J self conjugate operators in $[\mathfrak{X}]$, then R and J are uniquely determined. If in addition R and J commute, then $\|S\|_\sigma=\|S\|_W$.*

Proof. If R' and J' are self conjugate operators in $[\mathfrak{X}]$ and $S=R'+iJ'$, then $W(R-R')=iW(J'-J)=\{0\}$. Thus $R=R'$ and $J=J'$.

If R and J commute let $\lambda=|\lambda| \exp(i\theta)$. Then

$$\log \|\exp(\lambda S)\|/|\lambda| = \log \|\exp[|\lambda|(\cos(\theta)R-\sin(\theta)J)]\|/|\lambda|.$$

However Hilfssatz 3 of [21] shows that this is equal to the largest (real) number in $\sigma(\cos(\theta)R-\sin(\theta)J)$. Thus the lemma is proved by taking the supremum over all nonzero $\lambda \in \boldsymbol{C}$.

If $S=R+iJ$ with R and J self conjugate, we will call $\bar{S}=R-iJ$ the conjugate of S. Since $\|\cdot\|_W$ is always a norm on $[\mathfrak{X}]$ equivalent to the operator norm, this lemma shows that $\|\cdot\|_\sigma$ is a norm equivalent to $\|\cdot\|$ on any linear subspace of $[\mathfrak{X}]$ in which each element S can be written as $S=R+iJ$ with R and J commuting self conjugate operators in $[\mathfrak{X}]$. A further consequence is that such an operator is zero if its spectrum consists of zero alone (cf. Lemmas 14 and 15 of [18]).

2. **Commutative V^* algebras.** In this section we will study commutative algebras of operators in which every element satisfies the hypotheses of Lemma 1.6. In [21] I. Vidav studied a class of noncommutative algebras which satisfied this condition. Thus we follow Berkson [6] in making the following definition.

Definition 2.1. If every element S of a closed commutative algebra $\mathfrak{A} \subseteq [\mathfrak{X}]$ can be written in the form $S=R+iJ$ with R and J self conjugate operators in $\mathfrak{A}$ then $\mathfrak{A}$ together with the map $S=R+iJ \to \bar{S}=R-iJ$ will be called a commutative V^* algebra.

Although this definition is given in terms of an algebra of operators on $\mathfrak{X}$, it can be applied to an arbitrary commutative Banach algebra $\mathfrak{A}$ by considering $\mathfrak{A}$ as operating on itself through the left regular representation. In [21] it is shown that the conjugation $S \to \bar{S}$ in $\mathfrak{A}$ satisfies the usual properties of an involution.

It has recently been shown [13, 6] that every algebra of the type considered by Vidav is in fact a C^* algebra. A proof of the commutative case of this theorem will emerge from the work in this section.

First, some properties of self conjugate projection operators, and spectral measures composed of them, are needed. A self conjugate projection operator E on an arbitrary Banach space retains several of the nice properties of a self adjoint projection on Hilbert space. Its range and null space are "orthogonal" in the rather strong sense that the norm of the sum of two vectors, one from each subspace, is unaffected by multiplying each vector by a different complex number of absolute value 1. This is in fact merely the statement of self conjugacy for a projection, since $\|\exp(itE)\| = \|(I-E)+\exp(it)E\| = 1$. This in turn shows that $\|E\|=1$ unless $E=0$.

THEOREM 2.2. *Let E and F be self conjugate projection operators. If $E\mathfrak{X}=F\mathfrak{X}$, then $E=F$.*

Proof. By hypothesis $EF-FE=F-E$ has real numerical range. However $i(EF-FE)$ also has real numerical range by [21, Hilfssatz 2]. Thus $F=E$.

A spectral measure in $[\mathfrak{X}]$ with domain $(\mathfrak{M}, \mathscr{B})$ is a homomorphic map $E(\cdot)$ with $E(\mathfrak{M})=I$ from a Boolean algebra $\mathscr{B}$ of subsets of a set $\mathfrak{M}$ into a uniformly bounded Boolean algebra of projection operators in $[\mathfrak{X}]$. Let Γ be a linear manifold in $\mathfrak{X}^*$. Then a spectral measure $E(\cdot)$ in $[\mathfrak{X}]$ is called Γ countably additive if $x^*E(\cdot)x$ is countably additive for each $x^* \in \Gamma$ and $x \in \mathfrak{X}$. An $\mathfrak{X}^*$ countably additive spectral measure with a σ-field as domain is also countably additive in the strong operator topology [10, p. 325]. If $\mathfrak{M}$ is a locally compact Hausdorff topological space and $\mathscr{B}$ is the Borel field in $\mathfrak{M}$, then a spectral measure in $[\mathfrak{X}]$ with domain $(\mathfrak{M}, \mathscr{B})$ is called Γ regular if $x^*E(\cdot)x$ is regular for each $x^* \in \Gamma$ and $x \in \mathfrak{X}$. Thus if $\mathfrak{M}$ is a separable, locally compact, Hausdorff space, every Γ countably additive spectral measure on the Borel field in $\mathfrak{M}$ is Γ regular.

A linear manifold $\Gamma \subseteq \mathfrak{X}^*$ is said to be total iff, for any $x \in \mathfrak{X}$, $x^*x=0$ for all $x^* \in \Gamma$ implies $x=0$. Let Γ be a total linear manifold, and let $E(\cdot)$ be a Γ countably additive spectral measure with domain $(\mathfrak{M}, \mathscr{B})$ where $\mathscr{B}$ is a σ-field. If f is a $\mathscr{B}$ measurable complex valued function on $\mathfrak{M}$, then f is called E-essentially bounded if f is bounded on some set $\sigma \in \mathscr{B}$ with $E(\sigma)=I$. The set of (equivalence classes of) E-essentially bounded functions is denoted by $EB(E)$. The E-essential bound of $f \in EB(E)$ is

$$E\text{-ess sup}\,(f) = \inf_{E(\sigma)=I} \left\{\sup_{m\in\sigma} \{|f(m)|\}\right\}.$$

Then $\int f(m)E(dm) \in [\mathfrak{X}]$ can be defined as the limit in the norm topology of an approximating sequence of simple (finitely many valued) functions (cf. [11, pp. 899, 900]).

Let $S=\int \lambda E(d\lambda)$ for some spectral measure $E(\cdot)$ in $[\mathfrak{X}]$ with domain the Borel field in C and with $E(\sigma)=I$ for some bounded set σ. Let Γ be a total linear manifold in $\mathfrak{X}^*$ and let $E(\cdot)$ be Γ countably additive. Then S is called a bounded scalar type operator of class Γ and $E(\cdot)$ is called a resolution of the identity of class Γ for S. A bounded operator S is said to be of scalar type if it is a bounded scalar type operator of class Γ for some total linear manifold Γ.

A spectral measure $E(\cdot)$ with domain $\mathscr{B}$ is called self conjugate if $E(\sigma)$ is a self conjugate operator for each $\sigma \in \mathscr{B}$.

Theorem 5 of [10] is not valid for an arbitrary total linear manifold Γ, and consequently the proof of Theorem 6 of [10] fails also. Thus a scalar type operator of arbitrary class is not known to have a unique resolution of the identity. Nevertheless, the uniqueness of a self conjugate resolution of the identity can be proved. (The author wishes to thank the referee for bringing to his attention the problem discussed in this paragraph. This matter will be further discussed in a forthcoming paper by E. Berkson and H. R. Dowson to which the reader is referred for additional details. It should be noted that what is called a resolution of the identity here is called an s-resolution of the identity by Berkson and Dowson. Their Theorem 3.1 shows that the bounded scalar type operators of class Γ defined above are prespectral operators of class Γ in their terminology and are spectral operators of class Γ in the terminology of [10]. Thus they are scalar type operators in either terminology. Finally the proof of Theorem 2.3 below shows that (in their terminology) a prespectral operator has at most one self conjugate resolution of the identity. This strengthens their Theorem 3.3.)

THEOREM 2.3. *A scalar type operator has at most one self conjugate resolution of the identity.*

Proof. Theorem 4 of [10] shows that $E(\sigma)\mathfrak{X}$ is uniquely determined for any resolution of the identity $E(\cdot)$ of a given scalar type operator S and any closed set $\sigma \subseteq C$. Thus $E(\sigma)$ is uniquely determined for any self conjugate resolution of the identity and closed σ. However if $E(\cdot)$ is Γ countably additive and $x^* \in \Gamma$, $x \in \mathfrak{X}$, then $x^*E(\cdot)x$ is regular and hence determined by its values on the closed sets. Thus $E(\cdot)$ is uniquely determined when Γ is total.

The following lemma is an easy consequence of Theorem 2.1 of [6] and the Lebesgue dominated convergence theorem.

LEMMA 2.4. *Let $E(\cdot)$ be a Γ countably additive, self conjugate spectral measure in $[\mathfrak{X}]$ with domain $(\mathfrak{M}, \mathscr{B})$ where Γ is a total linear manifold in $\mathfrak{X}^*$ and $\mathscr{B}$ is a σ field. Then the map $f \to T(f)=\int f(m)E(dm)$ is an isometric *isomorphism between the B^* algebra $EB(E)$ of E-essentially bounded functions on $\mathfrak{M}$ and a commutative V^* subalgebra of $[\mathfrak{X}]$. Furthermore if $\{f_n\}$ is a uniformly bounded sequence of $\mathscr{B}$*

measurable complex functions on $\mathfrak{M}$ *converging pointwise to* f, *then* $\{T(f_n)\}$ *converges in the* Γ *operator topology to* $T(f)$.

THEOREM 2.5. *Let* $\mathfrak{X}$ *be* [(1) *weakly complete*; (2) *the adjoint of* $\mathfrak{Y}$]. *Let* $\mathfrak{A}$ *be a commutative* V^* *algebra in* $[\mathfrak{X}]$ *and let* $\mathfrak{M}$ *be a compact Hausdorff topological space. If* $T: C(\mathfrak{M}) \to \mathfrak{A}$ *is any continuous *isomorphism, then* T *is an isometry and there exists a unique* [(1) $\mathfrak{X}^*$; (2) $\mathfrak{Y}$] *regular countably additive spectral measure* $E(\cdot)$ *in* $[\mathfrak{X}]$ *which has the Borel field in* $\mathfrak{M}$ *as domain and satisfies*

$$T(f) = \int f(m)E(dm), \quad \forall f \in C(\mathfrak{M}).$$

Furthermore $E(\cdot)$ *is self conjugate and commutes with every operator in* $\mathfrak{A}$.

Proof. Since this proof is patterned after a well-known argument [10, Theorem 18], [11, pp. 887–902], we shall omit some of the tedious details.

If T is a continuous isomorphism, then

$$f \to y^*T(f)x, \qquad x \in \mathfrak{X}, \quad y^* \in \mathfrak{X}^* \quad [\text{resp. } y^* \in \mathfrak{Y}]$$

is a continuous linear functional on $C(\mathfrak{M})$. Thus by the Riesz representation theorem [11, p. 265], there is a unique regular countably additive measure $\mu(\cdot\,; y^*; x)$ on the Borel field $\mathscr{B}$ of $\mathfrak{M}$ such that

$$y^*T(f)x = \int f(m)\mu(dm; y^*; x)$$

and

$$|\mu(\sigma; y^*; x)| \leqq \|T\| \, \|y^*\| \, \|x\|$$

for all $\sigma \in \mathscr{B}$. By using the uniqueness of μ, it is easy to show that for any Borel set σ, $\mu(\sigma; y^*; x)$ is a continuous bilinear functional on y^* and x.

For $x \in \mathfrak{X}$, define $T_x: C(\mathfrak{M}) \to \mathfrak{X}$ by $T_x(f) = T(f)x$. For $\sigma \in \mathscr{B}$, define $\chi_\sigma \in C(\mathfrak{M})^{**}$ as the functional which assigns to each regular Borel measure on $\mathfrak{M}$ (i.e., element of $C(\mathfrak{M})^*$) its value at σ. Then $T_X^{**}\chi_\sigma y^* = \chi_\sigma T_X^* y^* = \mu(\sigma; y^*; x)$. If $\mathfrak{X}$ is weakly complete, then T_x is weakly compact [11, p. 494] and $T_X^{**}\chi_\sigma$ is then contained in the image of $\mathfrak{X}$ in $\mathfrak{X}^{**}$ [11, p. 482]. Thus in this case there is an element $E(\sigma)x \in \mathfrak{X}$ with $y^*E(\sigma)x = \mu(\sigma; y^*; x)$. Since we have already seen that $\mu(\sigma; y^*; x)$ depends continuously and linearly on x, $E(\sigma)$ is in fact an element of $[\mathfrak{X}]$.

If $\mathfrak{X} = \mathfrak{Y}^*$, then there is an element $E(\sigma)x$ of $\mathfrak{Y}^* = \mathfrak{X}$ such that $\mu(\sigma; y^*; x) = E(\sigma)xy^*$ for every $y^* \in \mathfrak{Y}$. Thus in either case, for each σ there is a unique operator $E(\sigma) \in [\mathfrak{X}]$ with $y^*T(f)x = \int f(m)y^*E(dm)x$ where y^* is an element of $\mathfrak{X}^*$ or $\mathfrak{Y}$ depending on which hypothesis is used.

If f is continuous on $\mathfrak{M}$ and, $x \in \mathfrak{X}$, and y^* contained in [(1) $\mathfrak{X}^*$; (2) $\mathfrak{Y}$] are arbitrary, then $y^*T(f)x = \int f(m)y^*E(dm)x = y^* \int f(m)E(dm)x$ and $T(f) = \int f(m)E(dm)$. Since T is not only linear but actually an isomorphism, this implies that $E(\cdot)$ is a spectral measure. (See [11, p. 897] for details.)

The regularity and countable additivity of each $\mu(\cdot\,; x; y^*)$ show that $E(\cdot)$ is $\mathfrak{X}^*$ or $\mathfrak{Y}$ regular and countably additive, depending upon the case.

Thus it only remains to show that $E(\cdot)$ is self conjugate. Suppose $x \in \mathfrak{X}$ and $x^* \in C(x)$ when $\mathfrak{X}$ is weakly complete. Then

$$\int \bar{f}(m)\mu(dm; x^*; x) = x^*T(\bar{f})x = x^*\overline{T(f)x} = \overline{x^*T(f)x} = \int \bar{f}(m)\overline{\mu(dm; x^*; x)}.$$

Thus by uniqueness $x^*E(\cdot)x = \mu(\cdot\,; x^*; x)$ is real and hence $E(\cdot)$ is self conjugate. If on the other hand $x^* \in \mathfrak{Y}$ and $x \in C(x^*)$ when $\mathfrak{X} = \mathfrak{Y}^*$, then the same argument still shows that $x^*E(\cdot)x$ is real and thus, by (8) of Lemma 1.5, $E(\cdot)$ is self conjugate. Lemma 2.4 then shows that T is isometric. Finally since each element of $\mathfrak{A}$ is an integral with respect to $E(\cdot)$, it commutes with each $E(\sigma)$.

The Gelfand representation ($\hat{\ }$) of a commutative V^* algebra $\mathfrak{A}$ into $C(\mathfrak{M})$, where $\mathfrak{M}$ is its carrier space, is a homeomorphic *isomorphism onto $C(\mathfrak{M})$ by Lemma 1.6 and the complex Stone-Weierstrass theorem [11, p. 274]. Furthermore the adjoint map is an isometric *isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}^{(*)} = \{S^* \mid S \in \mathfrak{A}\} \subseteq [\mathfrak{X}^*]$ by Lemma 1.5, (7). Thus the second case of Theorem 2.5 applied to the map $T: C(\mathfrak{M}) \to \mathfrak{A}^{(*)}$ defined by $T(\hat{S}) = S^*$ gives a new proof of Theorem 1.9 of [13] and Theorem 2.3 of [6] which we state as:

Corollary 2.6. *The Gelfand representation of a commutative V^* algebra $\mathfrak{A}$ is an isometric *isomorphism onto $C(\mathfrak{M})$ where $\mathfrak{M}$ is the carrier space of $\mathfrak{A}$.*

Lemma 2.7. *Let $\mathfrak{A}$ be a commutative V^* algebra on an arbitrary Banach space $\mathfrak{X}$. If $S \in \mathfrak{A}$ and $x \in \mathfrak{X}$, then $\|Sx\| = \|\bar{S}x\|$.*

Proof. Apply Theorem 2.5 to $\mathfrak{A}^{(*)}$ and T as defined above. Thus

$$S^* = \int \hat{S}(m)E(dm), \qquad \bar{S}^* = \overline{S^*} = \int \hat{S}^-(m)E(dm).$$

Let $f(m)$ be the Borel function defined as $\hat{S}^-(m)/\hat{S}(m)$ where $\hat{S}(m) \neq 0$ and as zero elsewhere. Then $\bar{S}^* = S^*U$ where $U = \int f(m)E(dm)$, and $\|U\| \leqq 1$ by Lemma 2.4. Taking suprema over $y^* \in \mathfrak{S}^*$ we get

$$\|\bar{S}x\| = \sup |y^*\bar{S}x| = \sup |S^*Uy^*x| = \sup |(Uy^*)Sx| \leqq \|Sx\|.$$

The equality follows by symmetry.

Theorem 2.8. *Let $\mathfrak{A}$ be a commutative but not necessarily closed subalgebra of $[\mathfrak{X}]$, where $\mathfrak{X}$ is an arbitrary Banach space. For every operator $S \in \mathfrak{A}$, let there exist self conjugate operators R and J contained in $\mathfrak{A}$ and such that $S = R + iJ$. Then the closure $\mathfrak{B}$ of $CI + \mathfrak{A}$ in the weak operator topology is a V^* algebra.*

Proof. The closure $\mathfrak{R}$ of $CI + \mathfrak{A}$ in the norm topology is a commutative V^* algebra by [6, Lemma 3.1].

The closure of an algebra in the weak operator topology coincides with its closure in the strong operator topology [2, Lemma 3.3]. Thus $\mathfrak{B}$ is the closure of $\mathfrak{R}$

in the strong operator topology. If $S \in \mathfrak{W}$, then S is the strong limit of a net $\{S_\alpha\}$ with $S_\alpha = R_\alpha + iJ_\alpha \in \mathfrak{N}$. By Lemma 2.7 $\|(\bar{S}_\alpha - \bar{S}_\beta)x\| = \|(S_\alpha - S_\beta)x\|$. Thus $\{\bar{S}_\alpha\}$ also converges strongly, and both $\{(S_\alpha + \bar{S}_\alpha)/2\}$ and $\{(S_\alpha - \bar{S}_\alpha)/2i\}$ converge to self conjugate operators R, J in $\mathfrak{W}$ for which $S = R + iJ$. Finally it is easy to check that the weak closure of a commutative algebra is a commutative algebra which is, of course, closed in the norm topology.

COROLLARY 2.9. *The weak closure of a commutative V^* algebra is a commutative V^* algebra.*

The three following theorems cover three cases in which a complete spectral resolution can be constructed for an operator (or at least its adjoint) when the operator belongs to a commutative V^* algebra. It will be seen later that the operators with which these theorems deal are exactly the normal operators in $[\mathfrak{X}]$, that the existence of these spectral resolutions generalize completely to the unbounded case, and that converse theorems exist.

THEOREM 2.10. *Let $\mathfrak{X}$ be weakly complete. Then every operator in a commutative V^* algebra $\mathfrak{A}$ of operators in $[\mathfrak{X}]$ is scalar of class $\mathfrak{X}^*$ with a self conjugate resolution of the identity. If $\mathfrak{W}$ is a weakly closed commutative V^* algebra of operators in $[\mathfrak{X}]$, then $\mathfrak{W}$ contains the range of the (self conjugate) resolution of the identity of each of its elements. If $\mathfrak{W}$ is the weak closure of $\mathfrak{A}$, then $\mathfrak{W}$ is the collection of all scalar operators the resolutions of the identity of which have ranges contained in the strong closure of the range of the spectral measure of Theorem 2.5 for the inverse of the Gelfand representation of $\mathfrak{A}$.*

Proof. Let $\mathfrak{A}$ be a commutative V^* algebra containing S, and let $\mathfrak{A}'$ be the smallest V^* subalgebra of $\mathfrak{A}$ containing S. Then the function corresponding to S in the Gelfand representation of $\mathfrak{A}'$ is a homeomorphism of the carrier space of $\mathfrak{A}'$ onto $\sigma(S)$. Thus the self conjugate spectral measure $E(\cdot)$ of Theorem 2.5 relative to the inverse of the Gelfand representation of $\mathfrak{A}'$ may be defined on the Borel sets of $\sigma(S)$ and extended to the Borel field in C in an obvious way. Then $S = \int \lambda E(d\lambda)$ is a scalar operator.

Let S be contained in $\mathfrak{W}$ and have $E(\cdot)$ as resolution of the identity. Let $\mathfrak{F}$ be the set of all bounded Borel functions f on C with $T(f) = \int f(\lambda)E(d\lambda)$ contained in $\mathfrak{W}$. $\mathfrak{F}$ contains any bounded continuous function f, since $T(f)$ lies in the smallest V^* subalgebra generated by S. But also, if $\{f_n\}$ is a uniformly bounded pointwise convergent sequence of functions in $\mathfrak{F}$, then its limit f is also in $\mathfrak{F}$ since $T(f_n)$ converges weakly to $T(f)$. Thus $\mathfrak{F}$ contains the class of bounded Baire functions. However on C each bounded Borel function is a Baire function. Thus $E(\sigma) \in \mathfrak{W}$ if σ is a Borel set.

Now let $\mathfrak{W}$ be the weak closure of $\mathfrak{A}$. Let $E(\cdot)$ be the spectral measure of Theorem 2.5 relative to the inverse of the Gelfand representation of $\mathfrak{A}$, and let $E(\mathscr{B})$ be its range. It is easy to check that the strong operator closure $E(\mathscr{B})_s$ of $E(\mathscr{B})$ is a Boolean algebra of self conjugate projection operators. The set of scalar operators with the ranges of their resolutions of the identity in $E(\mathscr{B})_s$ is just the closure $\mathfrak{R}$ in the norm topology of the set of finite linear combinations of projections in $E(\mathscr{B})_s$, and this is certainly a commutative V^* algebra contained in $\mathfrak{W}$. However since $\mathfrak{W}$ is a weakly closed commutative V^* algebra, every operator in $\mathfrak{W}$ has a resolution of the identity with range contained in $\mathfrak{W}$, and, by two theorems of Bade [3, Theorems 2.7, 2.8] the range is actually in $E(\mathscr{B})_s$.

Several of the subsequent proofs will depend on the weak * or $\mathfrak{Y}$-operator topology on $[\mathfrak{X}]$ where $\mathfrak{X}=\mathfrak{Y}^*$. The family of sets of the form

$$N(S; A^*; A) = \{T \mid |(S-T)x^*y| < 1, x^* \in A^*, y \in A\}$$

form a base for this topology where S is an element of $[\mathfrak{X}]$ and A^* and A are finite subsets of $\mathfrak{X}$ and $\mathfrak{Y}$ respectively. In the $\mathfrak{Y}$ operator topology, multiplication is in general only continuous with respect to the left factor, but is continuous with respect to the right factor if the left factor is an operator which is a continuous function from $\mathfrak{X}$ to $\mathfrak{X}$ in the $\mathfrak{Y}$ topology, i.e., if it is the adjoint of an operator in $[\mathfrak{Y}]$. In particular the $\mathfrak{Y}$ operator limit of a net of operators, each of which commutes with a given operator T, need not commute with T unless T is an adjoint. This is a source of difficulty. On the other hand, if $\{S_\alpha\}$ is a uniformly bounded net of operators and $S_\alpha x^* y$ converges for each $x^* \in \mathfrak{X}$ and $y \in \mathfrak{Y}$, then there is an $S \in [\mathfrak{X}]$ which is the limit of $\{S_\alpha\}$. We shall use these facts freely from now on.

THEOREM 2.11. *Let $\mathfrak{A}$ be a commutative V^* algebra in $[\mathfrak{X}^*]$. Every operator $S \in \mathfrak{A}$ is a scalar type operator of class $\mathfrak{X}$ with a self conjugate resolution of the identity. Let $\mathfrak{R}$ be the set of operators which commute with every operator in $\mathfrak{A}$ and belong to the closure in the $\mathfrak{X}$ operator topology of the set of self conjugate operators in $\mathfrak{A}$. Then $\mathfrak{W}^* = \mathfrak{R} + i\mathfrak{R}$ is a weakly closed commutative V^* algebra which contains the range of the self conjugate resolution of the identity for each of its elements.*

Proof. The first statement follows immediately from Theorem 2.5 and [10, Lemma 6]. Furthermore $\mathfrak{R}$ contains only self conjugate elements by Lemma 1.5, (9) and it clearly contains any real linear combination of its elements.

Next we will show that $\mathfrak{R}$ contains the product of any two of its elements and thus is a real algebra. Let R and J be elements of $\mathfrak{R}$ and let $\{R_\alpha\}$ and $\{J_\beta\}$ be nets of self conjugate operators in $\mathfrak{A}$ converging in the $\mathfrak{X}$ operator topology to R and J respectively. Then for fixed β, $R_\alpha J_\beta \to RJ_\beta$ and RJ_β commutes with all operators in $\mathfrak{A}$ since both R and J_β do. Thus $RJ_\beta = J_\beta R \in \mathfrak{R}$. Similarly JR is an element of $\mathfrak{R}$ since $J_\beta R \to JR$. Therefore $\mathfrak{R}$ is a real algebra. It is also commutative since $RJ-JR$ is self conjugate by the above argument while $i(RJ-JR)$ is self conjugate by [21, Hilfssatz 2]. Thus $RJ=JR$. If a sequence $\{R_n + iJ_n\}$ in $\mathfrak{R}+i\mathfrak{R}$ converges in the norm topology then so do $\{R_n\}$ and $\{J_n\}$. Therefore $\mathfrak{W}^* = \mathfrak{R}+i\mathfrak{R}$ is closed in the norm topology and is a commutative V^* algebra by what we have already proved. Its closures in the weak and strong operator topologies coincide. If $\{S_\alpha\}$ is a net in $\mathfrak{W}^*$

converging in the strong operator topology, then $\{R_\alpha\}$ and $\{J_\alpha\}$ both converge strongly by Lemma 2.7 and thus have limits in $\mathfrak{R}$. Therefore $\mathfrak{W}^*$ is weakly closed.

Let S be contained in $\mathfrak{W}^*$ and let $E(\cdot)$ be its self conjugate resolution of the identity. Let $\mathfrak{F}$ be the set of bounded Borel functions f such that $T(f) = \int f(m)E(dm)$ is contained in $\mathfrak{W}^*$. Then, as before, $\mathfrak{F}$ contains all bounded continuous functions. If $\{f_n\} \subseteq \mathfrak{F}$ is a uniformly bounded sequence converging pointwise to f, then $\{T(\mathrm{Re}\,(f_n))\}$ and $\{T(\mathrm{Im}\,(f_n))\}$ converge in the $\mathfrak{X}$ operator topology to $T(\mathrm{Re}\,(f))$ and $T(\mathrm{Im}\,(f))$, respectively. By Theorem 2.5, $E(\cdot)$ commutes with $\mathfrak{W}^*$, so these operators do also and hence belong to $\mathfrak{R}$. Therefore $T(f)$ is contained in $\mathfrak{W}^*$. Thus $\mathfrak{F}$ contains all bounded Borel functions and the theorem is proved.

Theorem 2.12. *Let $\mathfrak{X}$ be an arbitrary Banach space and let $\mathfrak{A}$ be a commutative V^* algebra in $[\mathfrak{X}]$. The adjoint $S^* \in [\mathfrak{X}^*]$ of every operator $S \in \mathfrak{A}$ is a scalar type operator of class $\mathfrak{X}$ with a self conjugate resolution of the identity. Let $\mathfrak{R}$ be the closure in the $\mathfrak{X}$ operator topology of the set of adjoints of self conjugate elements in $\mathfrak{A}$. Then $\mathfrak{W}^* = \mathfrak{R} + i\mathfrak{R}$ is a weakly closed commutative V^* algebra which contains the range of the self conjugate resolution of the identity of each of its elements.*

Proof. Apply Theorem 2.11 to $\mathfrak{A}^{(*)} = \{S^* \mid S \in \mathfrak{A}\}$. It is necessary to check that each element of $\mathfrak{R}$ as defined in this theorem commutes with each element of $\mathfrak{A}^{(*)}$ so that $\mathfrak{R}$ is in fact equal to the $\mathfrak{R}$ of Theorem 2.11. However since each element of $\mathfrak{A}^{(*)}$ is an adjoint, this follows from the general remarks above on $\mathfrak{X}$ operator convergence.

3. **Unbounded symmetric and self conjugate operators.** The results of this section are closely related to those of [19]. What we call a symmetric operator R on $\mathfrak{X}$ is, in the terminology of that article, an operator R such that iR and $-iR$ are both dissipative with respect to every semi-inner-product compatible with the norm of $\mathfrak{X}$. We shall use this relationship extensively.

In Hilbert space the closure of a symmetric operator is always a symmetric operator. Lemma 3.4 of [19] implies that this is also true in any space with a smooth unit ball. However in general we have only the following result.

Lemma 3.1. *The closure R_c of (the graph of) a densely defined symmetric operator R is an operator which satisfies*:

$$\|(\lambda - R_c)x\| \geqq |\mathrm{Im}\,(\lambda)|\,\|x\|, \qquad \forall x \in \mathfrak{D}(R_c),\ \forall \lambda \in \boldsymbol{C}. \tag{1}$$

Any operator which satisfies (1) *also satisfies*:

$$\lim_{t \to 0} \frac{\|(I + itR_c)x\| - \|x\|}{t} = 0, \qquad \forall x \in \mathfrak{D}(R_c^2). \tag{2}$$

Proof. Lemma 3.3 of [19] shows that R_c is an operator (i.e., is single valued). To prove (1) we note that for $x_n \in \mathfrak{D}(R)$ and $x_n^* \in C(x_n)$,

$$\|x_n\|\,\|(\lambda - R)x_n\| \geqq |\mathrm{Im}\,(x_n^*(\lambda - R)x_n)| \geqq |\mathrm{Im}\,(\lambda)|\,\|x_n\|^2.$$

If $x \in \mathfrak{D}(R_c)$, then $x = \lim x_n$ and $R_c x = \lim R x_n$ for some $x_n \in \mathfrak{D}(R)$ and thus x and R_c also satisfy (1). This proves

$$\liminf_{t \to 0+} \frac{\|(I+itR_c)y\| - \|y\|}{t} \geqq 0, \quad \limsup_{t \to 0-} \frac{\|(I+itR_c)y\| - \|y\|}{t} \leqq 0$$

for $y \in \mathfrak{D}(R_c)$ and *a fortiori* for $y \in \mathfrak{D}(R_c^2)$. Now if $x \in \mathfrak{D}(R_c^2)$, then $(I+itR_c)x$ is in $\mathfrak{D}(R_c)$, so

$$\|(I+itR_c)x\| \leqq \|(I-itR_c)(I+itR_c)x\| = \|x\| + o(t)$$

as $t \to 0$. This proves the other two inequalities needed to prove (2).

LEMMA 3.2. *A self conjugate operator is a densely defined closed maximal symmetric operator with real spectrum. Conversely a closed densely defined operator satisfying*

$$\|(\lambda - R)x\| \geqq |\mathrm{Im}\,(\lambda)|\,\|x\|, \qquad x \in \mathfrak{D}(R),\ \lambda \in C,$$

is either self conjugate or else its residual spectrum contains at least one of the nonreal half planes.

Proof. The generator iR of a strongly continuous group $\{U(t)\}$ is known to be closed and densely defined [11, p. 620]. If the group consists of isometries, then for any $x \in \mathfrak{D}(R) \cap \mathfrak{S}$ and any $x^* \in C(x)$ we have

$$\begin{aligned} \mathrm{Im}\,(x^*(\pm R)x) &= \mathrm{Re}\left[x^*\left(\lim_{t \to 0+} \frac{U(\mp t)x - x}{t}\right)\right] \\ &= \lim_{t \to 0+} \frac{\mathrm{Re}\,(x^* U(\mp t)x) - 1}{t} \leqq 0 \end{aligned}$$

since $\mathrm{Re}\,(x^* U(t)x) \leqq |x^* U(t)x| \leqq 1$. Thus R is symmetric. The spectrum of R is real by [11, VIII. 1. 11]. Since both nonreal half planes are subsets of $\rho(R)$, they would be subsets of the point spectrum of any proper extension R' of R. Therefore R' could not satisfy the inequality (1) of Lemma 3.1. Thus R is a maximal symmetric operator.

If R is a closed operator which satisfies this inequality, then any nonreal λ belongs either to $\sigma_r(R)$ or to $\rho(R)$ and does not belong to the boundary of the spectrum of R. Thus if any point of one of the nonreal half planes belongs to $\sigma_r(R)$, then the whole half plane does. On the other hand if both these half planes belong to $\rho(R)$ and R is densely defined, then this inequality allows us to apply a corollary of the Hille-Yosida theorem [11, p. 628] and to conclude that iR generates a strongly continuous group of isometries.

We wish to define commutativity for self conjugate operators. Since no entirely satisfactory definition of commutativity for two arbitrary unbounded operators has ever been given, we are free to define the concept so long as the definition is consistent with all similar definitions. As usual we will say that a closed unbounded operator S commutes with an operator $T \in [\mathfrak{X}]$ if $TS \subseteq ST$. For an unbounded self conjugate operator R we give the following definition which is justified by Theorem

3.4 and generalizes a similar concept for unbounded self adjoint operators on Hilbert space [20, p. 647].

DEFINITION 3.3. A self conjugate operator R will be said to commute with a closed operator T iff $U(t, R)T \subseteq TU(t, R)$ for all $t \in \boldsymbol{R}$.

In the following theorem the word commute is used only in the sense of this definition.

THEOREM 3.4. *Let R be self conjugate, and let T be a closed operator on the same space $\mathfrak{X}$. Then*:

(1) *If T is also self conjugate, then R commutes with T if and only if T commutes with R.*

(2) *If $R \in [\mathfrak{X}]$, then R commutes with T if and only if $RT \subseteq TR$.*

(3) *If $T \in [\mathfrak{X}]$, then R commutes with T if and only if $TR \subseteq RT$.*

(4) *R commutes with T if and only if*

$$(*) \qquad (\lambda - R)^{-1}T \subseteq T(\lambda - R)^{-1}$$

for all $\lambda \in \rho(R)$. This condition holds if $(*)$ holds for at least one value of λ from each nonreal half plane or for a single value of λ when $\sigma(R) \neq \boldsymbol{R}$.

(5) *If $\lambda \in \rho(T)$, then R commutes with T if and only if $(\lambda - T)^{-1}R \subseteq R(\lambda - T)^{-1}$.*

If R commutes with T, then $RTx = TRx$ whenever both are defined.

Proof. We shall prove only (3) and (4) since these proofs are typical of the rest. For $U(t, R)$, we write $U(t)$.

(3) Suppose $T \in [\mathfrak{X}]$ and $TU(t) = U(t)T$. Then for $x \in \mathfrak{D}(R)$,

$$\begin{aligned} TRx &= -iT \lim_{t \to 0} t^{-1}[U(t)x - x] \\ &= -i \lim_{t \to 0} t^{-1}[U(t)Tx - Tx] \\ &= RTx. \end{aligned}$$

On the other hand suppose $TR \subseteq RT$. Then Theorem 3.7 shows that for $x \in \mathfrak{G}(R)$ (see Definition 3.5)

$$TU(t)x = T \sum_{n=0}^{\infty} \frac{(itR)^n}{n!} x = U(t)Tx.$$

Since $TU(t)$ and $U(t)T$ are bounded and $\mathfrak{G}(R)$ is dense, $TU(t) = U(t)T$.

(4) If $TU(t) \supseteq U(t)T$, $x \in \mathfrak{D}(T)$ and $\pm \operatorname{Im}(\lambda) > 0$, then

$$\begin{aligned} (\lambda - R)^{-1}Tx &= \mp i \int_0^\infty \exp(\pm i\lambda t) U(\mp t)Tx\, dt \\ &= \mp iT \int_0^\infty \exp(\pm i\lambda t) U(\mp t)x\, dt \\ &= T(\lambda - R)^{-1}x \end{aligned}$$

where T can be brought out of the integral sign since it is closed. This proves the necessity of $(*)$ for all nonreal $\lambda \in \rho(R)$. However the subset of $\rho(R)$ on which

$(*)$ holds is clearly closed in $\rho(R)$, since T is closed, and since $(\lambda-R)^{-1} \to (\mu-R)^{-1}$ in the norm when $\lambda \to \mu$. Thus all $\lambda \in \rho(R)$ satisfy $(*)$ when R and T commute.

The set of $\lambda \in \rho(R)$ for which $(*)$ holds is also open since T is closed and $(\lambda-R)^{-1}$ may be expanded in a power series. Thus either of the conditions in the last sentence of part (4) of the lemma insure that $(*)$ is satisfied for all $\lambda \in \rho(R)$ and it remains only to show that this implies commutativity. This follows from [14, Theorem 11.6.5].

If R is a possibly unbounded scalar type operator [1, p. 379] on $\mathfrak{X}$ of class $\mathfrak{X}^*$ with resolution of the identity $E(\cdot)$, then the set of all $E(\sigma)x$, where σ ranges through bounded Borel sets and x through $\mathfrak{X}$, is a linear manifold which is dense in $\mathfrak{X}$. For any $E(\sigma)x$ contained in this set, $\lim \|R^nE(\sigma)x\|^{1/n}$ is bounded. A weakened form of this property first introduced in [12] (cf. [19]) will be crucial for the rest of our discussion. For later use we make Definition 3.5 and Lemma 3.6 refer to families of operators.

Definition 3.5. Let $\mathfrak{R}$ be a family of operators on $\mathfrak{X}$. Then $\mathfrak{S}(\mathfrak{R})$ is the set of all $x \in \mathfrak{X}$ satisfying the following two conditions:

(1) If M is any monomial of operators in $\mathfrak{R}$, then $x \in \mathfrak{D}(M)$.

(2) $\sup \{\|Mx\| \mid \text{total degree of } M \text{ is } n\}^{1/n} = o(n)$.

When $\mathfrak{R}$ consists of only a few operators we shall usually write $\mathfrak{S}(R)$ or $\mathfrak{S}(R, J)$ etc. It should be noted that $\mathfrak{S}(\mathfrak{R})$ is always a linear manifold possible consisting of 0 alone.

The next lemma is a generalization of Remark 2 of [19, p. 692].

Lemma 3.6. *Let $\mathfrak{R}$ be a finite family of commuting self conjugate operators. Then $\mathfrak{S}(\mathfrak{R})$ is dense in $\mathfrak{X}$.*

Proof. Let $\mathfrak{R} = \{R_1, R_2, \ldots, R_m\}$ and consider the set of elements of the form

$$x' = \pi^{-m/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=1}^{m} (\exp(-s_j^2) U(\delta_j s_j, R_j)) x \, ds_1 \, ds_2 \cdots ds_m$$

where the δ_j's are positive constants and x is any element of $\mathfrak{X}$. Because of the strong continuity of the groups involved, for any $x \in \mathfrak{X}$ the deltas may be chosen small enough so that $\|x - x'\|$ is as small as we please. Thus this set of elements is dense in $\mathfrak{X}$, and we need only show that it belongs to $\mathfrak{S}(\mathfrak{R})$. However because of the absolute convergence of the integral in the norm of $\mathfrak{X}$, and the commutativity of the groups, the integration may be performed in any order. Furthermore for any j and n, x' is contained in $\mathfrak{D}(R_j^n)$, and $R_j^n x'$ is δ_j^{-n} times the integral of the nth partial derivative of the integrand with respect to s_j. This follows easily from [11, Lemma VIII, 2.11]. Thus x' is in the domain of any monomial of R_j's. A simple estimate for the maximum of a Hermite polynomial (based on the fact that this maximum is attained beyond the last zero) shows that

$$\|Mx'\| \leqq (2)^{m/2}(4n/\delta^2 e)^{n/2}\|x\|$$

where $\delta = \min\{\delta_j\}$ and n is the total degree of M. Thus $\|Mx'\|^{1/n}$ is $o(n)$ uniformly in all monomials M, and $\mathfrak{G}(\mathfrak{R})$ is dense.

One weak consequence of this lemma is the fact that a linear combination of commuting self conjugate operators is always densely defined.

The following theorem extends Theorem 3.2 of [19] in several ways which will be needed below.

THEOREM 3.7. *Let R be an operator which has a closed extension and which satisfies $\|(\lambda - R)x\| \geqq |\mathrm{Im}\,(\lambda)|\;\|x\|$ for all $x \in \mathfrak{D}(R)$ and $\lambda \in \boldsymbol{C}$. Let $\mathfrak{E}(R)$ be dense. Then R_c is self conjugate and equal to $(R|\mathfrak{E}(R))_c$. Furthermore $U(t, R_c)$ is the closure of the operator $U(t)$ defined on $\mathfrak{E}(R)$ by*

$$(*) \qquad U(t)x = \sum_{n=0}^{\infty} \frac{(itR)^n x}{n!}, \qquad x \in \mathfrak{G}(R),$$

where the series converges absolutely for all $t \in \boldsymbol{R}$.

Proof. Since $\mathfrak{G}(R)$ is dense, $\mathfrak{G}(R_c)$ is also dense. Furthermore R_c satisfies (2) of Lemma 3.1. Thus the series $(*)$ with R replaced by R_c defines a strongly continuous group $U'(t)$ of isometries on $\mathfrak{G}(R_c)$ just as in the proof of [19, Theorem 3.2]. Also $U(t) = (U'(t)|\mathfrak{G}(R))$ so $\{(U(t))_c\}$ is a strongly continuous group of isometries. The proof that iR_c generates this group and that $R_c = (R|\mathfrak{G}(R))_c$ is again similar to that given in [19].

If R is a self conjugate operator on $\mathfrak{X}$, then the set of adjoints $U(t, R)^*$ of operators in its group forms a group of isometries which need not be strongly continuous. In order to study similar semigroups, R. S. Phillips has introduced an $\mathfrak{X}$ dense subspace $\mathfrak{X}^{\bigcirc}$ of $\mathfrak{X}^*$ on which the restriction $U^{\bigcirc}(t, R)$ of $U(t, R)^*$ is strongly continuous. In fact $\mathfrak{X}^{\bigcirc}$ can be defined for any densely defined symmetric operator R as the norm closed linear subspace spanned by $\mathfrak{D}(R^*)$. Also $R^{\bigcirc} = (R^*|\mathfrak{D}(R^{\bigcirc}))$ where $\mathfrak{D}(R^{\bigcirc}) = \{x^* \in \mathfrak{D}(R^*) \mid R^*x^* \in \mathfrak{X}^{\bigcirc}\}$. If R is self conjugate, then $iR^{\bigcirc}$ is the generator of $\{U^{\bigcirc}(t, R)\}$.If $\mathfrak{X}$ is reflexive $R^{\bigcirc} = R^*$. For further details see [14, Chapter XIV]. (N.B. In this reference these concepts are only defined for an operator the resolvent set of which is a neighborhood of $+\infty$. Since we do not require the corresponding condition, many results in the cited reference do not follow except when $\sigma(R) \subseteq \boldsymbol{R}$.)

THEOREM 3.8. *The following five conditions are sufficient and the last three are necessary in order for a densely defined symmetric operator R on $\mathfrak{X}$ to have a self conjugate closure.*

(1) *R is bounded.*

(2) *$\mathfrak{G}(R)$ is dense.*

(3) *$\mathfrak{G}(R_c)$ is dense.*

(4) *$i - R$ and $i + R$ have ranges dense in $\mathfrak{X}$.*

(5) *$R^{\bigcirc}$ is self conjugate.*

Proof. The sufficiency of (1) is merely a restatement of Lemma 1.5, (5). Lemma 3.1 and Theorem 3.7 applied to R_c prove the sufficiency of (3), hence of (2). In case (4), $(i-R_c)^{-1}=(i-R)_c^{-1}$ and $(i+R_c)^{-1}=(i+R)_c^{-1}$ are closed, bounded and everywhere defined, so again neither half plane is a subset of $\sigma_r(R_c)$.

In case (5), if $i\mp R$ does not have dense range, then we can find $x^* \in \mathfrak{X}^*$ such that $\pm ix^*y=x^*Ry$ for all $y \in \mathfrak{D}(R)$. Thus x^* is contained in $\mathfrak{D}(R^{\bigcirc})$ and $(\pm i-R^{\bigcirc})x^*=0$. This contradicts the inequality of Lemma 3.1 and shows that $R^{\bigcirc}$ is not self conjugate.

The necessity of (3) and (4) is proved by Lemmas 3.6 and 3.2, respectively. In the latter case we use the fact that $\mathfrak{D}((i\pm R_c)^{-1})$ is contained in the closure of the range of $i\pm R$. The necessity of (5) follows from the fact that $R^*=R_c^*$. Thus $R^{\bigcirc}=R_c^{\bigcirc}$ is the generator of the group $U^{\bigcirc}(t, R_c)=U(t, R_c^{\bigcirc})$ and hence is self conjugate.

We shall need to know that the closure of a real linear combination of commuting self conjugate operators is self conjugate. This is the major point of the following theorem. A similar property of commuting self adjoint operators in Hilbert space can be proved by constructing their commuting resolutions of the identity. Since self conjugate operators on a Banach space are usually not normal and certainly lack any semblance of a complete spectral resolution, no similar proof for this proposition could be given. On the other hand if the groups generated by the iR, $R \in \mathfrak{R}$ were all contained in a commutative V^* algebra, then this theorem could be extended to deal with an algebra rather than just a vector space.

THEOREM 3.9. *Let $\mathfrak{R}$ be an arbitrary family of commuting self conjugate operators. Then the set $\mathfrak{R}^c$ of all closures of real linear combinations of elements of $\mathfrak{R}$ is a family of commuting self conjugate operators which becomes a real vector space when the sum of two operators is redefined as the closure of the customary sum. If $\dotplus$ represents this new sum, then $U(t, aR\dotplus bJ)=U(at, R)U(bt, J)$.*

Proof. If $R_j \in \mathfrak{R}$, $a_j \in \mathbf{R}$ and $R=\sum_{j=1}^n a_jR_j$ (customary sum), then $\mathfrak{G}(R)\supseteq \mathfrak{G}(R_1, R_2, \ldots, R_n)$ which is dense. Since $W(R)$ is clearly real we can apply Theorem 3.8, (2) and conclude that R_c is self conjugate.

However $\{\prod_{j=1}^n U(t, a_jR_j)\}$ is easily seen to be a strongly continuous group of isometries and it is also easy to check that its generator contains iR and hence iR_c. Since a self conjugate operator lacks proper self conjugate extensions, iR_c is the generator of this group. Thus $\mathfrak{R}^c$ consists of commuting self conjugate operators.

If J_c with $J=\sum_{j=1}^m b_jJ_j$ is another arbitrary element of $\mathfrak{R}^c$, then the arguments above show that $i(R_c\dotplus J_c)$ is the generator of

$$\left\{\prod_{j=1}^{n} U(t, a_jR_j)\prod_{j=1}^{m} U(t, b_jJ_j)\right\}$$

and thus is $(R+J)_c$ which belongs to $\mathfrak{R}^c$ by definition. Thus $\mathfrak{R}^c$ is closed under $\dotplus$. The associativity of $(\dotplus)$ follows from the form of the group of isometries of a sum, and the rest of the properties of a vector space are easy to check.

4. **Normal operators.** In this section normal operators are defined in a relatively abstract form suitable for the purposes of §5. Then a variety of concrete necessary and sufficient conditions for normality are derived.

DEFINITION 4.1. An operator S is said to be normal if there exist self conjugate operators R and J such that

(1) $S=R+iJ$,

(2) $\{U(t, R)\}$ and $\{U(t, J)\}$ are contained in a commutative V^* algebra.

In §5 we will discover that the operators R and J of the definition are uniquely determined by S and shall call them the real and imaginary parts of S. Since no proof of uniqueness is given at this stage, the words "real part" and "imaginary part" must be interpreted as referring to some pair of self conjugate operators satisfying the conditions of Definition 4.1. It is convenient to agree that whenever S is a normal operator and we write $S=R+iJ$ without further description of R and J, then R and J satisfy Definition 4.1.

If R is an operator in $[\mathfrak{X}]$, the expressions

$$U(t, R) = \sum_{n=0}^{\infty} \frac{(itR)^n}{n!} \quad \text{and} \quad iR = \lim_{t\to 0} \frac{\exp(itR)-I}{t}$$

(where convergence is in the norm topology in both cases) show that $\{U(t, R)\}$ belongs to a commutative V^* algebra $\mathfrak{A}$ if and only if R does. Thus an operator S in $[\mathfrak{X}]$ is normal if and only if it belongs to a commutative V^* algebra. In this case $\|Sx\| = \|\bar{S}x\|$ for all $x \in \mathfrak{X}$ by Lemma 2.7. Theorem 4.2 extends this result to an unbounded normal operator and uses it to prove that such an operator is closed.

THEOREM 4.2. *A normal operator $S=R+iJ$ is closed and densely defined. Furthermore $U(t, R)U(s, J)$ is the conjugate of $U(-t, R)U(-s, J)$, and $\bar{S}=R-iJ$ satisfies $\|Sx\| = \|\bar{S}x\|$ for every $x \in \mathfrak{D}(S)=\mathfrak{D}(\bar{S})$.*

Proof. If $S=R+iJ$ is normal, then $\mathfrak{D}(S)$ is dense by Lemma 3.6.

Since a commutative V^* algebra is isomorphic to $C(\mathfrak{M})$ under its Gelfand representation, $(\hat{\ })$, the inverse and conjugate of $\hat{U}$ coincide if U is an operator with spectrum contained in the unit circle. Thus $U(t, R)U(s, J)$ is the conjugate of $U(-t, R)U(-s, J)$. Therefore $R_t=[U(t, R)-U(-t, R)]/2it$ and $J_t=[U(t, J)-U(-t, J)]/2it$ are both self conjugate. However for any $x \in \mathfrak{D}(S)=\mathfrak{D}(R) \cap \mathfrak{D}(J)$, $R_t x \to Rx$ and $J_t x \to Jx$ as t approaches zero. Thus by Lemma 2.7 $\|Sx\| = \lim \|(R_t+iJ_t)x\| = \lim \|(R_t-iJ_t)x\| = \|\bar{S}x\|$.

Now suppose $x_n \in \mathfrak{D}(S)$ and $x_n \to x$, $Sx_n \to y$ in the strong topology. Then $\bar{S}x_n$ is also a Cauchy sequence, which therefore converges to an element which we shall call z. But then $Rx_n=(Sx_n+\bar{S}x_n)/2$ converges to $(y+z)/2$. Since R is closed, x is contained in $\mathfrak{D}(R)$ and $Rx=(y+z)/2$. Similarly x is contained in $\mathfrak{D}(J)$ and therefore in $\mathfrak{D}(S)$. Thus S is closed since $Sx=y$.

THEOREM 4.3. *Let S, R, and J be operators on an arbitrary Banach space with $S=R+iJ$. Then the following are equivalent*:

(1) *S is normal with real and imaginary parts R and J, respectively.*

(2) *R and J are commuting self conjugate operators and $\|\int f(t)U(at, R)U(bt, J)dt\| \leqq \|\hat{f}\|_\infty$ for every $f \in L_1(\boldsymbol{R})$ and $a, b \in \boldsymbol{R}$ where $\|\hat{f}\|_\infty$ is the supremum norm of the Fourier transform of f.*

(3) *R and J are closed. For each pair of nonnegative integers n and m, $(R^nJ^m)_c$ is self conjugate, and all these operators commute.*

(4) *$(\pm i-R)^{-1}$ and $(\pm i-J)^{-1}$ are commuting operators in $[\mathfrak{X}]$ with dense range and for each pair of nonnegative integers n and m, R^nJ^m is symmetric.*

(5) *R and J are commuting self conjugate operators, and for all $s, t \in \boldsymbol{R}$ both*

$$C(t, s) = [U(t, R)U(s, J)+U(-t, R)U(-s, J)]/2$$

and

$$S(t, s) = [U(t, R)U(s, J)-U(-t, R)U(-s, J)]/2i$$

are symmetric.

(6) *Whenever* $\operatorname{Im}(\lambda)\neq 0$, $\operatorname{Im}(\mu)\neq 0$, *the operators $(\lambda-R)^{-1}$ and $(\mu-J)^{-1}$ belong to $[\mathfrak{X}]$, have dense range, commute, and satisfy $\|(\lambda-R)^{-1}\| \leqq |\operatorname{Im}(\lambda)|^{-1}$, $\|(\mu-J)^{-1}\| \leqq |\operatorname{Im}(\mu)|^{-1}$. Furthermore the conjugate of $A(\lambda, \mu)=(\lambda-R)^{-1}(\mu-J)^{-1}$ is $A(\bar{\lambda}, \bar{\mu})$.*

(7) *R and J are commuting self conjugate operators and*

$$\left\| \sum_{j=1}^{n} \alpha_j U(at_j, R)U(bt_j, J) \right\| \leqq \sup\left\{ \left| \sum_{j=1}^{n} \alpha_j \exp(it_js) \right| \,\middle|\, s \in \boldsymbol{R} \right\}$$

for all finite sets of $\alpha_j \in \boldsymbol{C}$ and $a, b, t_j \in \boldsymbol{R}$.

Proof. In cases (4) and (6) R and J are closed since $(\pm i-R)^{-1}$ and $(\pm i-J)^{-1}$ are closed, and thus are self conjugate by Lemma 3.2. They commute by Theorem 3.4, (4). Thus all six conditions imply that R and J are commuting self conjugate operators. The proof will be completed by showing the following implications: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1), (1) $\Leftrightarrow$ (6), and (1) $\Rightarrow$ (7) $\Rightarrow$ (5).

(1) $\Rightarrow$ (2): The integral $U\{f\}=\int f(t)U(at, R)U(bt, J)\,dt$ of condition (2) exists in the strong operator topology and thus is contained in the weakly closed commutative V^* algebra generated by $\{U(t, R)\}$ and $\{U(t, J)\}$. Thus $\|U\{f\}\| = \|U\{f\}\|_\sigma$. If $U(at, R)U(bt, J)=I$, the inequality of (2) is surely satisfied, and otherwise it is easy to see that $U\{f\}^n = U\{f^{(n)}\}$ where $f^{(n)}$ is the n-fold convolution of f with itself. Thus

$$\|U\{f\}\| = \lim \|U\{f^{(n)}\}\|^{1/n} \leqq \lim \left[\int |f^{(n)}(t)|\,dt\right]^{1/n}$$

$$= \sup\left\{ \left| \int f(t)\exp(its)\,dt \right| \,\middle|\, s \in \boldsymbol{R} \right\}$$

since the last two expressions are the spectral radius of f as an element of the convolution algebra $L_1(\boldsymbol{R})$ and the supremum norm of its Fourier transform, which are well known to be equal.

(2) $\Rightarrow$ (3): We denote $U(t, (aR+bJ)_c)=U(at, R)U(bt, J)$ by $U(t)$ (cf. Theorem 3.9).

We will show that the set $\mathfrak{A}$ of all $U\{f\}$'s with $f\in L_1(\boldsymbol{R})$ satisfies the hypotheses of Theorem 2.8 so that the weak closure $\mathfrak{W}$ of $CI+\mathfrak{A}$ is a commutative V^* algebra. Then $\mathfrak{W}$ contains $(\lambda-(aR+bJ)_c)^{-1}$ for nonreal λ by the usual integral representation for the resolvent of a group generator. Consequently $\mathfrak{W}$ also contains $U(t)$ since this is a strong limit of polynomials in the resolvent [14, Theorem 11.6.6.]. Note that $\mathfrak{W}$ depends on the choice of a and b.

If f is contained in $L_1(\boldsymbol{R})$, it can be written as $g+ih$ where

$$g(-t) = \overline{g(t)} \quad \text{and} \quad h(-t) = \overline{h(t)}.$$

Thus it is enough to show that $U\{g\}$ is self conjugate when g satisfies this condition. For any positive integer n let j_n be $n/2$ times the characteristic function of $[-1/n, 1/n]$. Because of the strong continuity of $U(t)$, it is easy to check that the sequence $u\{j_n\}$ converges strongly to I. Thus

$$\begin{aligned}\|I+iuU\{g\}\| &\leqq \liminf \|U\{j_n+iug\}\| \\ &\leqq \liminf \left[\sup\left\{\left|\int (j_n(t)+iug(t)) \exp(ist)\, dt\right| \mid s\in\boldsymbol{R}\right\}\right] \\ &\leqq |1+iu\|\hat{g}\|_\infty|\end{aligned}$$

and $U\{g\}$ is self conjugate by Lemma 1.5, (3).

If $x\in\mathfrak{D}((aR+bJ)_c^n)$, then

$$(aR+bJ)_c^n x = \lim_{t\to 0} t^{-n}\sum_{j=0}^{n}(-1)^j\binom{n}{j}U((n-2j)t)x.$$

However the operator on the right is self conjugate by Theorem 4.2. Therefore $(aR+bJ)_c^n$ has real numerical range. Since $(aR+bJ)_c^n$ is densely defined and has real spectrum (see [11, VIII. 2.9] and [11, VII. 9.10], respectively), it is self conjugate, and this is true for any $a, b\in\boldsymbol{R}$.

If $a', b'\in\boldsymbol{R}$ is any other choice of constants, we know $U(t)(a'R+b'J)_c\subseteq (a'R+b'J)_cU(t)$ by Theorem 3.9. Thus the inclusion holds true for powers of $(a'R+b'J)_c$ also. Using Theorem 3.4, (1), and then interchanging the roles of a', b' and a, b, we find that $(a'R+b'J)_c^n$ and $(aR+bJ)_c^m$ are commuting self conjugate operators. The vector space of all closures of real linear combinations of operators of the form $(aR+bJ)_c^n$ contains $(R^nJ^m)_c$ for any pair of nonnegative integers n and m, and thus Theorem 3.9 gives (3).

(3) $\Rightarrow$ (4): This is obvious by Lemma 3.2.

(4) $\Rightarrow$ (5): As noted above, R and J are commuting self conjugate operators so Theorem 3.9 guarantees that for any t and s, $(tR+sJ)_c$ is self conjugate and

$U(1, (tR+sJ)_c) = U(t, R)U(s, J)$. Therefore we can apply Theorem 3.7 and conclude that

$$C(t, s)x = \sum_{n=0}^{\infty} \frac{(-1)^n(tR+sJ)^{2n}x}{(2n)!}$$

for all $x \in \mathfrak{G}(R, J) \subseteq \mathfrak{G}((tR+sJ)_c)$. By Lemma 3.4

$$(tR+sJ)^{2n}x = \sum_{k=0}^{2n} \binom{2n}{k} t^k s^{2n-k} R^k J^{2n-k} x$$

for any $x \in \mathfrak{G}(R, J)$. Thus condition (4) implies that $(C(t, s)|\mathfrak{G}(R, J))$ has real numerical range. Since $\mathfrak{G}(R, J)$ is dense, $C(t, s)$ is the closure of this operator and is self conjugate by Lemma 1.5, (5). A similar argument shows that $S(t, s)$ is also self conjugate.

(5) $\Rightarrow$ (1): The algebra generated by $\{U(t, R)\}$ and $\{U(t, J)\}$ is the commutative algebra consisting of all complex linear combinations of $S(t, s)$ and $C(t, s)$ as $s, t \in \boldsymbol{R}$ vary. Thus, by Theorem 2.8, the weak closure of this algebra is a commutative V^* algebra, which proves that S is normal.

(1) $\Rightarrow$ (6): Since [11, p. 622]

$$(\lambda - R)^{-1} = \mp i \int_0^{\infty} \exp(\pm i\lambda t) U(\mp t, R)\, dt, \qquad \pm \operatorname{Im}(\lambda) > 0,$$

this operator is an element of the weakly closed commutative V^* algebra generated by $U(t, R)$ and $U(t, J)$. This equation together with Theorem 4.2 shows

$$\overline{(\lambda - R)^{-1}} = (\bar{\lambda} - R)^{-1}.$$

Similar results hold for $(\mu - J)^{-1}$. Thus

$$\overline{A(\lambda, \mu)} = A(\bar{\lambda}, \bar{\mu}).$$

(6) $\Rightarrow$ (1): Since

$$(\lambda_1 - R)^{-1}(\lambda_2 - R)^{-1} = [(\lambda_1 - R)^{-1} - (\lambda_2 - R)^{-1}]/(\lambda_2 - \lambda_1),$$

any product $\prod_{j=1}^{n} (\lambda_j - R)^{-1}$ can be written as the limit (if there are repeated factors) of a linear combination of resolvents. Since the same is true for J, the condition in (6) is sufficient to guarantee that the algebra $\mathfrak{A}$ generated by

$$\{(\lambda - R)^{-1} \mid \operatorname{Im}(\lambda) \neq 0\} \cup \{(\mu - J)^{-1} \mid \operatorname{Im}(\mu) \neq 0\}$$

satisfies Theorem 2.8. However if $\pm t > 0$, then $U(t, R)$ is the strong limit as $s \to \infty$ of $\exp(\mp t[s \pm is^2(\mp is - R)^{-1}])$ [11, p. 624]. Thus $\{U(t, R)\}$, and similarly $\{U(t, J)\}$, is included in the commutative V^* algebra which is the weak closure of $CI + \mathfrak{A}$.

(1) $\Rightarrow$ (7): If S is normal then the operator on the left of the inequality of condition (7) is contained in a commutative V^* algebra, and its spectral radius equals its norm. Powers of this operator are given by convolution as in the proof that (1) $\Rightarrow$ (2). Thus the norm of this operator is less than or equal to the spectral radius

of the atomic singular measure on $\boldsymbol{R}$ which attaches the values α_j to the points t_j, where this measure is considered as an element of $LA(-\infty, \infty)$. The right side is an expression for this number by [14, Theorems 4.20.1 and 4.20.4].

(7) $\Rightarrow$ (5): We apply (7) to

$$I+iuU(t, R)U(s, J)+iuU(-t, R)U(-s, J)$$

and to

$$I+uU(t, R)U(s, J)-uU(-t, R)U(-s, J).$$

This shows that both operators are $1+o(u)$ as $u \to 0$, and this is exactly condition (3) of Lemma 1.5.

A self adjoint operator on a Hilbert space is exactly a normal operator with real spectrum. On an arbitrary Banach space a normal operator with real spectrum resembles a self adjoint operator in Hilbert space much more closely than does a merely self conjugate operator. The next two corollaries specialize the results of Theorem 4.3 to this case. Theorem 5.1 and Lemmas 5.7 and 1.6 show that the imaginary part of a normal operator with real spectrum is zero.

COROLLARY 4.4. *Let R be a closed operator on $\mathfrak{X}$. Then the following are equivalent:*

(1) *R is a normal operator with real spectrum.*

(2) *R is densely defined, has a real spectrum, and for all positive integers n, R^n is symmetric.*

(3) *$\pm i-R$ has dense domain and range, and for all positive integers n, $x \in \mathfrak{D}(R^n)$ and $\lambda \in \boldsymbol{C}$*

$$\|(\lambda-R^n)x\| \geqq |\mathrm{Im}\,(\lambda)|\,\|x\|.$$

(4) *For all positive integers n, R^n is self conjugate.*

COROLLARY 4.5. *An operator R is a normal operator with real spectrum if and only if iR generates a strongly continuous group $\{U(t)\}$ and one (hence all) of the following conditions is satisfied:*

(1) *For all $t \in \boldsymbol{R}$, $\|U(t)\| \leqq 1$ and both $U(t)+U(-t)$ and $i[U(t)-U(-t)]$ are symmetric.*

(2) *For all finite sets of $\alpha_j \in \boldsymbol{C}$, and $t_j \in \boldsymbol{R}$*

$$\left\|\sum_{j=1}^{n} \alpha_j U(t_j)\right\| \leqq \sup\left\{\left|\sum_{j=1}^{n} \alpha_j \exp\,(it_j s)\right| \,\middle|\, s \in \boldsymbol{R}\right\}$$

(3) *For all $f \in L_1(\boldsymbol{R})$*

$$\left\|\int f(t)U(t)\,dt\right\| \leqq \|\hat{f}\|_\infty$$

where the right side is the supremum norm of the Fourier transform of f.

5. **Normal and scalar type operators.** The main purpose of this section is to define unbounded scalar operators of class Γ in terms of integration of unbounded functions with respect to a Γ countably additive spectral measure in $[\mathfrak{X}]$ and to prove the following theorem.

THEOREM 5.1. *Let Γ be a total linear manifold in $\mathfrak{X}^*$. Then a densely defined scalar type operator on $\mathfrak{X}$ of class Γ with a self conjugate resolution of the identity is normal. The converse is true if either $\mathfrak{X}$ is weakly complete and $\Gamma=\mathfrak{X}^*$, or $\mathfrak{X}=\mathfrak{Y}^*$ and $\Gamma=\mathfrak{Y}$. Furthermore an operator S on an arbitrary Banach space $\mathfrak{X}$ is normal if and only if it is closed and densely defined and S^* is a scalar type operator of class $\mathfrak{X}$ with a self conjugate resolution of the identity $E(\cdot)$ such that $R'=\int \operatorname{Re}(\lambda)E(d\lambda)$ and $J'=\int \operatorname{Im}(\lambda)E(d\lambda)$ are both closed in the $\mathfrak{X}$ topology.*

Before we can begin to prove Theorem 5.1 for unbounded operators, we need to define both the integral of an unbounded function with respect to a Γ countably additive measure in $[\mathfrak{X}]$ and the concept of an unbounded scalar operator of class Γ on $\mathfrak{X}$. The original definition [1] of unbounded integrals and scalar operators deals only with $\mathfrak{X}^*$ countably additive measures in $[\mathfrak{X}]$ and, while [4, p. 521] deals with an $\mathfrak{X}$ countably additive measure $E(\cdot)$ in $[\mathfrak{X}^*]$, it is assumed that $E(\cdot)$ arises as the adjoint of an $\mathfrak{X}^*$ countably additive measure in $[\mathfrak{X}]$. The definition given here generalizes these earlier concepts and was chosen to insure both that the integral of an unbounded function is a closed operator and that if $T\in[\mathfrak{X}]$ commutes with every $E(\sigma)$, then $T\int f(m)E(dm)\subseteq\int f(m)E(dm)T$.

The weak topology on $\mathfrak{X}$ defined by Γ, which we call the Γ topology, is the topology with a base consisting of the family of sets of the form

$$N(x;A) = \{y \mid |x^*(x-y)| < 1,\ x^*\in A\}$$

where x is an element of $\mathfrak{X}$ and A is a finite subset of Γ. The Γ operator topology on $[\mathfrak{X}]$ is defined in the obvious way.

The integral with respect to $E(\cdot)$ of any function $f\in EB(E)$ was defined following Theorem 2.2 as a limit in the norm topology of integrals of simple functions. If f is contained in $EB(E)$, we will write $T(f)$ for $\int f(m)E(dm)$. We denote by $M(E)$ the set of all $\mathscr{B}$ measurable complex functions on $\mathfrak{M}$ which are defined and finite valued on a set $\sigma\in\mathscr{B}$ for which $E(\sigma)=I$.

Let $f\in M(E)$. We wish to define the integral (which we shall denote by S) of f with respect to $E(\cdot)$. We shall do this in terms of a sequence of bounded functions f_n which converge to f. It might seem natural to define $\mathfrak{D}(S)$ as the set of all $x\in\mathfrak{X}$ for which $T(f_n)x$ converges in the Γ topology. However Γ convergence is too weak for our purposes, for if it were employed, then the important relationship $TS\subseteq ST$ might be false when T is an operator commuting with each $E(\sigma)$. Instead we will use convergence in the following topology.

DEFINITION 5.2. Let the terminology be as above. The set of all operators in $[\mathfrak{X}]$ which commute with each $E(\sigma)$, $\sigma\in\mathscr{B}$, will be denoted by $[E]$. The $E^*\Gamma$ topology on $\mathfrak{X}$ is the weak topology defined on $\mathfrak{X}$ by the set of all finite linear combinations of functionals of the form T^*y^* where T is an element of $[E]$ and y^* is contained in Γ.

Notice that if $g \in EB(E)$, then $T(g)$ is contained in $[E]$ and $T(g)T = TT(g)$ for any $T \in [E]$. Also the $E^*\Gamma$ topology is not stronger than the $\mathfrak{X}^*$ topology on $\mathfrak{X}$ but not weaker than the Γ topology. The net $\{x_\alpha\}$ converges to x in the $E^*\Gamma$ topology iff $y^*Tx_\alpha \to y^*Tx$ for every $T \in [E]$ and $y^* \in \Gamma$. Therefore if $x_\alpha \to x$ in the $E^*\Gamma$ topology and $T' \in [E]$, then $y^*T(T'x_\alpha) = y^*(TT')x_\alpha \to y^*TT'x$ and thus T' is a continuous operator in the $E^*\Gamma$ topology. We can define the $E^*\Gamma$ operator topology on $[\mathfrak{X}]$ in the usual way, i.e., $\{T_\alpha\}$ converges to T in the $E^*\Gamma$ operator topology iff $T_\alpha x \to Tx$ in the $E^*\Gamma$ topology for all $x \in \mathfrak{X}$. In this case $T'TV$ is the limit of $\{T'T_\alpha V\}$ for any $V \in [\mathfrak{X}]$ and $T' \in [E]$. In particular $E(\cdot)$ is actually countably additive in the $E^*\Gamma$ as well as in the Γ operator topology. We shall use all these facts without further comment.

Now we can complete the definition of S, the integral of f with respect to $E(\cdot)$. For any positive integer n let σ_n be the set of all m such that $|f(m)| \leqq n$. Let χ_n be the characteristic function of σ_n.

Definition 5.3. Let the notation be as defined above. Then $\mathfrak{D}(S)$ is the set of all $x \in \mathfrak{X}$ for which $T(f\chi_n)x$ converges in the $E^*\Gamma$ topology. If $x \in \mathfrak{D}(S)$ then Sx is the limit of this sequence.

Lemma 5.4. *Let the notation remain unchanged. Then*:

(1) *The domain of S contains the $E^*\Gamma$ dense set*

$$E\mathfrak{X} = \{E(\sigma)x \mid \sigma \in \mathscr{B};\ (f|\sigma) \text{ is bounded};\ x \in \mathfrak{X}\}.$$

(2) *If $T \in [E]$, then $TS \subseteq ST$.*

(3) *S is closed in the $E^*\Gamma$ and hence in the strong topology.*

(4) *The operator S remains unchanged if the sequence χ_n in its definition is replaced by any other increasing sequence χ_n^* of characteristic functions of sets $\sigma_n^* \in \mathscr{B}$ for which $(f|\sigma_n^*)$ is bounded and $E(\bigcup_{n=1}^\infty \sigma_n^*) = I$.*

(5) *If $f \in EB(E)$, then $S = T(f)$.*

Proof. (1) Suppose $\sigma \in \mathscr{B}$ and $(f|\sigma)$ is bounded. Then for all $n \geqq \sup\{|f(m)| \mid m \in \sigma\}$, $T(f\chi_n)E(\sigma)$ has the same value. Thus $E(\sigma)\mathfrak{X} \subseteq \mathfrak{D}(S)$. For later use we note that in particular $T(f\chi_n) = SE(\sigma_n)$ and $T(f\chi_n^*) = SE(\sigma_n^*)$ where σ_n, σ_n^*, χ_n, χ_n^* are as described in the definition and in (4) of this lemma. Since $E(\cdot)$ is $E^*\Gamma$ countably additive, $E(\sigma_n)x \to x$. Thus $E\mathfrak{X}$ is $E^*\Gamma$ dense.

(2) Let $T \in [E]$ and $x \in \mathfrak{D}(S)$. Then $T(f\chi_n)Tx = TT(f\chi_n)x \to TSx$ in the $E^*\Gamma$ topology, so $TS \subseteq ST$.

(3) Now suppose $x_\alpha \in \mathfrak{D}(S)$, $x_\alpha \to x$, and $Sx_\alpha \to z$ in the $E^*\Gamma$ topology. Then taking limits in the $E^*\Gamma$ topology with respect to α, $E(\sigma_n)z = \lim E(\sigma_n)Sx_\alpha = \lim SE(\sigma_n)x_\alpha = \lim T(f\chi_n)x_\alpha = T(f\chi_n)x$, since $E(\sigma_n)$ and $T(f\chi_n)$ are $E^*\Gamma$ continuous operators. Thus if we take limits in the $E^*\Gamma$ topology again, this time with respect

to n, then $z=\lim E(\sigma_n)z=\lim T(f\chi_n)x$ and so $x\in\mathfrak{D}(S)$ and $Sx=z$. Thus (the graph of) S is closed in the $E^*\Gamma$ topology and certainly in the stronger norm topology.

(4) Let S' be the operator which is defined in the same way as S except that χ_n^* is used in the place of χ_n. We choose $x\in\mathfrak{D}(S)$ and note that $T(f\chi_n^*)x=SE(\sigma_n^*)x=E(\sigma_n^*)Sx$ by what we have already proved. Thus taking limits in the $E^*\Gamma$ topology with respect to n, we have $\lim T(f\chi_n^*)x=\lim E(\sigma_n^*)Sx=Sx$, so S' is an extension of S. Therefore $\mathfrak{D}(S')\supseteq E(\sigma_n)\mathfrak{X}$ and $T(f\chi_n)=S'E(\sigma_n)$. If $x\in\mathfrak{D}(S')$, then $S'E(\sigma_m)x=\lim T(f\chi_n^*)E(\sigma_m)x=\lim E(\sigma_m)T(f\chi_n^*)x=E(\sigma_m)S'x$. Thus, as before, $\lim T(f\chi_n)x=\lim E(\sigma_n)S'x=S'x$, so $S'\subseteq S$ and $S=S'$.

(5) If $f\in EB(E)$, then $T(f\chi_n)$ is eventually equal to $T(f)$. Hence $\mathfrak{D}(S)=\mathfrak{X}$ and $S=T(f)$.

Because of this last result, we will now use $T(f)$ to represent the integral with respect to $E(\cdot)$ of any function f in $M(E)$, rather than just in $EB(E)$.

Many of the properties of unbounded scalar type operators in the sense of [1] can be shown to hold for unbounded scalar type operators of class Γ by use of the above results and the methods used previously for unbounded operators of class $\mathfrak{X}^*$ and bounded operators of class Γ. Thus we will list the next results and merely give references to the proofs which must be generalized.

The subscript c in the statement of the next lemma signifies closure in the $E^*\Gamma$ topology.

Lemma 5.5. *Let $E(\cdot)$ be as above and let $f, g\in M(E)$. Then*:

(1) $[T(f)+T(g)]_c=T(f+g)$,

(2) $[T(f)T(g)]_c=T(fg)$.

If in addition $g\in EB(E)$, then:

(3) $T(f)T(g)=T(fg)$.

Finally if $|f(m)|\leqq K|g(m)|$ for some K and all $m\in\sigma$ where $E(\sigma)=I$, then:

(4) $\mathfrak{D}(T(g))\subseteq\mathfrak{D}(T(f))$.

Proof. The proofs of Theorems 3.1 and 3.2 of [1] can be modified to give these results with (1) replaced by $T(f)+T(g)\subseteq T(f+g)$ and a similar change in the statement of (2). To obtain the results given here, let e_n be as defined in [1, p. 380]. Then for any $x\in\mathfrak{X}$, $E(e_n)x\to x$ in the $E^*\Gamma$ topology and $E(e_n)x\in\mathfrak{D}(T(f)+T(g))$. Thus if $x\in\mathfrak{D}(T(f+g))$, then $[T(f)+T(g)]E(e_n)x=E(e_n)T(f+g)x\to T(f+g)x$ and $x\in\mathfrak{D}([T(f)+T(g)]_c)$. A similar argument is used for (2).

Definition 5.6. An operator S is said to be a scalar operator of class Γ on $\mathfrak{X}$ iff there exists a Γ countably additive spectral measure $E(\cdot)$ defined on the Borel field $\mathscr{B}$ in C and such that $S=\int\lambda E(d\lambda)$. Any Γ countably additive spectral measure satisfying this equation is called a resolution of the identity for S. If S is a scalar operator of some class Γ, with Γ total, then S is said to be of scalar type.

It is clear from Lemmas 5.4, (5) and 5.7, (2) that this definition agrees with the definition quoted previously for a bounded scalar type operator of class Γ. Furthermore if $\Gamma=\mathfrak{X}^*$, it agrees with the definition given in [1]. For if S satisfies Definition

5.6 with $E(\cdot)$ as a resolution of the identity, then S is spectral with resolution of the identity $E(\cdot)$ in the sense of [1]. If S' is formed as an integral $\int \lambda E(d\lambda)$ in the sense of [1], where strong convergence is used, then S is clearly an extension of S', and hence equal to S' by [1, p. 378].

LEMMA 5.7. *Let S be a scalar type operator of class Γ on $\mathfrak{X}$ with resolution of the identity $E(\cdot)$. Then*:

(1) $\sigma(S|E(\sigma)\mathfrak{X}) \subseteq \bar{\sigma}, \quad \forall \sigma \in \mathscr{B}$.

(2) $E(\sigma(S)) = I$ *and* $\sigma(S) = ((\bigcup_{n=1}^{\infty} \sigma(S|E(\sigma_n)\mathfrak{X}))_c$ *where* $\sigma_n = \{\lambda \mid |\lambda| \leqq n\}$.

(3) *There is at most one self conjugate resolution of the identity for S.*

Proof. See [1, Lemma 3.1]. For part (3) see [10, Lemma 1 and Theorems 2 and 4]. Finally the argument of Theorem 2.3 above should be used.

LEMMA 5.8. *Let $E(\cdot)$ be a Γ countably additive spectral measure with domain $(\mathfrak{M}, \mathscr{B})$ and range in $[\mathfrak{X}]$. If $f \in M(E)$ then $T(f) = \int f(m)E(dm)$ is a scalar type operator of class Γ with a resolution of the identity E_f defined by*

$$E_f(\sigma) = E(f^{-1}(\sigma)) \qquad (\sigma \text{ is a Borel set in } \mathbf{C})$$

and spectrum defined by

$$\sigma(T(f)) = \bigcap \{f(\sigma)_c \mid E(\sigma) = I\}.$$

Furthermore if $\lambda \in \rho(T(f))$, then $(\lambda - T(f))^{-1} = T((\lambda - f(m))^{-1})$ is of scalar type and class Γ.

Proof. See [1, Theorem 3.3].

LEMMA 5.9. *A closed operator R on $\mathfrak{X}$ with nonempty resolvent set is a scalar operator of class Γ if and only if for one (hence all) $\lambda \in \rho(R)$, $(\lambda - R)^{-1}$ is a scalar operator of class Γ.*

Proof. See [1, p. 390, Corollary].

THEOREM 5.10. *Let R and J be two scalar type operators of class* [(1) $\mathfrak{X}^*$ *on a weakly complete Banach space* $\mathfrak{X}$ (2) $\mathfrak{Y}$ *on* $\mathfrak{X} = \mathfrak{Y}^*$]. *Let R and J have real spectrum, and resolutions of the identity $F(\cdot)$ and $G(\cdot)$, respectively, which together generate a Boolean algebra of self conjugate projection operators. Then $S = R + iJ$ is scalar of the same class with a self conjugate resolution of the identity $E(\cdot)$ which satisfies $R = \int \mathrm{Re}\,(\lambda)E(d\lambda)$ and $J = \int \mathrm{Im}\,(\lambda)E(d\lambda)$.*

Proof. By Theorem 2.8 the weakly closed algebra generated by $F(\cdot)$ and $G(\cdot)$ is a V^* algebra. In case (1) let $\mathfrak{W}$ represent this algebra, and in case (2) let $\mathfrak{W}$ represent the algebra $\mathfrak{W}^*$ formed from it as described in Theorem 2.11. Then $\mathfrak{W}$ contains the self conjugate resolution of the identity for any operator which it contains.

Define $R' = \int \arctan\,(\mathrm{Re}\,(\lambda))F(d\lambda)$, $J' = \int \arctan\,(\mathrm{Re}\,(\lambda))G(d\lambda)$.

Then $R'+iJ' \in \mathfrak{W}$ and thus has a self conjugate resolution of the identity $E'(\cdot)$ of class [(1) $\mathfrak{X}^*$, (2) $\mathfrak{Y}$]. Furthermore

$$R' = \int \mathrm{Re}\,(\lambda)E'(d\lambda), \qquad J' = \int \mathrm{Im}\,(\lambda)E'(d\lambda),$$

by Lemmas 1.6 and 2.4. Thus for each Borel set σ

$$E'(\{\lambda \mid \mathrm{Re}\,(\lambda) \in \sigma\}) = F(\{\lambda \mid \arctan\,(\mathrm{Re}\,(\lambda)) \in \sigma\}),$$
$$E'(\{\lambda \mid \mathrm{Im}\,(\lambda) \in \sigma\}) = G(\{\lambda \mid \arctan\,(\mathrm{Re}\,(\lambda)) \in \sigma\}),$$

by Lemmas 5.8 and 5.7, (5). Therefore Lemmas 5.7 and 5.8 show

$$R = \int \lambda F(d\lambda) = \int \mathrm{Re}\,(\lambda)F(d\lambda) = \int \tan\,(\mathrm{Re}\,(\lambda))E'(d\lambda),$$

$$J = \int \lambda G(d\lambda) = \int \mathrm{Re}\,(\lambda)G(d\lambda) = \int \tan\,(\mathrm{Im}\,(\lambda))E'(d\lambda).$$

By Lemma 5.5, (1), $S \subseteq \int g(\lambda)E'(d\lambda)$ where $g(\lambda) = \tan\,(\mathrm{Re}\,(\lambda)) + i \tan\,(\mathrm{Im}\,(\lambda))$. However by Lemma 5.5, (4), the domain of the integral of g is contained in both $\mathfrak{D}(R)$ and $\mathfrak{D}(J)$ thus proving equality. Therefore finally, S is of scalar type and class [(1) $\mathfrak{X}^*$, (2) $\mathfrak{Y}$] with a self conjugate resolution of the identity $E(\cdot) = E_0'(\cdot)$ as defined in Lemma 5.8 and

$$R = \int \mathrm{Re}\,(\lambda)E(d\lambda), \qquad J = \int \mathrm{Im}\,(\lambda)E(d\lambda).$$

Proof of Theorem 5.1. Suppose $S = \int \lambda E(d\lambda)$ for some Γ countably additive self conjugate spectral measure $E(\cdot)$. We will use Theorem 4.3, (6) to show that S is normal. Let $R = \int \mathrm{Re}\,(\lambda)E(d\lambda)$ and $J = \int \mathrm{Im}\,(\lambda)E(d\lambda)$. Then $S = R + iJ$ by Lemma 5.5, (1) and (4). Also $\mathfrak{D}(R)$ and $\mathfrak{D}(J)$ are both dense since they contain $\mathfrak{D}(S)$. If $\mathrm{Im}\,(\lambda) \neq 0$, then $(\lambda - R)^{-1} = \int (\lambda - \mathrm{Re}\,(\mu))^{-1}E(d\mu)$ is everywhere defined and bounded by $|\mathrm{Im}\,(\lambda)|^{-1}$. The same is true for $(\mu - J)^{-1}$ which commutes with $(\lambda - R)^{-1}$. Finally $A(\lambda, \mu)$ and $A(\bar{\lambda}, \bar{\mu})$ can be written as integrals of complex conjugate bounded functions and are thus conjugate by Lemma 2.4.

Now suppose S is normal, and let $\mathfrak{A}$ be the weakly closed commutative V^* algebra generated by $\{U(t, R)\}$ and $\{U(t, J)\}$. Then $\mathfrak{A}$ contains

$$(i-R)^{-1} = -i \int_0^\infty \exp\,(-t)U(t, R)\,dt$$

since the integral converges in the strong operator topology. If $\mathfrak{X}$ is weakly complete then R, and similarly J, have $\mathfrak{X}^*$ countably additive resolutions of the identity in $\mathfrak{A}$ by Theorem 2.10 and Lemma 5.8. Similarly if $\mathfrak{X}$ is the adjoint of $\mathfrak{Y}$, then the $\mathfrak{Y}$ countably additive, self conjugate resolutions of the identity of R and J lie in the algebra $\mathfrak{W}^*$ of Theorem 2.11. Finally if $\mathfrak{X}$ is arbitrary, $(i-R^*)^{-1}$ and $(i-J^*)^{-1}$, hence R^* and J^* have $\mathfrak{X}$ countably additive resolutions of the identity in the algebra $\mathfrak{W}^*$ of Theorem 2.12. Since every projection in a commutative V^* algebra is self conjugate, Theorem 5.10 shows that S in the first two cases or R^*+iJ^* in the last case is a scalar type operator of the correct class with a self conjugate resolution of the identity.

In the last case R^*+iJ^* is an $E^*\mathfrak{X}$ closed restriction of S^*. In order to show that $R^*+iJ^*=S^*$ we introduce the operators

$$U_\delta = \pi^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-s^2-t^2)U(\delta s, R)U(\delta t, J)\, ds\, dt$$

(where the integrals are defined in the strong operator topology (cf. proof of Lemma 3.6)). The operators U_δ approach the identity as $\delta \to 0$. Also for any $\delta \neq 0$ they map $\mathfrak{X}$ into $\mathfrak{D}(R)\cap\mathfrak{D}(J)=\mathfrak{D}(S)$, and satisfy $U_\delta R \subseteq RU_\delta \in [\mathfrak{X}]$ and $U_\delta J \subseteq JU_\delta \in [\mathfrak{X}]$. Furthermore if $\pm t>0$, $x^*\in\mathfrak{X}^*$, and $x\in\mathfrak{X}$ then

$$\begin{aligned} x^*U(t, R)x &= \lim_{s\to\infty} x^*\exp(\mp t[s\pm is^2(\mp is-R)^{-1}])x \\ &= \lim_{s\to\infty}\int \exp(\mp t[s\pm is^2(\mp is-\operatorname{Re}(\lambda))^{-1}])E(d\lambda)x^*x \\ &= \int \exp(it\operatorname{Re}(\lambda))E(d\lambda)x^*x. \end{aligned}$$

Thus $U_\delta^* = \int \exp(-\delta^2|\lambda|^2/4)E(d\lambda)$ commutes with any operator in $[E]$. Therefore if $x^*\in\mathfrak{X}^*$ then $U_\delta^*x^*\in\mathfrak{D}(R^*+iJ^*)$ and $U_\delta^*x^*\to x^*$ in the $E^*\mathfrak{X}$ topology. If $x^*\in\mathfrak{D}(S^*)$ then also $(R^*+iJ^*)U_\delta^*x^*=S^*U_\delta^*x^*=U_\delta^*S^*x^*\to S^*x^*$ in the $E^*\mathfrak{X}$ topology so $S^*=R^*+iJ^*$. Note that the R' and J' in the statement of the theorem are just R^* and J^* and hence are closed in the $\mathfrak{X}$ topology.

In order to prove the sufficiency of the last condition let S be a closed densely defined operator with $S^*=\int \lambda E(d\lambda)$ where $E(\cdot)$ is a self conjugate $\mathfrak{X}$ countably additive spectral measure. Since R' and J' are closed and densely defined in the $\mathfrak{X}$ topology by hypothesis and by Lemma 5.4, (1), there are strongly closed and densely defined operators R and J such that $R^*=R'$ and $J^*=J'$.

Theorem 4.3, (6) will be used again to prove that $R+iJ$ is normal. Since $[(\lambda-R)^{-1}]^*=\int(\lambda-\operatorname{Re}(\mu))^{-1}E(d\mu)$ for $\operatorname{Im}(\lambda)\neq 0$, and a similar equation holds for J, the first sentence of (6) is verified. Furthermore $A(\lambda,\mu)^*$ and $A(\bar\lambda,\bar\mu)^*$ are seen to be conjugate when written as integrals with respect to $E(\cdot)$. Thus $A(\lambda,\mu)$ and $A(\bar\lambda,\bar\mu)$ are conjugate by Lemma 1.5, (7).

COROLLARY 5.11. *The real and imaginary parts of a normal operator are uniquely determined.*

Proof. If $S=R+iJ$ is normal, then Theorem 5.1 shows the existence of a necessarily unique self conjugate resolution of the identity $E(\cdot)$ for S^*. However $R^*=\int\operatorname{Re}(\lambda)E(d\lambda)$ and $J^*=\int\operatorname{Im}(\lambda)E(d\lambda)$. Since these operators have domains dense in the $\mathfrak{X}$ topology, R and J are uniquely determined.

A number of necessary and sufficient conditions for an operator to be scalar can be obtained by combining Theorems 4.3 and 5.1 and a method due to Berkson [5, p. 367] for making a spectral measure self conjugate. Let $E(\cdot)$ be a spectral measure defined on $(\mathfrak{M}, \mathscr{B})$ which is not necessarily countably additive in any topology. Let $x \in \mathfrak{X}$ and define

$$\|x\|_E = \sup\left\{\sum_{j=1}^{n} |y^*E(\sigma_j)x|\right\}$$

where the supremum is over all $y^* \in \mathfrak{S}^*$ and all finite families $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ of disjoint sets in $\mathscr{B}$. Then $\|\cdot\|_E$ is a norm on $\mathfrak{X}$ equivalent to $\|\cdot\|$ and each $E(\sigma)$, $\sigma \in \mathscr{B}$ is self conjugate relative to $\|\cdot\|_E$.

THEOREM 5.12. *An operator S on a weakly complete Banach space $\mathfrak{X}$ is a scalar type operator of class $\mathfrak{X}^*$ iff there exist operators R and J which satisfy $S=R+iJ$ and one (hence all) of the following conditions:*

(1) *There is an equivalent norm on $\mathfrak{X}$ relative to which S is normal with real and imaginary parts R and J.*

(2) *For each pair n, m of nonnegative integers iR^nJ^m, generates a strongly continuous group $\{U_{nm}(t)\}$. All these groups commute, and there is a common bound M for the norm of any finite product of operators from these groups.*

(3) *iR and iJ generate commuting strongly continuous groups $\{U(t, R)\}$ and $\{U(t, J)\}$ and there is a common bound M for the norm of any finite product of operators of the form*

$$\sum_{j=1}^{n} \alpha_j U(at_j, R)U(bt_j, J)$$

where $\alpha_j \in \boldsymbol{C}$ and $a, b, t_j \in \boldsymbol{R}$ satisfy

$$\sup\left\{\left|\sum_{j=1}^{n} \alpha_j \exp(it_js)\right| \,\Big|\, s \in \boldsymbol{R}\right\} \leqq 1.$$

(4) *iR and iJ generate commuting strongly continuous groups $\{U(t, R)\}$ and $\{U(t, J)\}$ and there is a common bound M for the norm of any finite product of operators of the form $\int f(t)U(at, R)U(bt, J)\,dt$ where $f \in L_1(\boldsymbol{R})$ has Fourier transform bounded by 1 and $a, b \in \boldsymbol{R}$ are arbitrary.*

Proof. If S is scalar of class $\mathfrak{X}^*$ then it is densely defined by Lemma 5.4. Let $E(\cdot)$ be a resolution of the identity for S. Then S becomes normal when $\mathfrak{X}$ is renormed with the equivalent norm $\|\cdot\|_E$. On the other hand if (1) is satisfied, $S=R+iJ$ is scalar of class $\mathfrak{X}^*$ by Theorem 5.1, since this property is unaffected by an equivalent renorming. Thus condition (1) is necessary and sufficient.

If $S=R+iJ$ is normal, then (2), (3), and (4) hold with $M=1$ by Theorem 4.3 (3), (7), and (2), respectively. Thus they hold for some M when (1) is satisfied.

If (2), (3), or (4) hold, let $|||x|||=\sup\{\|Tx\|\}$ where T ranges over the bounded set of products indicated. It is easy to check that $|||\cdot|||$ is an equivalent norm satisfying $\|x\| \leqq |||x||| \leqq M\|x\|$ and that $|||T||| \leqq 1$ for each T. Theorem 4.3 implies (1).

Notice that (1), . . . , (4) are necessary conditions for any densely defined scalar type operator on any Banach space.

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