ON A SIMILARITY INVARIANT FOR COMPACT OPERATORS

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DEDICATED TO THE MEMORY OF HAZLETON MIRKIL

Let $\mathscr{H}$ be a Hilbert space, and $\mathscr{K}$ the algebra of all compact operators acting on $\mathscr{H}$. If $K \in \mathscr{K}$, then $K = WA$, where $A = (K^*K)^{1/2}$ is compact and positive, and W is a partial isometry mapping the range of A isometrically onto the range of K. If k_n and a_n are the nth eigenvalues, counted with multiplicities and arranged in order of decreasing magnitude, of K and A, respectively, then $0 \leqq |k_n| \leqq a_n$, and $a_n \downarrow 0$ as $n \uparrow \infty$.

For each $K \in \mathscr{K}$ and p, $0 < p \leqq \infty$, put

$$(1)\qquad \begin{aligned} \|K\|_p = \|A\|_p &= \left(\sum_{n=1}^{\infty} a_n^p\right)^{1/p}, && 0 < p < \infty, \\ &= \sup\{a_n : 1 \leqq n < \infty\}, && p = \infty. \end{aligned}$$

Then $0 \leqq \|K\|_p \leqq \infty$, and $\|K\|_p \downarrow$ as $p \uparrow$. Moreover,

LEMMA 1. *If $K, M \in \mathscr{K}$ and B, C are bounded operators on $\mathscr{H}$, then*

$$(2)\qquad \begin{aligned} \|K+M\|_p &\leqq 2^{1/p}\{\|K\|_p^p + \|M\|_p^p\}^{1/p}, && 0 < p \leqq 1, \\ &\leqq 2^{1/p}\{\|K\|_p + \|M\|_p\}, && 1 \leqq p \leqq \infty, \end{aligned}$$

$$(3)\qquad \|KM\|_p \leqq 2^{1/p}\|K\|_r\|M\|_s, \quad \textit{where } 1/p = 1/r + 1/s,$$

$$(4)\qquad \|BKC\|_p \leqq \|B\|\,\|K\|_p\|C\|,$$

$$(5)\qquad \|K^*\|_p = \|K\|_p.$$

Proof. See [2, Lemma 9, p. 1093].

Now for each $K \in \mathscr{K}$, put

$$(6)\qquad \tau(K) = \text{glb}\{p : \|K\|_p < \infty\} = \text{glb}\{p : A^p \in \text{trace class}\}.$$

Then $0 \leqq \tau(K) \leqq \infty$, and from Lemma 1 we have

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Received by the editors June 23, 1967.

[1] The results obtained here were suggested primarily by the work of R. M. Dudley [1], preprints of which we gratefully acknowledge.

COROLLARY 2. *If K, $M \in \mathscr{K}$ and B, C are bounded operators on $\mathscr{H}$, then*

$$\tau(K+M) \leqq \max\{\tau(K), \tau(M)\}, \tag{7}$$

$$\tau(KM) \leqq \tau(K)\tau(M)/(\tau(K)+\tau(M)), \tag{8}$$

$$\tau(BKC) \leqq \tau(K), \tag{9}$$

$$\tau(K^*) = \tau(K). \tag{10}$$

Proof. See Lemma 1.

In particular, it follows from (9) that if B is a bounded invertible operator, then

$$\tau(BKB^{-1}) = \tau(K). \tag{11}$$

Hence τ is a *similarity invariant* for the class $\mathscr{K}$ of compact operators. It is clear from the definitions that if K is of finite rank, trace class, or Hilbert-Schmidt class, then $\tau(K)=0$, $\leqq 1$, or $\leqq 2$, respectively. Moreover, we have

LEMMA 3. *If K, $M \in \mathscr{K}$, and if, for all $\phi \in \mathscr{H}$,*

$$\|K\phi\| \leqq \text{const } \|M\phi\|, \tag{12}$$

then $\tau(K) \leqq \tau(M)$.

Proof. If $\|K\phi\|^2=(K^*K\phi, \phi) \leqq \text{const } \|M\phi\|^2=(M^*M\phi, \phi)$ for all $\phi \in \mathscr{H}$, then $K^*K \leqq \text{const } M^*M$. If a_n and b_n are the nth eigenvalues of $(K^*K)^{1/2}$ and $(M^*M)^{1/2}$, counted with multiplicities and arranged in order of decreasing magnitude, then it follows that $a_n \leqq \text{const } b_n$ [2, p. 909]. Hence if $\sum b_n^p < \infty$, for any p, $0<p<\infty$, then $\sum a_n^p < \infty$, and $\tau(K) \leqq \tau(M)$.

Thus $\tau(K)$ provides a measure of the "size" of K. In this paper we propose to explore this idea by introducing various other measures of the "size" of K and relating them to $\tau(K)$.

All of our measures of the "size" of K are given in terms of the asymptotic behavior of certain positive sequences or functions associated with K. If $\{b_n\}$ and $\{c_n\}$ are arbitrary monotone-increasing positive sequences, with b_n, $c_n \uparrow \infty$ as $n \uparrow \infty$, then the asymptotic behavior of b_n may be compared with that of c_n by introducing the *relative order of growth* γ, defined by the formula

$$\begin{aligned} \gamma &= \text{glb}\{\mu > 0 : b_n \leqq \text{const } c_n^\mu\}, \\ &= \infty \quad \text{if no such } \mu > 0 \text{ exists.} \end{aligned} \tag{13}$$

Then clearly $0 \leqq \gamma \leqq \infty$, and if $0<\gamma-\varepsilon<\gamma<\gamma+\varepsilon<\infty$, we have $b_n \leqq \text{const } c_n^{\gamma+\varepsilon}$ for all n, and $b_n \geqq \text{const } c_n^{\gamma-\varepsilon}$ for arbitrarily large n. The computation of the relative order of growth is facilitated by the formula

$$\gamma = \limsup_{n\to\infty} \frac{\log b_n}{\log c_n}. \tag{14}$$

Similarly, if $\{b_n\}$ and $\{c_n\}$ are monotone-decreasing sequences, with $b_n, c_n \downarrow 0$ as $n \downarrow \infty$, then we introduce the *relative order of decay* δ, defined by the formula

$$\begin{aligned} \delta &= \operatorname{lub}\{\mu > 0 : b_n \leqq \text{const } c_n^{\mu}\}, \\ &= \infty \quad \text{if no such } \mu > 0 \text{ exists.} \end{aligned} \tag{15}$$

Then $0 \leqq \delta \leqq \infty$, and if $0 < \delta - \varepsilon < \delta < \delta + \varepsilon < \infty$, we have $b_n \leqq \text{const } c_n^{\delta-\varepsilon}$ for all n, and $b_n \geqq \text{const } c_n^{\delta+\varepsilon}$ for arbitrarily large n. The computation of δ is given by

$$\delta = \liminf_{n \to \infty} \frac{\log b_n}{\log c_n}. \tag{16}$$

When $c_n = n$ (or $1/n$) we call γ (or δ) simply the *order of growth* (or *order of decay*, respectively) of b_n.

From now on let $K \in \mathscr{K}$ be a fixed compact operator acting on $\mathscr{H}$, and $A = (K^*K)^{1/2}$. Let $\mathscr{B}$ be the unit ball in $\mathscr{H}$, and $\mathscr{E}$ the (compact, convex, symmetric) image of $\mathscr{B}$ under K. The "size" of K is reflected in the "size" of $\mathscr{E}$, which can be measured in several different ways. Among them we cite the following:

DEFINITION 4 (THE METRIC VOLUME [1]). Let $\mathscr{E}$ be any compact convex symmetric subset of $\mathscr{H}$, and $\mathscr{H}_n$ any n-dimensional subspace of $\mathscr{H}$. Let $|\mathscr{E} \cap \mathscr{H}_n|$ denote the n-dimensional Lebesgue volume of $\mathscr{E} \cap \mathscr{H}_n$, and put $V_n = \sup |\mathscr{E} \cap \mathscr{H}_n|$, the supremum taken over all the $\mathscr{H}_n$ in $\mathscr{H}$. Thus V_n is the least upper bound of the volumes of the n-dimensional sections of $\mathscr{E}$, and is called the *n-dimensional metric volume* of $\mathscr{E}$.

Since $\mathscr{E}$ is compact, $V_n \downarrow 0$ as $n \uparrow \infty$. The rate of decrease of V_n can be effectively compared with that of the volume B_n of the unit n-ball $\mathscr{B}_n = \mathscr{B} \cap \mathscr{H}_n$. Put

$$\begin{aligned} \beta(\mathscr{E}) &= \operatorname{lub}\{\mu > 0 : V_n \leqq \text{const } (B_n)^{\mu}\}, \\ &= 0 \quad \text{if no such } \mu > 0 \text{ exists.} \end{aligned} \tag{17}$$

Then $\beta(\mathscr{E})$ is the *order of decay* of the metric volume of $\mathscr{E}$ relative to that of the unit ball $\mathscr{B}$. When $\mathscr{E} = K(\mathscr{B})$, we shall write $\beta(\mathscr{E}) = \beta(K)$.

DEFINITION 5 (THE METRIC WIDTH [5]). Let $\mathscr{E}$ be any compact convex symmetric subset of $\mathscr{H}$, and $\mathscr{H}_n$ any n-dimensional subspace of $\mathscr{H}$. Let $d(\mathscr{E}, \mathscr{H}_n)$ denote the maximal orthogonal distance from $\mathscr{E}$ to $\mathscr{H}_n$, and put $w_n = \inf d(\mathscr{E}, \mathscr{H}_n)$, the infinum taken over all the $\mathscr{H}_n$ in $\mathscr{H}$. Then w_n is the greatest lower bound of the distance from $\mathscr{E}$ to the n-dimensional subspaces of $\mathscr{H}$, and is called the *n-dimensional metric width* of $\mathscr{E}$.

Since $\mathscr{E}$ is compact, $w_n \downarrow 0$ as $n \uparrow \infty$. The rate of decrease of w_n can be effectively compared with that of the sequence $1/n$. Put

$$\begin{aligned} \omega(\mathscr{E}) &= \operatorname{lub}\{\mu > 0 : w_n \leqq \text{const } (1/n)^{\mu}\}, \\ &= 0 \quad \text{if no such } \mu > 0 \text{ exists.} \end{aligned} \tag{18}$$

Then $\omega(\mathscr{E})$ is the *order of decay* of the metric width of $\mathscr{E}$ (relative to $1/n$). When $\mathscr{E} = K(\mathscr{B})$, we write $\omega(\mathscr{E}) = \omega(K)$.

DEFINITION 6 (THE METRIC ENTROPY [5]). Let $\mathscr{E}$ be any compact convex symmetric subset of $\mathscr{H}$, and $\varepsilon > 0$. Let $\mathscr{U}(\varepsilon)$ be any finite covering of $\mathscr{E}$ by open balls of radius ε, and let card $\mathscr{U}(\varepsilon)$ denote the number of balls in $\mathscr{U}(\varepsilon)$. Put $N(\varepsilon) = \inf \operatorname{card} \mathscr{U}(\varepsilon)$, the infimum taken over all finite coverings $\mathscr{U}(\varepsilon)$ of $\mathscr{E}$. Put further $H(\varepsilon) = \log N(\varepsilon)$. Then $H(\varepsilon)$ is a measure of the size of ε-covering required by $\mathscr{E}$, and is called the *ε-entropy* of $\mathscr{E}$.

Clearly $H(\varepsilon) \uparrow \infty$ as $\varepsilon \downarrow 0$. Here the rate of increase of $H(\varepsilon)$ can be effectively compared with that of $1/\varepsilon$. Put

$$\begin{aligned}\rho(\mathscr{E}) &= \operatorname{glb}\{\mu > 0 : H(\varepsilon) \leqq \text{const}\,(1/\varepsilon)^{\mu}\} \\ &= \infty \quad \text{if no such } \mu > 0 \text{ exists.}\end{aligned} \tag{19}$$

Then $\rho(\mathscr{E})$ is the *order of growth* of the ε-entropy of $\mathscr{E}$. The value of $\rho(\mathscr{E})$ can evidently be computed from the formula

$$\rho(\mathscr{E}) = \limsup_{\varepsilon \to \infty} \frac{\log H(\varepsilon)}{\log 1/\varepsilon}; \tag{20}$$

when $\mathscr{E} = K(\mathscr{B})$ we write $\rho(\mathscr{E}) = \rho(K)$.

Other measures of the "size" of K may be obtained in various other ways. Among them we cite the following:

DEFINITION 7 (THE EIGENVALUE SEQUENCE). As before, let k_n be the nth eigenvalue, counted with multiplicities and arranged in order of decreasing magnitude, of the operator K. Since K is compact, $|k_n| \downarrow 0$ as $n \uparrow \infty$. The rate of decrease of $|k_n|$ can be effectively compared with that of $1/n$. Put

$$\begin{aligned}\kappa(K) &= \operatorname{lub}\{\mu > 0 : |k_n| \leqq \text{const}\,(1/n)^{\mu}\}, \\ &= 0 \quad \text{if no such } \mu > 0 \text{ exists.}\end{aligned} \tag{21}$$

Then $\kappa(K)$ is the *order of decay* of the eigenvalue sequence of K.

DEFINITION 8 (BEHAVIOR ON ORTHONORMAL BASES). Let $\Phi = \{\phi_n\}$ be any orthonormal basis for $\mathscr{H}$, and put $l_n = \|K\phi_n\|$. Assume the l_n are arranged in order of decreasing magnitude. Since K is compact, $l_n \downarrow 0$ as $n \uparrow \infty$. Put

$$\begin{aligned}\lambda(K, \Phi) &= \operatorname{lub}\{\mu > 0 : l_n \leqq \text{const}\,(1/n)^{\mu}\}, \\ &= 0 \quad \text{if no such } \mu > 0 \text{ exists.}\end{aligned} \tag{22}$$

Then $\lambda(K, \Phi)$ is the *order of decay* of the sequence $\|K\phi_n\|$.

DEFINITION 9 (THE FREDHOLM DETERMINANT [2, p. 1106ff]). Now let K be a *normal* compact operator, and assume $\tau(K) < \infty$. Let $k = [\tau(K)]$ be the greatest integer in $\tau(K)$, and z be any complex number. For each integer $j > k$, put $\sigma_j = \operatorname{trace}(K^j)$, and form

$$d(z, K) = \det_k(I - zK) = \exp\left\{-\sum_{j=k+1}^{\infty} \frac{\sigma_j z^j}{j}\right\}; \tag{23}$$

then $d(z, K)$ is the (*generalized*) *Fredholm determinant* of $I-zK$. We know that $d(z, K)$ is a well-defined complex-valued function of z, analytic in the whole z-plane [2, p. 1106]. Let $M(r, d)$ be the maximum modulus of $d(z, K)$ on the circle $|z|=r$,

$$M(r, d) = \max_{|z|=r} |d(z, K)| \tag{24}$$

then $M(r, d) \uparrow \infty$ as $r \uparrow \infty$. The rate of growth of $\log M(r, d)$ can be effectively compared with that of r. Put

$$\gamma(K) = \text{glb}\,\{\mu > 0 : \log M(r, d) \leqq \text{const}\, r^{\mu}\}. \tag{25}$$

Then $\gamma(K)$ is the *exponential order of growth* of $d(z, K)$. To compute $\gamma(K)$, we use

$$\gamma(K) = \limsup_{r\to\infty} \frac{\log\log M(r, d)}{\log r}. \tag{26}$$

Now for any z for which z^{-1} lies in the resolvent set of K, let $R(z^{-1}, K) = z(I-zK)^{-1}$ be the resolvent of K, and put

$$D(z, K) = d(z, K)R(z^{-1}, K); \tag{27}$$

then $D(z, K)$ is the (*generalized*) *Fredholm minorant* of K. It is known that $D(z, K)$ is a well-defined operator-valued function of z, which admits an analytic extension to the whole z-plane [2, p. 1112]. If we define the maximum modulus by

$$M(r, D) = \max_{|z|=r} \|D(z, K)\|, \tag{28}$$

then $M(r, D) \uparrow \infty$ as $r \uparrow \infty$. The rate of growth of $\log M(r, D)$ is then given by

$$\Gamma(K) = \text{glb}\,\{\mu > 0 : \log M(r, D) \leqq \text{const}\, r^{\mu}\}. \tag{29}$$

Then $\Gamma(K)$ is the *exponential order of growth* of $D(z, K)$. To compute $\Gamma(K)$, we use

$$\Gamma(K) = \limsup_{r\to\infty} \frac{\log\log M(r, D)}{\log r}. \tag{30}$$

DEFINITION 10 (THE FREDHOLM COEFFICIENTS). Again let K be a *normal* compact operator and $d(z, K)$ and $D(z, K)$ the entire functions introduced in Definition 9. Write

$$d(z, K) = \sum_{n=0}^{\infty} d_n(K)z^n; \tag{31}$$

here $d_n(K)$ is the nth Taylor coefficient of $d(z, K)$ in the Taylor series expansion about the origin. Since $d_n(z, K)$ is entire, we must have $d_n \to 0$ as $n \to \infty$. The rate of decrease of the d_n can be effectively compared with that of $1/n!$. Put

$$\delta(K) = \text{lub}\,\{\mu > 0 : |d_n(K)| \leqq \text{const}\,(1/n!)^{\mu}\}. \tag{32}$$

Then $\delta(K)$ is the *order of decay* of the Fredholm coefficients d_n relative to $1/n!$.

Similarly, we have

$$D(z, K) = \sum D_n(K)z^n \tag{33}$$

where $D_n(K)$ is the nth Taylor coefficient of $D(z, K)$. Since $D(z, K)$ is entire, we must have $\|D_n(K)\| \to 0$ as $n \to \infty$. Now put

$$\Delta(K) = \text{lub}\,\{\mu > 0 : \|D_n(K)\| \leqq \text{const}\,(1/n!)^\mu\} \tag{34}$$

then $\Delta(K)$ is the corresponding *order of decay* of the $D_n(K)$ relative to $1/n!$.

DEFINITION 11 (THE RESOLVENT). Again let K be a *normal* compact operator, and assume $\tau(K) < \infty$. For z any complex number with z^{-1} in the resolvent set of K, let $R(z^{-1}, K) = z(I - zK)^{-1}$. Then $R(z^{-1}, K)$ is an operator-valued function of z, meromorphic in the whole z-plane. In fact, we have $R(z^{-1}, K) = D(z, K)/d(z, K)$.

To estimate the rate of growth of $R(z^{-1}, K)$ as $|z| \to \infty$, we must replace the maximum modulus with something a little less sensitive to the presence of poles. For this purpose we introduce the *characteristic function of Nevanlinna*, defined as follows (cf. [3, p. 4]).

Let $n(r, R)$ denote the number of poles of $R(z^{-1}, K)$ (i.e., the number of zeros of $d(z, K)$) lying inside the circle $|z| = r$, and define (note that $n(0, R) = 0$)

$$N(r, R) = \int_0^r n(t, R)\frac{dt}{t}. \tag{35}$$

Furthermore, for $x > 0$, put $\log^+ x = \max\{\log x, 0\}$, and define

$$m(r, R) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \|R(r^{-1}e^{-i\theta}, K)\|\, d\theta. \tag{36}$$

Finally, define

$$T(r, R) = N(r, R) + m(r, R). \tag{37}$$

Then $T(r, R)$ is Nevanlinna's characteristic function, designed to play the role of $\log M(r, D)$ for R. Clearly $m(r, R)$ is a weighted average of the modulus of $R(z^{-1}, K)$ on the circle $|z| = r$, and $N(r, R)$ counts the number of poles inside this circle. We shall see that $T(r, R)$ is finite for all r, and that $T(r, R) \uparrow \infty$ as $r \uparrow \infty$ (cf. [3, p. 8]). The rate of growth of $T(r, R)$ is measured by

$$\zeta(K) = \text{glb}\,\{\mu > 0 : T(r, R) \leqq \text{const}\, r^\mu\}. \tag{38}$$

Clearly we have

$$\zeta(K) = \limsup_{r \to \infty} \frac{\log T(r, R)}{\log r}. \tag{39}$$

This completes our enumeration of possible measures of the "size" of K. We now propose to show that they are essentially all the same.

THEOREM 12. *Let K be any compact operator acting on $\mathscr{H}$, and $A=(K^*K)^{1/2}$. Assume $\tau(K)<\infty$. Then (see Definitions 4–11)*

$$2/(\beta(K)-1) = 1/\omega(K) = \rho(K) = \tau(K), \tag{40}$$

$$\gamma(A) = \Gamma(A) = 1/\delta(A) = 1/\Delta(A) = \zeta(A) = 1/\kappa(A) = \tau(A) = \tau(K). \tag{41}$$

If K is normal, then

$$\gamma(K) = \Gamma(K) = 1/\delta(K) = 1/\Delta(K) = \zeta(K) = 1/\kappa(K) = \tau(K). \tag{42}$$

If $2 \leqq \tau(K)<\infty$, then $\lambda(K, \Phi)=\lambda(K)$ is independent of Φ, and

$$\lambda(K) = \tau(K). \tag{43}$$

The proof of (40) depends on the following observations: Let $\{k_n\}$ be the eigenvalue sequence of K, counted with multiplicities and arranged in order of decreasing magnitude. For each $\varepsilon>0$, let $n(\varepsilon)=\max\{n : |k_n| \geqq \varepsilon\}$. Define

$$\begin{aligned}
\kappa_1(K) &= \operatorname{lub}\{\mu : |k_n| \leqq \text{const}\,(1/n)^\mu\},\\
\kappa_2(K) &= \operatorname{lub}\left\{\mu : \left|\prod_{i=1}^{n} k_i\right| \leqq \text{const}\,(1/n!)^\mu\right\},\\
\tau_1(K) &= \operatorname{glb}\{\mu : n(\varepsilon) \leqq \text{const}\,(1/\varepsilon)^\mu\},\\
\tau_2(K) &= \operatorname{glb}\left\{\mu : \sum |k_n|^\mu < \infty\right\}.
\end{aligned} \tag{44}$$

LEMMA 13. *With κ_1, κ_2 and τ_1, τ_2 as defined in* (44), *we have*

$$\tau_1 = \tau_2 = 1/\kappa_1 = 1/\kappa_2. \tag{45}$$

Proof. That $\tau_1 = \tau_2$ is classic, and is proved e.g. in [4, p. 10]. That $\tau_1=1/\kappa_1$ is proved as follows (cf. [1]):

For any $\mu>\kappa_1$, we have $|k_n| \leqq \text{const}\,(1/n)^\mu$ for all n. Now given $\varepsilon>0$, choose $n=n(\varepsilon)$, and note $\varepsilon \leqq |k_{n(\varepsilon)}| \leqq \text{const}\,(1/n(\varepsilon))^\mu$. Hence $n(\varepsilon) \leqq (\text{const}/\varepsilon)^{1/\mu}$ for all $\varepsilon>0$, and so $\tau_1 \leqq 1/\mu$. Conversely, if $\mu<\kappa_1$, we have $|k_n| \geqq \text{const}\,(1/n)^\mu$ for arbitrarily large n. Given such an n, choose $\varepsilon=|k_n|$, and note $\varepsilon=|k_n| \geqq \text{const}\,(1/n(\varepsilon))^\mu$. Hence $n(\varepsilon) \leqq (\text{const}/\varepsilon)^{1/\mu}$ for arbitrarily small ε, and so $\tau_1 \geqq 1/\mu$. Since μ is arbitrary, we have proved $\tau_1 \leqq 1/\kappa_1 \leqq \tau_1$.

To show that $\kappa_1=\kappa_2$, note first that for any $\mu<\kappa_1$, $|k_n| \leqq \text{const}\,(1/n)^\mu$ for all n. Hence $|\prod_{i=1}^n k_i| \leqq (\text{const})^n\,(1/n!)^\mu \leqq \text{const}\,(1/n!)^\nu$ for any $\nu<\mu$. Hence $\kappa_1 \leqq \kappa_2$. Similarly, for any $\mu>\kappa_1$, $|k_n| \geqq \text{const}\,(1/n)^\mu$ for arbitrarily large n, and so $|\prod_{i=1}^n k_i| \geqq (\text{const})^n\,(1/n)^{n\mu} \geqq \text{const}\,(1/n!)^\nu$ for any $\nu>\mu$. Hence $\kappa_2 \leqq \kappa_1$, and so $\kappa_1=\kappa_2$.

To prove (40) we now simply observe that the image $\mathscr{E}=K(\mathscr{B})$ of the unit ball $\mathscr{B}$ under K is a *compact ellipsoid*, whose semiaxes are just the eigenvalues a_n of A

(see [6]). It then follows directly that the n-dimensional metric volume V_n is given by

$$V_n = B_n \prod_{i=1}^{n} a_i, \tag{46}$$

while the n-dimensional metric width w_n is given by

$$w_n = a_n.$$

Hence

$$\beta(K) = \liminf_{n\to\infty} \frac{\log V_n}{\log B_n}$$

$$= \liminf_{n\to\infty} \frac{2\log \prod_{i=1}^{n} a_i}{\log (1/n!)} + 1 \tag{47}$$

$$= 2\kappa_2(A)+1,$$

so $(\beta(K)-1)/2=\kappa_2(A)$. Here we have used the known fact that

$$\lim_{n\to\infty} (\log B_n/\log 1/n!) = 1/2.$$

Since $w_n=a_n$ for all n, we have $\omega(K)=\kappa_1(A)$.

Furthermore, we know that the number $N(\varepsilon)$ of elements in an optimal ε-covering of $\mathscr{E}$ is bounded above and below by (see [6])

$$\prod_{i=1}^{n(2\varepsilon)} \frac{a_i}{\varepsilon} \leqq N(\varepsilon) \leqq \prod_{i=1}^{n(\varepsilon/4\sqrt{2})} \frac{4\sqrt{2}a_i}{\varepsilon}. \tag{48}$$

Since $a_1 \geqq a_2 \geqq \cdots \geqq a_n \geqq 0$, we have

$$2^{n(2\varepsilon)} \leqq N(\varepsilon) \leqq (4\sqrt{2}/\varepsilon)^{n(\varepsilon/4\sqrt{2})}. \tag{49}$$

Hence

$$n(2\varepsilon)\log 2 \leqq H(\varepsilon) \leqq n(\varepsilon/4\sqrt{2}) \log (4\sqrt{2}/\varepsilon). \tag{50}$$

By dividing through by $\log(1/\varepsilon)$ and taking the limit supremum as $\varepsilon \to 0$, we get

$$\tau_1(A) \leqq \beta(K) \leqq \tau_1(A). \tag{51}$$

From Lemma 13 we have $1/\kappa_1(A)=1/\kappa_2(A)=\tau_1(A)=\tau_2(A)=\tau(A)=\tau(K)$. Hence

$$2/(\beta(K)-1) = 1/\omega(K) = \beta(K) = 1/\kappa(A) = \tau(A) = \tau(K). \tag{52}$$

Moreover, if K is normal, then clearly $|k_n|=a_n$, and so $\kappa(K)=\kappa(A)$.

The proof of (41) depends on the following result: Let $f(z)$ be an entire function of z, of finite genus. Let d_n be the nth coefficient of the Taylor series for f computed at the origin, and let z_n be the nth zero of f, counted with multiplicities and arranged in order of increasing magnitude. Define

$$\begin{aligned} \delta &= \text{lub}\,\{\mu : |d_n| \leqq \text{const}\,(1/n!)^{\mu}\}, \\ \gamma &= \text{glb}\,\{\mu : |f(z)| \leqq \exp \text{const}\,|z|^{\mu}\}, \\ \tau &= \text{glb}\left\{\mu : \textstyle\sum |z_n|^{-\mu} < \infty\right\}. \end{aligned} \tag{53}$$

LEMMA 14. *With γ, δ and τ as defined above, we have*

$$1/\delta = \gamma = \tau. \tag{54}$$

Proof. The fact that $\gamma=\tau=\limsup_{n\to\infty}(n \log n/\log 1/|d_n|)$ is classic (see[4, Chapter 1]). Here we need only observe that

$$\delta = \liminf_{n\to\infty} \frac{\log |d_n|}{\log (1/n!)}. \tag{55}$$

Hence

$$1/\delta = \limsup_{n\to\infty;} \frac{\log n!}{\log 1/|d_n|} = \gamma = \tau. \tag{56}$$

To prove (41), we observe that, if K is normal, and $\tau(K)<\infty$, then $d(z, K) = \det_k (I-zK)$ is given by

$$\begin{aligned} d(z, K) &= \exp \operatorname{tr}\left\{-\sum_{j=k+1}^{\infty} \frac{(zK)^j}{j}\right\} \\ &= \prod_{n=1}^{\infty}\left\{(1-zk_n) \exp \sum_{j=1}^{k} \frac{(zk_n)^j}{j}\right\} \end{aligned} \tag{57}$$

(see [2, p. 1106]). Hence $d(z, K)$ is an entire function of z, of finite genus, whose zeros are $z_n=1/k_n$, and whose Taylor coefficients are $d_n(K)$. It follows immediately from Lemma 14 that

$$1/\delta(K) = \gamma(K) = \tau(K). \tag{58}$$

If now K is arbitrary, then A is normal, and

$$1/\delta(A) = \gamma(A) = \tau(A) = \tau(K). \tag{59}$$

A similar argument holds for $D(z, K)$. Assume K is normal, and $\tau(K)<\infty$. Then for any eigenvalue k_n of K and any z with $|z|>1$ we have

$$|z/(1-zk_n)| \geqq 1/(1+|k_n|) \geqq 1/(1+|k_1|). \tag{60}$$

It follows that the resolvent $R(z^{-1}, K)$ of K satisfies

$$\|R(z^{-1}, K)\| \geqq 1/(1+\|K\|) \tag{61}$$

for all z with $|z|>1$. Hence $D(z, K)=d(z, K)R(z^{-1}, K)$ satisfies

$$\|D(z, K)\| \geqq |d(z, K)|/(1+\|K\|) \tag{62}$$

for all z with $|z|>1$.

It follows that

$$\Gamma(K) \geqq \gamma(K). \tag{63}$$

On the other hand, if $\mu>\tau(K)$, then we know that

$$\|D(z, K)\| \leqq |z| \exp\{\text{const } |z|^{\mu}\|K\|_{\mu}^{\mu}\} \tag{64}$$

(see [2, p. 1112]). Hence $\Gamma(K)\leqq\mu$, and so $\Gamma(K)\leqq\tau(K)=\gamma(K)$.

The proof that $\Delta(K)=1/\Gamma(K)$ is the operator analogue of the proof that $\delta(K)=1/\gamma(K)$, and will not be presented here (see [4, p. 4]).

Thus when K is normal, we have $1/\Delta(K)=\Gamma(K)=\tau(K)$. When K is arbitrary, we have $1/\Delta(A)=\Gamma(A)=\tau(A)=\tau(K)$.

We note in passing that the order of decay of the Fredholm coefficients is of some interest in the problem of computing approximants to $R(z^{-1}, K)=D(z, K)/d(z, K)$. The asymptotic accuracy of the approximants can be estimated from the values of $\delta(K)$ and $\Delta(K)$, which in turn can be determined from the value of the invariant $\tau(K)$.

For the resolvent, we argue as follows: With $N(r, R)$, $m(r, R)$ and $T(r,R)$ defined as in (35), (36) and (37), note that $n(r, R)$ is the number of poles of $R(z^{-1}, K)$ inside $|z|=r$, i.e., the number of zeros of $d(z, K)$ inside $|z|=r$, which is just the number of eigenvalues k_n of K with $|k_n|\geqq 1/r$. By Lemma 13, then, $n(r, R)$ has order of growth $\tau(K)$. It follows that

$$N(r, R) = \int_0^r n(t, R)\frac{dt}{t}$$

also has order of growth $\tau(K)$.

To compute the order of growth of $m(r, R)$, first note that

$$\|R(z^{-1}, K)\| = |d(z, K)^{-1}|\ \|D(z, K)\|.$$

Hence

$$\log^+ \|R(z^{-1}, K)\| \leqq \log^+ |d(z, K)^{-1}|+\log^+ \|D(z, K)\|.$$

Thus $m(r, R)\leqq m(r, 1/d)+m(r, D)$.

Now $m(r, D)\leqq \log^+ M(r, D)=\log M(r, D)$ for r sufficiently large. Here $M(r, D)$ is the maximum modulus of $D(z, K)$. The order of growth $\Gamma(K)$ of $\log M(r, D)$ we have shown to be equal to $\tau(K)$.

For $m(r, 1/d)$, we observe that from Jensen's Theorem we have (cf. [3, p. 4])

$$m(r, 1/d)+N(r, 1/d) = m(r, d)+N(r, d). \tag{65}$$

But since $d(z, K)$ is entire, $N(r, d)=0$. Moreover, for large r, $m(r, d)\leqq \log M(r, d)$, whose order of growth $\gamma(K)$ is equal to $\tau(K)$. Finally $N(r, 1/d)=N(r, R)$ has order of growth $\tau(K)$, as shown above. It follows that $m(r, 1/d)$ has order of growth at most $\tau(K)$.

Hence $T(r, R)=m(r, R)+N(r, R)$ has order of growth equal to the maximum of that of $m(r, R)$ and $N(r, R)$, which is just $\tau(K)$, as required.

We have shown that if K is normal, then $\zeta(K)=\tau(K)$. When K is arbitrary, then A is normal, and we have $\zeta(A)=\tau(A)$.

It remains to prove (43). Let K be any compact operator with $2\leqq \tau(K)<\infty$, and $\Phi=\{\phi_n\}$ any orthonormal basis for $\mathscr{E}$. We know that if $2\leqq \lambda(K, \Phi)<\mu$, then

$\sum \|K\phi_n\|^\mu < \infty$ and so $\|K\|_\mu < \infty$ (cf. [2, p. 1106]) and so $\lambda(K) < \mu$. Hence $\lambda(K) \leqq \lambda(K, \Phi)$.

Conversely, if $2 \leqq \tau(K) < \mu$, then we know that $\|K\|_\mu < \infty$, and so

$$\sum \|K\phi_n\|^\mu = \sum (A^2\phi_n, \phi_n)^{\mu/2} < \text{const}\,(\|A^2\|_{\mu/2}) = \text{const}\,(\|K\|_\mu)^\mu < \infty$$

(cf. [2, p. 1138]). Thus $\lambda(K, \Phi) < \mu$, and so $\lambda(K, \Phi) \leqq \tau(K)$.

Note that this result is independent of the choice of basis Φ.

The argument also proves that if $\tau(K) \leqq 2$, then $\lambda(K, \Phi) \leqq 2$, and if $\lambda(K, \Phi) \leqq 2$ then $\tau(K) \leqq 2$. In these cases, however, $\lambda(K, \Phi)$ is no longer independent of the basis Φ and equals $\tau(K)$ only for bases "sufficiently close" to the eigenfunctions of A.

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