HANKEL FORMS, TOEPLITZ FORMS AND MEROMORPHIC FUNCTIONS(1)

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1. The purpose of this note is to recast in a meromorphic function-theoretic setting some results from a paper on the spectral theory of Hankel matrices [2]. Specifically, we will obtain a generalization of the following theorem of Carathéodory and Fejér [1].

THEOREM 1. (a) *Let* $S=S(a_0, a_1, \dots, a_n)$ *be the class of functions* $F(z)$, *analytic in* $|z|<1$, *with power series of the form*

$$F(z) = a_0+a_1z+\cdots+a_nz^n+z^{n+1}\sum_{\nu=0}^{\infty} b_\nu z^\nu.$$

Let $(\mu^{(n)})^2$, $\mu^{(n)}>0$, *be the largest eigenvalue of the Hermitian matrix* (h_{pq}), $p, q=0, 1, \dots, n$, *where*

$$h_{pq} = a_p\bar{a}_q+a_{p-1}\bar{a}_{q-1}+\cdots+a_0\bar{a}_{q-p} \quad \text{for } p \leqq q. \tag{1.1}$$

Then

$$\mu^{(n)} = \inf_{F(z)\in S}\sup_{|z|<1}|F(z)|. \tag{1.2}$$

Equality holds in (1.2) *if and only if* $F(z)$ *has the form*

$$F(z) = \mu^{(n)}e^{i\gamma}\prod_{k=1}^{m}\frac{z+w_k}{1+\bar{w}_kz}, \quad \gamma \text{ real}, \ |w_k| < 1,$$

where m *and* w_k *are uniquely determined.*

(b) *Let* $S'=S(c_0, 2c_1, \dots, 2c_n)$, *where* c_0 *is real, and set* $c_{-k}=\bar{c}_k$. *Denote by* $\lambda^{(n)}$ *the least eigenvalue of the Toeplitz matrix* (c_{q-p}), $p, q=0, 1, \dots, n$. *Then*

$$\lambda^{(n)} = \sup_{G\in S'}\inf_{|z|<1}\operatorname{Re} G(z). \tag{1.3}$$

Equality holds in (1.3) *if and only if* $G(z)$ *has the form*

$$G(z) = \lambda^{(n)}+\sum_{k=1}^{m}\rho_k\frac{1+\varepsilon_kz}{1-\varepsilon_kz}, \quad 1 \leqq m \leqq n, \ \rho_k > 0, \ |\varepsilon_k| = 1,$$

where m, ρ_k, ε_k *are uniquely determined.*

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See [3] for a proof of Theorem 1 and some related theorems. In the Hankel matrix content in which we will consider Theorem 1, 1(a) is the basis for the proof of the theorem of Nehari [5].

For the purpose of our generalization, let $f(z)=a_0+a_1z+\cdots+a_nz^n$. Denote by $S_k(f)=S_k(a_0, a_1, \ldots, a_n)$ the class of functions $F(z)$ meromorphic in $|z|<1$ and of the form $F(z) = f(z)+z^{n+1}g(z)/[(z-\alpha_1)\cdots(z-\alpha_k)]$, where $g(z)\in H^2$ and $|\alpha_\nu|<1$, $\nu=1, \ldots, k$. There will be proved the following

THEOREM 2. (a) *Let* $(\mu_j^{(n)})^2$, $\mu_j^{(n)}\geqq 0$ *be the* $(j+1)$*th largest eigenvalue of the matrix* (1.1). *Then*

$$\mu_j^{(n)} = \inf_{F\in S_j(f)} \sup_{|z|=1} |F(z)|. \tag{1.4}$$

Equality holds in (1.4) *if and only if* $F(z)$ *has the form*

$$F(z) = \mu_j^{(n)}e^{i\gamma}\prod_{k=1}^{m}\frac{z+w_k}{1+\bar{w}_kz}\prod_{k=1}^{r}\frac{1+\bar{v}_kz}{z+v_k}, \quad |w_k|<1, |v_k|<1, \tag{1.5}$$

where $m\leqq n$, $r\leqq j$. *Furthermore, each class* S_j *contains a function of the form* (1.5) *with* $m+r\leqq n$.

(b) *Let* $S_j'=S_j(c_0, 2c_1, \ldots, 2c_n)$. *Denote by* $\lambda_j^{(n)}$ *the* $(j+1)$*th smallest eigenvalue of the Toeplitz matrix* $(c'_{q-p}), p, q=0, 1, \ldots, n$ $(c_p'=c_p,\ p>0,\ c'_{-p}=\bar{c}_p,\ c_0'=\operatorname{Re} c_0)$, *and suppose that* $c_0'\neq 0$. *Then*

$$\lambda_j^{(n)} = \sup_{G\in S_j'}{}' \inf_{|z|=1} \operatorname{Re} G(z), \tag{1.6}$$

where the sup$'$ *is taken over the set of* $G\in S_j'$ *for which* $G(z)$ *is bounded for* $|z|=1$.

By noting that the matrix (h_{pq}) defined by (1.1) is unitarily equivalent to $\mathscr{H}\mathscr{H}^*$ (§2) where $\mathscr{H}$ is the Hankel matrix (a_{n-j-k}), $j, k=0, \ldots, n$ $(a_k=0$ for $k<0)$, we see that one of the main results of [2, Theorem 2.2] is essentially Theorem 2(a) above with the additional assumption that $a_0, a_1, \ldots, a_n$ be real. Theorem 2(a) also appears in [2, Theorem 2.1], in the case $j=n+1$, even without the assumption that $a_0, a_1, \ldots, a_n$ be real. In §§2 and 3 below, we extend our theory from [2] to the context of Theorem 1(a).

Since Carathéodory and Fejér [1, p. 232], the standard way of proving theorems of type 1(b) from 1(a) is to apply the Möbius transformation $Wz=(1-z)(1+z)^{-1}$ to the class S. In [6], Schur gave an elaboration of this method, which we shall adapt in §4 to the present situation; cf. also Gronwall [4]. We will deal, however, with the more general Möbius transformation $W_Mz=(1-Mz)(1+Mz)^{-1}$, and prove the following theorem, from which 2(b) follows easily.

THEOREM 2. (c) *Let* $S_j^\gamma=S_j^\gamma(c_0, 2c_1, \ldots, 2c_n)$, c_0 *real, denote the class of functions* $G(z)$ *belonging to* $S_j(c_0, 2c_1, \ldots, 2c_n)$ *and such that the image of* $|z|=1$ *under* $G(z)$ *is contained in one of the disks* $D_{\zeta,\delta}=\{|z-\zeta|\leqq\delta\}$, *for some real* ζ, $\delta<\gamma$. *Let* $\lambda_j^{(n)}$ *be as in Theorem* 2(b). *Then*

$$\lambda_j^{(n)} = \sup_{G\in S_j^\gamma} \sup\,(\zeta-\delta)\ -(1-M)(1+M)^{-1}, \tag{1.7}$$

where $M = M_\gamma = \gamma^{-1}((1+\gamma^2)^{1/2}-1)$ (<1) *and the second supremum is over all* $D_{\zeta,\delta}$ *containing* G $(|z|=1)$. *Equality holds in* (1.7) *if and only if the function* $G(z)$ *has the form*

$$G(z) = (1-MF(z))(1+MF(z))^{-1}, \tag{1.8}$$

with F *given by* (1.5). *Furthermore, each class* S_j^γ *contains a function of the form* (1.8) *with* F *given by* (1.5) *with* $m+r \leqq n$.

A word or two of comment on the difference between Theorems 1(b) and 2(b) is in order. Since, in Theorem 2(b), we are considering functions which are in general unbounded in $|z|<1$, the introduction of $|z|=1$ instead of $|z|<1$ in (1.4) and (1.6) is essential. It is this which necessitates the boundedness requirement in (1.6). Actually, even (1.3), with $|z|<1$ replaced by $|z|=1$, is false without the boundedness requirement of (1.6). To see this, apply Theorem 1(b) with $c_0, c_1, \ldots, c_n$ replaced by $-c_0, -c_1, \ldots, -c_n$ and obtain a maximizing function $G(z)$ which maps the unit circle onto the line $\operatorname{Re} z = -\lambda_n^{(n)}$. But then $-G(z) \in S_0(c_0, 2c_1, \ldots, 2c_n)$ and, for $|z|=1$, $\inf \operatorname{Re}(-G(z)) = \lambda_n^{(n)} > \lambda_0^{(n)}$, if $c_0, \ldots, c_n$ are suitably chosen. This shows that the boundedness requirement is necessary even for (1.3) if the infimum is taken only over $|z|=1$. Furthermore, since the maximizing functions of Theorem 1(b) are unbounded, there is clearly no analogue of the representation in Theorem 2(b).

Finally, we mention that the proof of Theorem 2(c) could be carried out with W_M replaced by a variety of Möbius transformations. The results, however, would amount to nothing more than translations and rotations of the results of Theorems 2(b) and 2(c).

2. Let $a_0, a_1, \ldots$ be a square summable sequence of complex numbers and let $f(z) = \sum a_\nu z^\nu$. Fixing n once and for all, we introduce the following nonstandard notation: $\mathscr{H}(f(e^{i\varphi}))$ denotes the Hankel matrix (a_{n-j-k}), $(j, k = 0, \ldots, n$, $a_\nu = 0$ if $\nu < 0)$, so that $\mathscr{H}(f)$ is the triangular Hankel matrix usually denoted by $\mathscr{H}(e^{-in\varphi}f(e^{i\varphi}))$. Thus, if $x(e^{i\varphi}) = \sum_{\nu=0}^n x_\nu e^{i\nu\varphi}$, $y(e^{i\varphi}) = \sum_{\nu=0}^n y_\nu e^{i\nu\varphi}$ satisfy

$$f(e^{i\varphi})x(e^{i\varphi}) = y(e^{i\varphi}) + e^{i(n+1)\varphi}h(e^{i\varphi}), \tag{2.1}$$

where h is analytic in $|z|<1$, we have $\mathscr{H}(f)(x_0, x_1, \ldots, x_n) = (y_n, y_{n-1}, \ldots, y_0)$. We will also write $\mathscr{H}(f)(x(e^{i\varphi})) = z(e^{i\varphi}) \equiv e^{in\varphi}y(e^{-i\varphi})$ when (2.1) holds. $\mathscr{H}(f)$ may be considered as acting on either H_n^2, the set of polynomials in $e^{i\varphi}$ of degree at most n, with $L^2(0, 2\pi)$ norm, or on H^2, the $L^2(0, 2\pi)$ closure of polynomials in $e^{i\varphi}$, by defining $\mathscr{H}(f)x = 0$ for $x \in H^2 \ominus H_n^2$. Whether H_n^2 or H^2 is intended will always be either clear from the context or unimportant.

The Toeplitz matrix (c_{q-p}), $p, q = 0, 1, \ldots, n$, will be denoted by $\mathscr{T}(g(e^{i\varphi})) = \mathscr{T}_n(g(e^{i\varphi}))$, where $g(e^{i\varphi}) = \sum_{\nu=-n}^n c_\nu e^{i\nu\varphi}$. Furthermore, $\mathscr{T}_n(g(e^{i\varphi}))$ always acts on H_n^2.

Now let

$$\mathscr{C}(f) = \mathscr{H}(f)\mathscr{H}(f)^* \tag{2.2}$$

and note that $\mathscr{C}(f)$ satisfies

$$\mathscr{C}(f) = \mathscr{U}(h_{pq})\mathscr{U}, \tag{2.3}$$

where h_{pq} is given by (1.1) and $\mathscr{U}=\mathscr{U}_n$ is the unitary operator on H_n^2:

$$\mathscr{U}(x_0, x_1, \ldots, x_n) = (x_n, x_{n-1}, \ldots, x_0).$$

Since $\mathscr{C}(f)$ is unitarily equivalent to (h_{pq}), we will consider only $\mathscr{C}(f)$ in the proof of Theorem 2(a).

Consider the generalized Blaschke product

$$B_j(z) = B(z, \alpha_1, \ldots, \alpha_n) = \prod_{\nu=1}^{j} (z-\alpha_\nu)/(1-\bar{\alpha}_\nu z), \qquad |\alpha_\nu| \neq 1,$$

and for $B(z)=B(z, \alpha_1, \ldots, \alpha_n)$ write $B^*(z)=B(z, \bar{\alpha}_1, \ldots, \bar{\alpha}_n)$. Our proof of Theorem 2(a) is based upon the formal circumstance that, if $\mathscr{C}(f)$ is considered as acting on H^2,

$$(\mathscr{C}(Bf)x, y) = (\mathscr{C}(f)B^*x, B^*y). \tag{2.4}$$

(2.4) follows from (2.2) and from the Hankel matrix relation

$$(\mathscr{H}(g)x, y) = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\varphi}g(e^{i\varphi})x(e^{i\varphi})\bar{y}(e^{-i\varphi})\, d\varphi.$$

(Cf. §10 of [2] for details.)

Turning to the proof of Theorem 2(a), note that, if $\mu_0^2, \mu_1^2, \ldots, \mu_n^2$ are the non-trivial eigenvalues of $\mathscr{C}(f)$ acting on H^2 : $\mu_0 \geqq \mu_1 \geqq \cdots \geqq \mu_n \geqq 0$, then, by the minimum-maximum principle,

$$\mu_j^2 = \inf_{\dim M^\perp = j} \sup_{x,y\in M, \|x\|=\|y\|=1} |(\mathscr{C}(f)x, y)|,$$

and thus (2.4) implies that

$$\mu_j^2 \leqq \|\mathscr{C}(B_j f)\|, \tag{2.5}$$

where $B_j(z)=B(z, \alpha_1, \ldots, \alpha_j)$, $|\alpha_\nu|<1$. For $\dim (B_jH^2)^\perp = \dim (B_j^*H^2)^\perp = j$ [2, Lemma 1.1], if BH^2 denotes the subspace of H^2 of functions of the form Bx, $x\in H^2$.

From (2.5) it is easy to see that (1.4) holds with equality replaced by "$\leqq$". In fact, if $g(z)$ is bounded and analytic in $|z|<1$, Theorem 1(a) implies $\mu_j \leqq \|\mathscr{C}(B_j f)\|^{1/2} \leqq \|B_j f + e^{in\varphi}g(e^{i\varphi})\|_\infty = \|f+\bar{B}_j(e^{i\varphi})e^{in\varphi}g(e^{i\varphi})\|_\infty$, and $f(z)+B_j(z)^{-1}z^n g(z)$ is a typical element of the class $S_j(f)$.

3. For the proof of the remainder of Theorem 2(a), we require a device from [2]. The proof is very similar to the corresponding proof of Theorem 2.2 in [2] and will only be sketched here.

LEMMA 3.1. *Let $\mu_0^2, \mu_1^2, \ldots, \mu_n^2$ denote the nontrivial eigenvalues of $\mathscr{C}(f)$; suppose that $\mu_0 \geqq \mu_1 \geqq \cdots > \mu_r = \mu_{r+1} = \cdots = \mu_{r+s} > \cdots \geqq \mu_n \geqq 0$ and let $u_r(e^{i\varphi})$ denote a solution (if any) of the equation*

$$\mathscr{H}(f)u_r(e^{i\varphi}) = \pm \mu_r \bar{u}_r(e^{-i\varphi}). \tag{3.1}$$

Then $u_r(z)$ has at least r and at most $r+s$ zeros in $|z|<1$.

Proof. If $u_r(z) = \prod_{\nu=1}^{n}(z-\alpha_\nu)$, where $|\alpha_\nu|<1$ for $\nu=1,\ldots,k$, $|\alpha_\nu| \geqq 1$, for $\nu=k+1,\ldots,n$, then since u_r satisfies

$$f(e^{i\varphi})u_r(e^{i\varphi}) = \mu_r e^{in\varphi}\bar{u}_r(e^{i\varphi}) + e^{i(n+1)\varphi}y(e^{i\varphi}),$$

where $y(z)$ is analytic in $|z|<1$, we have

$$f(e^{i\varphi}) = \mu_r \prod_{\nu=1}^{n}(1-\bar{\alpha}_\nu e^{i\varphi})/(e^{i\varphi}-\alpha_\nu) + e^{i(n+1)\varphi}y_1(e^{i\varphi}) \Big/ \prod_{\nu=1}^{n}(e^{i\varphi}-\alpha_\nu), \tag{3.2}$$

where $y_1(e^{i\varphi}) = y(e^{i\varphi})/\prod_{\nu=k+1}^{n}(e^{i\varphi}-\alpha_\nu) \in H^2$. In fact, solving (3.2) for y_1 shows that $y_1 \in L^2$, and $y_1 \in H^2$ follows from $|\alpha_\nu| \geqq 1$ for $\nu=k+1,\ldots,n$. Thus $f(e^{i\varphi}) - e^{i(n+1)\varphi}y_1(e^{i\varphi})/\prod_{\nu=1}^{k}(e^{i\varphi}-\alpha_\nu) = f_2 \in S_k(f)$ and $|f_2| = \mu_r$. By §2 above, this would lead to a contradiction if k were less than r. Thus u_r has at least r zeros in $|z|<1$.

In order to prove the reverse inequality, note that, if $a_0 \neq 0$ (or if n is reduced, and $\{a_j\}$ renumbered so that $a_0 \neq 0$) and if $1/f(z) = p(z) + z^{n+1}G(z)$, where p is a polynomial and G is analytic at 0, then $\mathscr{H}(f)^{-1} = \mathscr{U}\mathscr{H}(p)\mathscr{U}$ [2, Corollary 2.1]. From $\mathscr{H}(f)^{-1}\bar{u}_r = \mu_r^{-1}u_r$, i.e., $\mathscr{H}(p)\mathscr{U}\bar{u}_r = \mu_r^{-1}\mathscr{U}u_r$, it follows that $\mathscr{U}u_r$ has at least $n-(s+r)$ zeros in $|z|<1$. Thus u_r has at least $n-(s+r)$ zeros outside $|z|<1$, and hence at most $s+r$ zeros in $|z|<1$. This proves Lemma 3.1.

To apply the lemma, let $p^*(e^{i\varphi})$ denote $\bar{p}(e^{-i\varphi})$ for $p \in L^2$. Thus $f(e^{i\varphi})x(e^{i\varphi}) = y_1(e^{i\varphi}) + y_2(e^{i\varphi})e^{in\varphi}$, x a polynomial of degree $n, y_1, y_2 \in H^2$, implies that $\bar{f}(e^{-i\varphi})\bar{x}(e^{-i\varphi}) = \bar{y}_1(e^{-i\varphi}) + \bar{y}_2(e^{-i\varphi})e^{in\varphi}$, so that $(\mathscr{H}(f)x(e^{i\varphi}))^* = \mathscr{H}(f)^*(x(e^{i\varphi}))^*$. Thus, if

$$\mathscr{C}(f)x(e^{i\varphi}) = \mu^2 x(e^{i\varphi}), \tag{3.3}$$

let $y(e^{i\varphi}) = \mathscr{H}(f)^*x(e^{i\varphi}) + \mu x(e^{i\varphi})^*$. Then, if $y \not\equiv 0$ (i.e., if $\mathscr{H}(f)x^* \neq -\mu x$), we have $\mathscr{H}(f)y(e^{i\varphi}) = \mathscr{C}(f)x(e^{i\varphi}) + \mu(\mathscr{H}(f)^*x(e^{i\varphi}))^* = \mu[\mu x(e^{i\varphi}) + (\mathscr{H}(f)^*x(e^{i\varphi}))^*] = \mu y(e^{i\varphi})^*$. Thus (3.3) implies (3.1) and hence (3.2) with $r \leqq k \leqq r+s$, so that there is a function in one of the classes $S_r(f), S_{r+1}(f), \ldots, S_{r+s}(f)$ for which equality holds in (1.4).

To show that there is such a minimizing function in each class, it suffices to show that there is one in $S_r(f)$, since $S_r(f) \subset S_{r+1}(f) \subset \cdots \subset S_{r+s}(f)$. This is true if $s=0$, and by an obvious modification of [2, Lemma 3.1], $\mathscr{C}(f)$ is the limit of matrices $\mathscr{C}(f_m)$, where $\mathscr{C}(f_m)$ has a simple spectrum. But $S_r(f_m)$ contains a minimizing function for $\mathscr{C}(f_m)$:

$$h_m(e^{i\varphi}) + f_m(e^{i\varphi}) = \mu_r \prod_{\nu=1}^{n}(e^{i\varphi}-\alpha_\nu^{(m)})/(1-\bar{\alpha}_\nu^{(m)}e^{i\varphi})$$

where $h_m(z)=z^{n+1}g(z)/\prod_{v=1}^{r}(z-\beta_v^{(m)})$, $|\beta_v^{(m)}|<1, g\in H^2$. Taking the limit of a subsequence of the (uniformly bounded) sequence $\{h_m(e^{i\varphi})+f_m(e^{i\varphi})\}$, we obtain a minimizing function for $\mathscr{C}(f)$ in $S_r(f)$.

To complete the proof of Theorem 2(a), it suffices to prove that every minimizing function has the desired form. Let $g(z)\in S_r(f)$ be a minimizing function, $B_j(e^{i\varphi})$ $(j\leqq r)$ the Blaschke product whose zeros are the poles of g. Then it is clear that $B_j(z)g(z)$ is a minimizing function for $\mathscr{C}(B_jf)$ of class $S_0(b_0, b_1, \ldots, b_n)$, where $B_jg=\sum b_v z^v$. Thus, by Theorem 1(a), $B_j(z)g(z)$ may be written in the form

$$B_j(z)g(z) = \mu\prod_{v=1}^{n}(e^{i\varphi}-\alpha_v)/(1-\bar{\alpha}_v e^{i\varphi}), \quad |\alpha_v| < 1.$$

Theorem 2(a) follows easily.

4. One of the main results of Schur's paper [6] is the following fact: *Let $F(z)$, $G(z)$ be analytic in $|z|<1$, $G(z)\neq 0$ in $|z|<1$. Then* 1 *is the norm of the matrix $\mathscr{C}(F/G)$ if and only if the matrix $\mathscr{C}(G)-\mathscr{C}(F)$ is nonnegative definite and* 0 *is an eigenvalue.* (Actually, Schur considered the unitarily equivalent matrix (h_{pq}) of (1.1) in place of C.) He applied this fact with $F(z)=1-f(z)$, $G(z)=1+f(z)$ ($f(z)$ analytic and $|f(z)|\leqq 1$ in $|z|<1$). In this case it is easy to show (as we indicate below) that the matrix $\mathscr{C}(G)-\mathscr{C}(F)$ is essentially the Toeplitz matrix involved in Theorem 1(b). We shall make use of the following generalization of Schur's theorem.

LEMMA 4.1. *Let F, G be analytic in $|z|<1$, $G(0)\neq 0$ and $F(z)/G(z)=K(z)+z^{n+1}K_1(z)$, $K(z)$ a polynomial of degree n and $K_1(z)$ analytic at* 0. *Then*

$$\mathscr{I}-\mathscr{C}(K) = \mathscr{U}_n\mathscr{H}(G)^{-1}[\mathscr{C}(G)-\mathscr{C}(F)](\mathscr{H}(G)^{-1})^*\mathscr{U}_n \tag{4.1}$$

and hence $1\geqq\mu_j$, the $(j+1)$th largest eigenvalue of $\mathscr{C}(K)$, if and only if $0\leqq\lambda_j$, the $(j+1)$th smallest eigenvalue of $\mathscr{C}(G)-\mathscr{C}(F)$.

Proof. The last statement follows easily from (4.1) since the inequality $1\geqq\mu_j$ is equivalent to the existence of a subspace V of H_n^2 of dimension $j-1$ such that $\mathscr{I}-\mathscr{C}(K)$ is nonnegative definite on $H_n^2\ominus V$ and negative definite on V. But such a V exists by (4.1) if and only if $\mathscr{C}(G)-\mathscr{C}(F)$ is negative definite on $V'=\mathscr{H}(G)^{-1}\mathscr{U}_nV$ and nonnegative definite on $H_n^2\ominus V'$. It is clear that this last statement is equivalent to $0\leqq\lambda_j$.

For the proof of (4.1), it suffices, by (2.2), to prove

$$\mathscr{U}_n\mathscr{C}(K)\mathscr{U}_n = \mathscr{H}(G)^{-1}\mathscr{C}(F)(\mathscr{H}(G)^{-1})^*,$$

i.e., to prove

$$\mathscr{U}_n\mathscr{C}(K)\mathscr{U}_n = \mathscr{H}(G)^{-1}\mathscr{H}(F)\mathscr{H}(F)^*(\mathscr{H}(G)^{-1})^*. \tag{4.2}$$

Corollary 2.1 of [2] states that $\mathscr{H}(F)=\mathscr{H}(G)\mathscr{U}_n\mathscr{H}(K)$, i.e., $\mathscr{H}(G)^{-1}\mathscr{H}(F)=\mathscr{U}_n\mathscr{H}(K)$. Taking adjoints, we have $\mathscr{H}(F)^*(\mathscr{H}(G)^{-1})^*=\mathscr{H}(K)^*\mathscr{U}_n$ and multiplying these last two relations gives (4.2). This proves Lemma 4.1.

To prove Theorem 2(c), we use Lemma 4.1 with $F=1-M(f-\eta)$ and $G=1+M(f-\eta)$, where $f(z)=c_0+2c_1z+\cdots+2c_nz^n$ and η is an unspecified real constant for the moment. We wish to compute $\mathscr{C}(1-M(f-\eta))-\mathscr{C}(1+M(f-\eta))$. Applying (2.2) and noting that $\mathscr{H}(1+M(f-\eta))=\mathscr{H}(1)+M\mathscr{H}(f-\eta)=\mathscr{U}_n+M\mathscr{H}(f-\eta)$, an easy computation yields

$$\mathscr{C}(1+M(f-\eta))-\mathscr{C}(1-M(f-\eta)) = 2M[\mathscr{H}(f-\eta)\mathscr{U}+\mathscr{U}\mathscr{H}(f-\eta)].$$

For our choice of $f(z)$, $\mathscr{H}(f-\eta)\mathscr{U}+\mathscr{U}\mathscr{H}(f-\eta)=2(\mathscr{T}_n(\operatorname{Re} f(e^{i\varphi}))-\eta\mathscr{I})$, where $\mathscr{T}=\mathscr{T}_n(\operatorname{Re} f(e^{i\varphi}))$ is the finite Toeplitz matrix $\mathscr{T}=(c'_{q-p}), p, q=0, 1, \ldots, n$ ($c'_p=c_p, p>0, c'_{-p}=\bar{c}_p, c'_0=\operatorname{Re} c_0$). According to Lemma 4.1, if

$$(1-M(f-\eta))/(1+M(f-\eta)) = K(z)+K_1(z)z^{n+1}$$

(K a polynomial, K_1 analytic at $z=0$), 1 is the jth largest eigenvalue of $\mathscr{C}(K)$ if and only if 0 is the jth smallest eigenvalue of $\mathscr{T}-\eta\mathscr{I}$.

We turn, now, to the class $S_k^\gamma(c_0, 2c_1, \ldots, 2c_n)$. Let $\lambda=\sup \sup (\zeta-\delta)$ for $D_{\zeta,\delta}\supset h(|z|=1)$ and $h(z)\in S_k^\gamma$. Then, if $\varepsilon>0$ is sufficiently small, and $\eta=\lambda-(1-M)(1+M)^{-1}-\varepsilon$, the class $S_k^\gamma(c_0-\eta, 2c_1, \ldots, 2c_n)$ contains an element $G(z)$ which satisfies

$$\inf \operatorname{Re} G(e^{i\varphi}) = (1-M)(1+M)^{-1}+\varepsilon/2.$$

Thus, reducing ε again, if necessary, we may assume that G maps $|z|=1$ into the disk $D=\{|z-(\gamma+(1-M)(1+M)^{-1})|\leqq\gamma\}$. But the Möbius transformation $W_Mz=(1-Mz)(1+Mz)^{-1}$ maps the unit disk to D and therefore

$$F(z) = W_M^{-1}G(z) = M^{-1}(1-G(z))/(1+G(z))$$

satisfies $|F(z)|\leqq 1$ for $1\geqq|z|$ near 1. We may also suppose (by adjusting ε) that $G(0)\neq -M^{-1}$ so that $F(z)$ is analytic at 0. Let $F(z)=\sum b_\nu z^\nu$ in a neighborhood of 0. We claim that $F(z)\in S_k(b_0, b_1, \ldots, b_n)$. It suffices to show that $F(z)$ has at most k poles in $|z|<1$. Actually, since $W_MF=G\in S_k^\gamma(c_0, 2c_1, \ldots, 2c_n)$, it suffices to show that F has as many $(-M^{-1})$-points as poles in $|z|<1$. But $M^{-1}>1$ and $|F(z)|\leqq 1$ imply that $F(z)+M^{-1}$ lies in a sector: $\arg w<\pi/2-\varepsilon_0$ for $|z|=1$. Thus, for $|z|=1-\delta$, for sufficiently small $\delta>0$, $F(z)+M^{-1}$ lies in a sector $\arg w<\pi/2-\varepsilon_0/2$. We conclude that $\Delta \arg (F(z)+M^{-1})=0$ for $|z|=1-\delta$, and hence $F(z)+M^{-1}$ has as many zeros as poles in $|z|<1-\delta$, by the argument principle. We conclude that $F(z)\in S_k(b_0, b_1, \ldots, b_n)$, so that, by Theorem 2(a), $\mu_j^{(n)}\leqq 1$. Lemma 4.1 implies that the $(j+1)$st smallest eigenvalue of $\mathscr{T}(\operatorname{Re} f)-\eta\mathscr{I}$ is nonnegative. That is, $\lambda_j^{(n)}\geqq\eta$. Thus we have proved that $\lambda-(1-M)(1+M)^{-1}\geqq\eta$ implies that $\lambda_j^{(n)}\geqq\eta$ and hence $\lambda_j^{(n)}\geqq\lambda-(1-M)(1+M)^{-1}$, where λ denotes the right side of (1.7).

To prove the reverse inequality, let $\eta=\lambda_j^{(n)}-\varepsilon$, and set $W_M^{-1}(G(z)-\eta)=h(z)=\sum a_\nu z^\nu$ for some element $G(z)\in S_k^\gamma(c_0, 2c_1, \ldots, 2c_n)$ for which $h(z)$ is analytic at 0. By Lemma 4.1, $\mu_j^{(n)}<1$ for the matrix $\mathscr{C}(h(e^{i\varphi}))$ and so, by Theorem 2(a), the class

$S_k(a_0, a_1, \ldots, a_n)$ contains elements $F(z)$ of the form (1.5). By another application of the argument principle, it is easily seen that the number of poles of F equals the number of $-M^{-1}$ points. Furthermore, it is easily seen that f_1 and f_2 satisfy $f_1 - f_2 = z^{n+1}g(z)$, $g(z)$ mereomorphic in $|z| < 1$, analytic at 0, if and only if $W_M f_1 - W_M f_2 = z^{n+1} g_2(z)$, g_2 analytic at 0. Thus $W_M F(z) \in S_k(c_0 - \eta, 2c_1, \ldots, 2c_n)$. $W_M F(z) \in S_k^\gamma$ if $\varepsilon > 0$ is sufficiently small follows since W_M maps $|z| < 1 - \delta_1$ into a circle of radius δ_2, $\delta_2 < \gamma$. Thus $S_k^\gamma(c_0 - \eta, 2c_1, \ldots, 2c_n)$ contains an element $G(z)$ with $\inf \operatorname{Re} G(z) \geqq (1-M)(1+M)^{-1}$, and we have that $\sup \inf \operatorname{Re} G(z) > \eta + (1-M)(1+M)^{-1} = -\varepsilon + \lambda_j^{(n)} + (1-M)(1+M)^{-1}$. Since ε may be taken arbitrarily small, this proves (1.7).

Repeating the above argument with $\varepsilon = 0$ shows that an element of the form (1.8) is an extremal element of S_k^γ. The converse of this statement follows easily by taking W_M^{-1} and applying the argument principle once again.

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