AN IDENTITY FOR ELLIPTIC EQUATIONS
WITH APPLICATIONS

BY
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1. Introduction. An elementary identity involving a linear elliptic partial
differential operator $L$ and its associated hermitian form will be used to obtain new
comparison theorems, oscillation theorems, and lower bounds for eigenvalues.
Comparison theorems will be obtained for both subsolutions and complex-valued
solutions in unbounded domains of Euclidean space, generalizing earlier results of
Hartman and Wintner [4], Protter [8], and the author [11], [12]. Oscillation
theorems of Kreith’s type [6] will be extended to (i) unbounded domains; (ii) non-
self-adjoint operators; and (iii) subsolutions.

Lower bounds for the eigenvalues of $L$ arise naturally from the basic identity in
the case of bounded domains, and are extended to unbounded domains when the
coefficients of $L$ satisfy suitable conditions. The form of the lower bounds is the
same as that obtained by Protter and Weinberger [9], [10] for bounded domains.

2. The main lemma. The linear elliptic differential operator $L$ defined by

\[
Lv = \sum_{i,j=1}^{n} D_i(A_{ij}D_jv) + 2 \sum_{i=1}^{n} B_iD_iv + Cv
\]

will be considered on unbounded domains $R$ in $n$-dimensional Euclidean space $E^n$.
The boundary $P$ of $R$ is supposed to have a piecewise continuous unit normal vector
at each point. As usual, points in $E^n$ are denoted by $x=(x_1, x_2, \ldots, x_n)$ and differ-
entiation with respect to $x_i$ is denoted by $D_i$, $i=1, 2, \ldots, n$. The coefficients
$A_{ij}$, $B_i$, and $C$ are assumed to be real and continuous in $R \cup P$ and the matrix
$(A_{ij})$ positive definite in $R$ (ellipticity condition). The domain $\mathcal{D}_L = \mathcal{D}_L(R)$ of $L$ is
defined to be the set of all complex-valued functions $v \in C^4(R \cup P)$ such that all
derivatives of $v$ involved in $Lv$ exist and are continuous at every point in $R$.

Let $T_a$ denote the $n$-disk $\{x \in E^n : |x-x_0| < a\}$ and let $S_a$ denote the bounding
$(n-1)$-sphere, where $x_0$ is a fixed point in $E^n$. Define

\[
Ra = R \cap T_a, \quad Pa = P \cap T_a, \quad Ca = R \cap S_a.
\]

Clearly there exists a positive number $a_0$ such that $R_a$ is a bounded domain with
boundary $Pa \cup Ca$ for all $a \geq a_0$.

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Let $Q[z]$ be the hermitian form in $n+1$ variables $z_1, z_2, \ldots, z_{n+1}$ defined by

\[ Q[z] = \sum_{i,j=1}^{n} A_{ij} z_i z_j - \sum_{i=1}^{n} B_i (z_i z_{n+1} + z_{n+1} z_i) + G|z_{n+1}|^2 \]

where $G$ is any continuous function in $R$ satisfying the inequality

\[ G \det (A_{ij}) \geq \sum_{i=1}^{n} B_i B^*_i, \]

$B^*_i$ denoting the cofactor of $-B_i$ in the matrix associated with $Q[z]$. Condition (4) is known to be necessary and sufficient for $Q[z]$ to be positive semidefinite [2], [12].

Let $M_a$ be the quadratic functional defined by

\[ M_a[u] = \int_{R_a} F[u] \, dx, \]

where

\[ F[u] = \sum_{i,j} A_{ij} D_i u D_j \bar{u} - 2 \text{Re} \left( \sum_i B_i D_i \bar{u} \right) + (G - C)|u|^2. \]

Define $M[u] = \lim_{a \to \infty} M_a[u]$ (whenever the limit exists). The domain $\mathcal{D}_M = \mathcal{D}_M(R)$ of $M$ is defined to be the set of all complex-valued functions $u \in C^1(R \cup P)$ such that $M[u]$ exists and $u$ vanishes on $P$.

Define

\[ [u, v] = \int_{C_a} u \sum_{i,j} A_{ij} n_i D_i v \, ds, \]

where $(n_i)$ denotes the unit normal to $C_a$, and define

\[ [u, v]_{a} = \lim_{a \to \infty} [u, v]_{a}, \]

whenever the limit on the right side exists. The notation $M[u; R]$ will be used for $M[u]$ and $[u, v; R]$ will be used for $[u, v]$ in $\S 5$ when different domains are under consideration.

An $L$-subsolution (supersolution) is a real-valued function $v \in \mathcal{D}_L(R)$ which satisfies $L v \leq 0$ ($L v \geq 0$) at every point in $R$.

The following are extensions of results in [12] to subsolutions and supersolutions, and to complex-valued functions $u \in \mathcal{D}_M(R)$.

**Lemma 1.** For every $u \in C^1(R)$ and every real $v \in \mathcal{D}_L(R)$ which does not vanish in $R$, the following identity is valid at each point in $R$:

\[ \sum_{i,j} A_{ij} X_i X_j - 2 \text{Re} \left( \sum_i B_i X_i \right) + G|u|^2 + \sum_i D_i(|u|^2 Y_i) = F[u] + |u|^2 v^{-1} L v, \]

where

\[ X_i = v D_i(u/v), \quad Y_i = v^{-1} \sum_{j=1}^{n} A_{ij} D_j v, \quad i = 1, 2, \ldots, n. \]

The proof is a direct calculation similar to that given in [12].
Theorem 1. If there exists \( u \in \mathcal{D}_M(R) \) not identically zero such that \( M[u] < 0 \), then there does not exist an \( L \)-subsolution (supersolution) \( v \) satisfying \( \|u^2/v, v\| \geq 0 \) which is positive (negative) everywhere in \( R \cup P \). In particular, every real solution of \( Lv = 0 \) satisfying \( \|u^2/v, v\| \geq 0 \) must vanish at some point of \( R \cup P \). In the self-adjoint case \( B_i = 0, i = 1, 2, \ldots, n, \) and \( G = 0 \), the same conclusions are valid when the hypothesis \( M[u] < 0 \) is weakened to \( M[u] \leq 0 \).

Proof. Suppose to the contrary that there exists such a positive \( L \)-subsolution. Then integration of (9) over \( R_a \) yields

\[
\int_{R_a} F[u] \, dx \geq \int_{R_a} \sum_{i} D_i(|u|^2 Y_i) \, dx
\]

since the first three terms on the left side of (9) constitute a positive semidefinite form by the hypothesis (4). Since \( u = 0 \) on \( P_a \), by the definition of \( \mathcal{D}_M \), it follows from Green's formula that the right side of (10) is equal to

\[
\int_{P_a \cup C_a} \sum_{i} |u|^2 n_i Y_i \, ds = \int_{C_a} \frac{|u|^2}{v} \sum_{i,f} A_{i,f} n_i D_f v \, ds = \|u^2/v, v\|.
\]

Thus (7), (10), and the hypothesis \( \|u^2/v, v\| \geq 0 \) imply that

\[
M[u] = \lim_{a \to \infty} \int_{R_a} F[u] \, dx \geq 0.
\]

The contradiction proves that a positive \( L \)-subsolution satisfying \( \|u^2/v, v\| \geq 0 \) cannot exist. The analogous statement for a negative \( L \)-supersolution \( v \) follows from the fact that \(-v\) would then be a positive \( L \)-subsolution.

To prove the second statement of Theorem 1, suppose to the contrary that there exists a real solution \( v \neq 0 \) in \( R \cup P \). Then \( v \) would be either a positive \( L \)-subsolution or a negative \( L \)-supersolution in \( R \cup P \).

The proof in the self-adjoint case is similar to that given in [12, p. 281] and will be omitted.

We remark that the condition \( \|u^2/v, v\| \geq 0 \) of Theorem 1 is a mild "boundary condition at \( \infty \)" generalizing the usual condition \( v \neq 0 \) on the boundary of bounded domains.

3. Lower bounds for eigenvalues. Let \( \mathcal{D} \) be the Hilbert space \( \mathcal{L}_2(R) \), with inner product \( \langle u, v \rangle = \int_R u(x) \overline{v(x)} \, dx \) and norm \( ||u|| = \langle u, u \rangle^{1/2} \). Let \( \mathcal{D} \) be the set of all complex-valued functions \( u \in \mathcal{D}_L \cap \mathcal{D} \) such that \( u \) vanishes on \( P \). In this section the elliptic operator (1), with domain \( \mathcal{D} \), is assumed to have the self-adjoint form

\[
Lv = \sum_{i,f} D_i(A_{i,f} D_f v) - Cv,
\]

under the conditions described below (1). In the case of the Schrödinger operator \(-L = -\Delta + C(x)\), it is well-known [1], [3, p. 146] that the lower part of the spectrum contains only eigenvalues of finite multiplicity if \( C(x) \) is bounded from below.
In the self-adjoint elliptic case, an assumption on the coefficients $A_{ij}$ is needed as well.

Let $A^+(x)$ denote the largest eigenvalue of $(A_{ij}(x))$ and define

$$
\alpha(r) = \max_{1 \leq |x| \leq r} A^+(x),
$$

$$
\alpha_0(r) = \max_{1 \leq |x| \leq r} \left[ \alpha(1), \max_{1 \leq |x| \leq r} |x|^{-2} A^+(x) \right],
$$

which are nondecreasing functions of $r$. The following assumptions are special cases of those given by Ikebe and Kato [5].

ASSUMPTIONS. (i) $C(x)$ is bounded from below;
(ii) $\int_{r_0}^{\infty} [\alpha(r)\alpha_0(r)]^{-1/2} \, dr = \infty$.

It follows in particular from (i) and (ii) that the conditions $u \in \mathcal{S}$, $Lu \in \mathcal{S}$ imply that $[u, u] = 0$ [5].

Our purpose is to obtain a useful lower bound for the eigenvalues (if any) of $-L$. In the case of bounded domains, Protter and Weinberger [10] recently obtained results of this type by using a general form of the maximum principle. It will be shown here in the case of unbounded domains that a lower bound is available as an easy consequence of Lemma 1.

**Theorem 2.** Let $\lambda$ be the lowest eigenvalue and $u$ be an associated normalized eigenfunction of the problem $-Lu = \lambda u$, $u \in \mathcal{D}$. If $v$ is any function in $\mathcal{D}_L$ such that $v(x) > 0$ for $x \in R \cup P$ and $[|u|^2/v, v] \geq 0$, then

$$
\lambda \geq \inf_{x \in R} [-Lv(x)/v(x)].
$$

**Proof.** With $B_i = 0$, $i = 1, 2, \ldots, n$ and $G = 0$, integration of (9) over $R_\alpha$ yields

$$
M_a[u] + \int_{R_\alpha} |u|^2 v^{-1} L v \, dx \geq \sum_{i=1}^{n} D_i [u]^2 Y_i \, dx
$$

where the positive-definiteness of $(A_{ij})$ has been taken into account. Since $u = 0$ on $P_\alpha$, it follows from Green's formula that

$$
M_a[u] = - \int_{R_\alpha} \bar{u} L u \, dx + [u, u]_a.
$$

However, $\lim [u, u]_a = 0 (a \to \infty)$ is a general consequence of $u \in \mathcal{S}$ and $Lu \in \mathcal{S}$ under the above assumptions [5], and therefore

$$
M[u] = \lim_{a \to \infty} M_a[u] = \lambda \|u\|^2 = \lambda.
$$

As in the proof of Theorem 1, the right member of (12) has the limit $[|u|^2/v, v]$ as $a \to \infty$, which is nonnegative by hypothesis. Thus

$$
\lambda + \int_R |u|^2 v^{-1} L v \, dx \geq 0,
$$

which implies (11).
In the bounded case, the condition \([|u|^2/v, v|\geq 0\) is vacuous and Theorem 2 reduces to a well-known result [9]. However, the proof given here is especially easy. We remark that the extra condition \([|u|^2/v, v|\geq 0\) in the unbounded case is a condition on the asymptotic behavior of \(v\) as \(|x| \to \infty\); it is roughly equivalent to the usual hypotheses for bounded domains that \(u = 0\) on the boundary, \(v > 0\) in \(R \cup P\), and \(v \in C^1(R \cup P)\). In the case of the Schrödinger operator \(-\Delta + C(x)\), it is known [3, p. 179] that \(|u(x)| < Ke^{-\mu|x|}\), where \(K\) and \(\mu\) are constants, for every eigenfunction \(u\), and hence various exponential functions can serve as the test functions \(v\). As an easy example, consider the one-dimensional harmonic oscillator problem

\[
-\frac{d^2u}{dx^2} + x^2u = \lambda u, \quad 0 \leq x < \infty,
\]

\(u(0) = 0\).

The test function \(v = \exp(-x^2/2)\) yields the lower bound 1 whereas the exact lowest eigenvalue is known to be 3.

4. Comparison theorems. Consider, in addition to (1), a second elliptic operator \(l\) defined by

\[
l u = \sum_{i,j=1}^n a_{ij} D_i D_j u + 2 \sum_{i=1}^n b_i D_i u + cu
\]

in which the coefficients satisfy the same conditions as the coefficients in (1). In addition to (5) consider the quadratic functional defined by

\[
m_a[u; Q] = \int_{\Omega \cap \tau_a} \left[ \sum_{i,j} a_{ij} D_i D_j u - 2 \Re \left( u \sum_{i=1}^n b_i D_i \bar{u} \right) - c|u|^2 \right] dx
\]

for every subdomain \(Q \subset R\), and let \(m[u; Q] = \lim_{a \to \infty} m_a[u; Q]\). The domain \(\mathcal{D}(Q)\) of \(m\) is the analogue of \(\mathcal{D}_m(Q)\) (defined in §2). The variation of \(L\) relative to the domain \(Q\) is defined as \(V[u; Q] = m[u; Q] - M[u; Q]\), that is

\[
V[u; Q] = \int_Q \left[ \sum_{i,j} (a_{ij} - A_{ij}) D_i u D_j \bar{u} - 2 \Re \left( u \sum_{i=1}^n (b_i - B_i) D_i \bar{u} \right) \right] + (C - c - G)|u|^2 \right] dx,
\]

with domain \(\mathcal{D}_V(Q) = \mathcal{D}_m(Q) \cap \mathcal{D}_M(Q)\).

The analogues of (7), (8) for the operator \(l\) relative to the domain \(Q\) are

\[
\{u, v; Q\}_a = \int_{\Omega \cap \tau_a} \sum_{i,j} a_{ij} n_i \Re (u D_j v) \, ds;
\]

\[
\{u, v; Q\} = \lim_{a \to \infty} \{u, v; Q\}_a.
\]

When \(Q = R\) is the only domain under consideration, the abbreviations \(V[u]\), \(\{u, v\}\) will be used for \(V[u; R]\), \(\{u, v; R\}\), respectively.

The following comparison theorems of Sturm's type are easy extensions of those
Theorem 3. Suppose $G$ is a continuous function in $R$ satisfying the inequality (4). If there exists a nontrivial solution $u \in \mathcal{D}_V(R)$ of $lu=0$ such that $\{u, u\} \leq 0$ and $V[u] > 0$ then there does not exist an $L$-subsolution (-supersolution) which is positive (negative) everywhere in $R \cup P$ and satisfies $|u|^2/v, v] \geq 0$. In particular, every real solution of $Lv=0$ satisfying $|u|^2/v, v] \geq 0$ must vanish at some point of $R \cup P$. The same conclusions hold if the hypotheses $V[u] > 0$, $|u|^2/v, v] \geq 0$ are replaced by $V[u] \geq 0$, $|u|^2/v, v] \geq 0$, respectively.

Theorem 4. With $G$ as in Theorem 3, if there exists a positive $l$-supersolution $u \in \mathcal{D}_V(R)$ such that $\{u, u\} \leq 0$ and $V[u] > 0$, then the conclusions of Theorem 3 are valid.

Theorem 5 (Self-adjoint case). Suppose $b_i = B_i = 0$, $i=1, 2, \ldots, n$ in (1) and (13) and $G=0$. If there exists either (i) a nontrivial complex-valued solution $u \in \mathcal{D}_V(R)$ of $lu=0$, or (ii) a positive $l$-supersolution $u \in \mathcal{D}_V(R)$, such that $\{u, u\} \leq 0$ and $V[u] \geq 0$, then an $L$-subsolution (-supersolution) $v$ satisfying $|u|^2/v, v] \geq 0$ cannot be everywhere positive (negative) in $R \cup P$. In particular, every real solution of $Lv=0$ satisfying $|u|^2/v, v] \geq 0$ must vanish at some point of $R \cup P$.

Proof of Theorem 3. Since $u=0$ on $P_a$, it follows from Green's formula that

$$m_a[u] = -\int_{R_a} \text{Re} (ulu) \, dx + \{u, u\}. a.$$ 

Since $lu=0$ and $l$ has real-valued coefficients, also $lu=0$. Since $\{u, u\} \leq 0$, we obtain in the limit $a \to \infty$ that $m[a] \leq 0$. The hypothesis $V[u] > 0$ is equivalent to $M[u] < M[u]$. Hence $M[u] < 0$ and Theorem 1 shows an $L$-subsolution (-supersolution) cannot be everywhere positive (negative) in $R \cup P$ under the hypothesis $|u|^2/v, v] > 0$. The second statement of Theorem 3 also follows from Theorem 1. The last statement follows upon obvious modifications of the inequalities.

If $u$ is a positive $l$-supersolution in $R$ such that $\{u, u\} \leq 0$, it follows again from [17] that $m[u] \leq 0$. The proof of Theorem 4 is then completed in the same way as that of Theorem 3. The proof of Theorem 5 follows similarly from the statement in Theorem 1 relative to the self-adjoint case.

It follows from (14) by partial integration that

$$V[u; Q] = \int_Q \left[ \sum_{i,j} (a_{ij} - A_{ij}) D_i u D_j \bar{u} + \delta |u|^2 \right] \, dx + \Omega(Q),$$

where

$$\delta = \sum_{i=1}^n D_i (b_i - B_i) + C - c - G,$$

and

$$\Omega(Q) = \lim_{a \to \infty} \int_{Q \cap S_a} \sum_{i} (B_i - b_i) |u|^2 n_i \, ds,$$ 

whenever the limit exists.

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$L$ is called a strict Sturmian majorant of $l$ in $Q$ when the following conditions are fulfilled: (i) $(a_{ij} - A_{ij})$ is positive semidefinite and $\delta \geq 0$ in $Q$; (ii) $\Omega(Q) \geq 0$; and (iii) either $\delta > 0$ at some point in $Q$ or $(a_{ij} - A_{ij})$ is positive definite and $c \neq 0$ at some point. A function defined in $Q$ is said to be of class $C^{2,1}(Q)$ when all of its second partial derivatives exist and are Lipschitzian in $Q$.

**Theorem 6.** Suppose that $L$ is a strict Sturmian majorant of $l$ and that all the coefficients $a_{ij}$ involved in $l$ are of class $C^{2,1}(R)$. If there exists a nontrivial solution $u \in \mathcal{D}_v(R)$ of $lu = 0$ such that $\{u, u\} \leq 0$, then no $L$-subsolution (supersolution) $v$ satisfying $|u|^2/v, v| \geq 0$ can be everywhere positive (negative) in $R \cup P$. In particular, every real solution of $Lv = 0$ satisfying $|u|^2/v, v| \geq 0$ must vanish at some point of $R \cup P$.

**Theorem 7 (Self-adjoint case).** Suppose $b_i = B_i = 0$, $i = 1, 2, \ldots, n$ in (1) and (13), $G = 0, C \geq c$, and $(a_{ij} - A_{ij})$ is positive semidefinite in $R \cup P$. If there exists either (i) a nontrivial complex-valued solution $u \in \mathcal{D}_v(R)$ of $lu = 0$, or (ii) a positive $l$-super-solution $u \in \mathcal{D}_v(R)$, such that $\{u, u\} \leq 0$, then the conclusion of Theorem 6 is valid.

Since the pointwise conditions $G = 0, C \geq c$, and $(a_{ij} - A_{ij})$ positive semidefinite obviously imply that $V[u] \geq 0$, Theorem 7 is an immediate consequence of Theorem 5. The fact that the hypotheses of Theorem 6 imply $V[u] > 0$ was demonstrated in [12, p. 283], and consequently the conclusion of Theorem 6 follows from Theorems 3 and 4.

In the special case of the Schrödinger operator $-l = -\Delta + c(x)$ with $c(x)$ bounded from below in $R$, the hypothesis $\{u, u\} \leq 0$ of Theorems 5 and 7 can be replaced by $u \in \mathcal{D}$ and $lu \in \mathcal{D}$ since these conditions imply that $\{u, u\} = 0$ [3, p. 56]. In the self-adjoint elliptic case, the same statement holds under quite general conditions on the coefficients, e.g. those stated prior to Theorem 2, as shown by Ikebe and Kato [5]. Also, the conclusion of Theorem 7 is valid even if $(A_{ij})$ is only positive semidefinite provided $L$ is a strict Sturmian majorant of $l$ and all the coefficients $a_{ij}$ are of class $C^{2,1}(R)$ [12, p. 283].

5. Oscillation theorems. In [6] Kreith obtained oscillation theorems for self-adjoint elliptic equations of the form $Lv = 0$ in the case that one variable $x_n$ is separable. He considered the case of bounded domains for which part of the boundary is singular. Here we shall obtain oscillation theorems of a general nature on unbounded domains by appealing to the comparison Theorems 3–7.

Let $T'_a$ denote the complement of $T_a$ in $E^n$. A function $u$ is said to be oscillatory in $R$ at $\infty$, or simply oscillatory in $R$, whenever $u$ has a zero in $R \cap T'_a$ for all $a > 0$.

A domain (not necessarily bounded) $Q \subset R$ is called a nodal domain of a function $u$ if $u = 0$ on $\partial Q$ and $\{u, u; Q\} \leq 0$. If $Q$ is bounded, the latter condition is understood to be void, and the definition reduces to the standard definition of a nodal domain. If $-l$ is the Schrödinger operator with potential $c(x)$ bounded from below, sufficient
The conditions for \( Q \) to be a nodal domain of \( u \in D(Q) \) are \( u = 0 \) on \( \partial Q \), \( u \in \mathcal{D} \), and \( lu \in \mathcal{D} \) \([3, p. 56]\). A function \( u \) is said to have the nodal property in \( R \) whenever \( u \) has a nodal domain \( Q \subset R \cap T^*_a \) for all \( a > 0 \).

The following results are immediate consequences of Theorems 3–7.

Theorem 8. Suppose \( G \) is a continuous function in \( R \) satisfying (4). Suppose there exists either (i) a nontrivial complex-valued solution \( u \) of \( lu = 0 \), or (ii) a positive \( l \)-supersolution \( u \), with the nodal property in \( R \) such that \( V[u; Q] > 0 \) for every nodal domain \( Q \). Then every real solution of \( Lv = 0 \) is oscillatory in \( R \) provided \( |u|^2 / v, \nu; Q| \geq 0 \) for every \( Q \). If the nodal domains are all bounded, every solution of \( Lv = 0 \) is oscillatory in \( R \). In the self-adjoint case \( b_i = B_i = 0 \), \( i = 1, 2, \ldots, n \), the same conclusions hold under the weaker condition \( V[u; Q] \geq 0 \) for every nodal domain \( Q \).

Theorem 9. Suppose that \( L \) is a strict Sturmian majorant of \( I \) and that all the coefficients involved in \( I \) are of class \( C^{2,1}(R) \). If there exists a nontrivial complex-valued solution of \( lu = 0 \) with the nodal property in \( R \), then every real solution of \( Lv = 0 \) is oscillatory in \( R \) provided \( |u|^2 / v, \nu; Q| \geq 0 \) for every nodal domain \( Q \). If the nodal domains are all bounded, every solution of \( Lv = 0 \) is oscillatory in \( R \). In the self-adjoint case \( b_i = B_i = 0 \), \( i = 1, 2, \ldots, n \), the same conclusions hold under the weaker hypotheses \( G = 0 \), \( C \geq c \), and \((a_i - A_i)\) positive semidefinite in \( R \cup P \).

Kreith has shown [6] that equations of the form

\[
D_n[a(x_n)D_nu] + \sum_{i,j=1}^{n-1} D_i[a_{ij}(x)D_ju] + c(x_n)u = 0, \quad \vec{x} = (x_1, x_2, \ldots, x_{n-1}),
\]

have bounded nodal domains in the form of cylinders, under suitable hypotheses, when \( R \) is a bounded domain with an \((n-1)\)-dimensional singular boundary. We shall show that the analogous construction for unbounded domains is valid provided \( R \) is limit cylindrical, i.e. contains an infinitely long cylinder. Without loss of generality we can assume that \( R \) contains a cylinder of the form

\[ G \times \{x_n : 0 \leq x_n < \infty\}, \]

where \( G \) is a bounded \((n-1)\)-dimensional domain.

Let \( \mu \) be the smallest eigenvalue of the boundary problem

\[
-\sum_{i,j=1}^{n-1} D_i[a_{ij}(\vec{x})D_j\phi] = \mu \phi \quad \text{in } G, \\
\phi = 0 \quad \text{on } \partial G.
\]

Theorem 10. If there exists a positive number \( b \) such that

\[
\int_b^\infty \frac{dt}{a(t)} = \infty \quad \text{and} \quad \int_b^\infty [c(t) - \mu] \, dt = \infty,
\]

then equation (18) has a solution \( u \) with the nodal property in \( R \). If \( V[u; Q] \geq 0 \) for every nodal domain \( Q \), every solution of \( Lv = 0 \) is oscillatory in \( R \). In particular,
every solution of the self-adjoint equation $L v = 0$ is oscillatory provided $C \geq c$ and $(a_i - A_{ij})$ is positive semidefinite in $R \cup P$.

**Proof.** The hypotheses (20) imply that the ordinary differential equation

$$D_n[a(x_n)Dnw] + [c(x_n) - \mu]w = 0$$

is oscillatory at $x_n = \infty$ on account of well-known theorems of Leighton [7] and Wintner [13]. Let $w$ be a solution with zeros at $x_n = \delta_1, \delta_2, \ldots, \delta_m, \ldots$, where $\delta_m \uparrow \infty$. If $\phi$ is an eigenfunction of (19) corresponding to the eigenvalue $\mu$, then the function $u$ defined by $u(x) = w(x_n)\phi(x)$ is a solution of (18) by direct calculation, with nodal domains in the form of cylinders

$$G_m = G \times \{x_n : \delta_m < x_n < \delta_{m+1}\}, \quad m = 1, 2, \ldots$$

Thus $u$ has a nodal domain $G_m \subset R \cap T'_a$ for all $a > 0$. In fact, given $a > 0$, choose $m$ large enough so that $\delta_m \geq a$. Then $x \in G_m$ implies $|x| \geq |x_n| > a$ so $x \in T'_a$. Hence (18) has a solution $u$ with the nodal property. The second statement of Theorem 10 follows from Theorem 8 and the last statement follows from Theorem 9.

**References**


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