CONTINUITY OF METRIC PROJECTIONS

BY

D. E. WULBERT[1]

I. **Introduction and notation.** Let M be a Tchebychev set in a Banach space X, and let P represent the metric projection associated with M. This paper centers about the relationship between properties of M and the continuity of P. It is known, for example, that P may be discontinuous even if M is a linear subspace of X, [3], [6], and [9]. The present paper characterizes the locally compact Tchebychev subsets of a smooth Banach space which have continuous metric projections. One necessary and sufficient condition, for instance, is that the Tchebychev set be convex. Other relations between properties of P and the structure of M are found in [1], [6], and [7]. In §IV short proofs are presented for two known results on the continuity of the rational approximation operator.

For terminology on Banach spaces we will follow M. M. Day [4]. Throughout this paper $(X, \|\cdot\|)$ will denote a Banach space. For $r>0$, and x in X, $S[r, x]$ and $S(r, x)$ will denote respectively the closed and open sphere in X, of radius r and centered at x. For a subset K of X, bd K and cl K represent the boundary of K and the closure of K respectively. The Banach space of continuous functions on $[0, 1]$ topologized with the supremum norm is denoted by $C[0, 1]$.

II. **Bounded connectedness.** A subset of X is termed *boundedly connected* if its intersection with every open sphere in X is a connected set.

The proofs for the propositions in this section exhibit no novelty and have been omitted.

1. PROPOSITION.

(a) *A convex set in X is boundedly connected.*

(b) *If X is Euclidean three space, the boundary of a circular cylinder or of a circular cone is boundedly connected. Also the set difference of a solid sphere centered at the origin and a solid circular cone with vertex at the origin is boundedly connected.*

(c) *If $X=C[0, 1]$, then the classical rational functions are boundedly connected. (This result will be proved in* §IV.)

Received by the editors June 6, 1966 and, in revised form, July 10, 1967.

[1] This paper contains results from the author's dissertation written while he held a National Science Foundation Cooperative Fellowship. This research was also sponsored in part by the Air Force Office of Scientific Research under grant AFOSR 1109-66. The author is grateful to Professor E. Ward Cheney who supervised this study, and to the University of Lund where the author was a guest during the preparation of this manuscript.

2. PROPOSITION. *If M is a boundedly connected subset of X, then M is both connected and locally connected. In addition if b is a scalar and x is in X, then bM and $\{x\}+M$ are boundedly connected.*

Examples can be constructed in Euclidean two space to show that the sum of two compact, boundedly connected sets is not generally boundedly connected, even if one of the sets is convex(²).

III. **Tchebychev sets and metric projections.** For a subset M of X and a point x in X we define

$$d(M, x) = \inf\{\|m-x\| : m \text{ in } M\}.$$

If for each x in X there exists precisely one point Px in M such that $\|Px-x\| = d(M, x)$, then M is called a *Tchebychev set*. The mapping P is termed the *metric projection associated with M*.

For the remainder of the paper M will denote a Tchebychev subset of X, and P will denote the metric projection associated with M.

3. THEOREM. *Let M be a Tchebychev set in a Banach space X. Then in the following list* (a) *implies* (b), *and* (c) *implies* (d). *If M is locally compact then* (a), (b), (c), *and* (d) *are equivalent and* (e) *implies* (f). *If, moreover, X is a smooth space, all the properties below are equivalent.*

(a) *P is continuous,*

(b) *M is boundedly connected,*

(c) *M is a sun (i.e., for every $c>0$ and every x in X*

$$P(Px+c[x-Px]) = Px),$$

(d) *M admits no proper local minimums (i.e., if y is in M and $d([M \cap S(y, r)], x) = d(y, x)$ for some $r>0$, then $y=Px$),*

(e) *M is convex,*

(f) *M is boundedly compact (i.e., $S[r, x] \cap M$ is compact for each x in X and each $r>0$).*

Proof. The last part of the theorem is an easy corollary to the preceding parts and the result of Efimov and Stechkin (see also L. P. Vlasov [11]) that a Tchebychev set, in a smooth Banach space, which is a sun is convex.

We now show that (a) implies (b). Suppose that M is not boundedly connected, but P is continuous. There is a point b_1 in X and an r greater than zero such that the intersection K of M and $S(r, b_1)$ is not connected. There must exist disjoint open sets U and V in X, each of which meets K, and which contains K in their union. Suppose Pb_1 is in U. Let b_0 be a point in the intersection of V and K. We define

$$b_c = cb_1+(1-c)b_0, \quad \text{for } 0 \leqq c \leqq 1.$$

Let

$$c' = \inf\{c : Pb_c \text{ is in the intersection of } K \text{ and } U\}.$$

(²) A more detailed study of boundedly connected sets by L. P. Vlasov [12] has appeared subsequent to their introduction here.

Now suppose $Pb_{c'}$ is in U. By the continuity of P, $Pb_{(c'-(1/n))}$ converges to $Pb_{c'}$ as n approaches ∞ (we have assumed that n is greater than $1/c'$). However this is not possible since none of these points is in U.

If $Pb_{c'}$ is in V, we can choose a sequence $\{c_i\}$ such that $c'<c_i\leqq 1$ for each $i=1, 2, \ldots$, Pb_{c_i} is in U and c_i converges to c'. Since Pb_{c_i} is never in $V\cap K$, Pb_{c_i} can not converge to $Pb_{c'}$. Thus we have contradicted the continuity assumption, provided that Pb_c is contained in K for $0\leqq c\leqq 1$.

Suppose x is in X, and $\|x-b_c\|<\|b_0-b_c\|$. We have

$$\|x-b_c\|+\|b_c-b_1\| < \|b_0-b_c\|+\|b_c-b_1\|.$$

But

$$\|b_0-b_c\|+\|b_c-b_1\| = \|b_0-b_1\| < r.$$

Hence x is in K and (a) implies (b).

We next show that (c) implies (d). Suppose y is a local best approximation to x. There is an $r>0$ such that if t is in M and $\|t-y\|\leqq r$ then $\|t-x\|\geqq\|y-x\|$. Choose $c>0$ small enough that $\|x_c-y\|<r/2$, where $x_c=cx+(1-c)y$. One now verifies that y is a best approximation to x_c among points in $S(r, y)\cap M$. If t is not in $S(r, y)$ then $\|t-x_c\|>r/2$. Hence $y=Px_c$. Since M is a sun $y=Px$.

For the remainder of the proof we will assume that M is locally compact.

(d) implies (a). Suppose that x_n converges to x. Let U be a compact neighborhood of Px. We will show that Px_n is eventually in U. Since bd U is compact

$$d(\text{bd } U, x) > \|x-Px\|.$$

Hence for n sufficiently large

$$d(\text{bd } U, x_n) > \|x_n-Px_n\|.$$

Since U is compact, U admits a best approximation u_n to each x_n. From the above calculation u_n is not contained in bd U. Thus u_n is a local best approximation to x_n. Hence $Px_n=u_n$ is eventually contained in U.

To prove that (b) implies (a) we will show that if M is boundedly connected then for each x there is an $r>d(M, x)$ such that $S[r, x]$ intersects M in a compact set.

Let U be a compact neighborhood of Px in M. Since Px is not contained in the compact set bd U, and since $n(y)=\|y-x\|$ is a continuous function, there is a real number h such that $d(\text{bd } U, x)>h>d=d(M, x)$. Now K, the intersection of $S(h, x)$ and M, is connected and does not meet bd U. Hence K is either contained in U or in the complement of U. Since Px is in K, K is contained in U. It follows that any number r, such that $h>r>d$, satisfies the previous statement. It follows that (b) implies (a).

We may assume the existence of a vector x such that for c greater than 1, $P(cx+(1-c)Px)\neq Px$. From the proof of (b) implies (a) there is an $h>d=d(M, x)$

such that $S[h, x]$ intersects M in a compact set, say K. Let $r=(1/2)[h-d]$. For any y in $S[r, x]$, $\|y-Px\| \leqq (1/2)[h+d]$, also for any p not in $S[h, x]$, $\|y-p\| > (1/2)[h+d]$. We conclude that Py is in K. Let f be the function defined on $S[r, x]$ by $f(y)=x+(r/h)[x-Py]$. Since P is assumed to be continuous f carries $S[r, x]$ continuously into itself. Furthermore the range of f has compact closure, since P carries $S[r, x]$ onto a set with compact closure.

A continuous mapping of a closed convex set in a Banach space into a compact subset of itself has a fixed point [4, p. 83, Corollary 2]. Let y be a fixed point for f. We have that

$$x = (h/(h+r))y+(1-h/(h+r))Py.$$

Since $h/(h+r)$ is a positive number less than one, $Px=Py$. Therefore

$$y = (h+r)x/h+(1-(h+r)/h)Px.$$

However this contradicts our choice of x, and we have completed the proof of (b) implies (c).

To prove that (e) implies (f). Let K be the intersection of M with some closed sphere of radius r. For convenience we may assume that 0 is in K. Since M is locally compact there is a positive number h such that $S[h, 0]$ has compact intersection with M. Let U denote this intersection. Since M is convex, K is contained in the compact set $(3r/h)U$. Hence K is compact.

The known result that (f) implies (a) follows readily from the fact that the function $n(x)=\|Px-x\|$ is continuous. The proof of the theorem is completed.

The implication (f) implies (c) in Theorem 3 was first proved by L. P. Vlasov [11].

The following corollary is a direct consequence of Theorem 3, Proposition 2, and the Hahn-Mazurkiewicz theorem which states that the continuous images of [0, 1] are precisely the compact, metrizable spaces which are connected and locally connected.

4. Corollary. *A compact Tchebychev set is the continuous image of* [0, 1].

5. Proposition. *A closed subset of a (uniformly convex, strictly convex, smooth, respectively) Banach space X is convex if and only if it is boundedly connected in every equivalent (uniformly convex, strictly convex, smooth, respectively) norm topology on X.*

Proof. Since the intersection of convex sets is connected, the necessity is obvious.

Suppose that K is not convex. We may assume the existence of two points x and y in K, each of norm one such that $(1/2)[x+y]=0$, and 0 is not in K. Let Y denote the one-dimensional space spanned by x and y. Let Z be a complement of Y in X. Let r be a positive number less than $d(K, 0)$. Let T denote the homeomorphism of X onto itself defined by $T(z+g)=z+rg$ for z in Z and g in Y. Let $p(x)=\|T(x)\|$ for all x in X. The proof is completed by showing that $p(\cdot)$ defines an equivalent norm on X which is uniformly convex, strictly convex, or smooth with $\|\cdot\|$, and

that K is not boundedly connected in $(X, p(\cdot))$. We will show that K is not boundedly connected in $(X, p(\cdot))$. Let U and V be the disjoint open subsets of X defined by

$$U = \{z+cx : z \text{ in } Z \text{ and } c \text{ greater than } 0\},$$

and

$$V = \{z+cy : z \text{ in } Z \text{ and } c \text{ greater than } 0\}.$$

Let h be a number such that $d(K, 0) > h > r$, and let

$$S = \{x \text{ in } K : p(x) \text{ is less than } h\}.$$

Since Z has empty intersection with S, S is contained in the union of U and V. Since x and y are contained in the intersections of S with U and V respectively, S is not connected. Thus K is not boundedly connected in $(X, p(\cdot))$. This completes our proof of the proposition.

The outline for the above proof was suggested by the referee and replaces a considerably more tedious argument.

The following corollaries are consequences of Proposition 5 and Theorem 3.

6. COROLLARY. *A closed subset of a uniformly convex space X is convex if and only if in every equivalent uniformly convex topology on X, it is a Tchebychev set which admits a continuous metric projection.*

7. COROLLARY. *A compact subset of a strictly convex Banach space X is convex if and only if it is a Tchebychev set in every equivalent strictly convex norm topology on X.*

IV. **Application to rational approximation.** Let X be the Banach space $C[0, 1]$. Let P_n denote the polynomials of degree less than or equal to n. Put

$$Q_n = \{p \text{ in } P_n : p(x) > 0 \text{ for } 0 \leqq x \leqq 1 \text{ and } \|p\| = 1\},$$

and

$$R_m^n = \{p/q : p \text{ in } P_n \text{ and } q \text{ in } Q_m\}.$$

An irreducible function in R_m^n is defined to be *normal* if it is not contained in the set R_{m-1}^{n-1}. It is known that R_m^n is a Tchebychev set, and that, if f is a normal function in R_m^n, then there is a neighborhood of f which has compact closure in R_m^n and which contains only normal points.

If R_m^n is boundedly connected it will follow from the proof of Theorem 3 that the metric projection associated with R_m^n is continuous at any point which has a normal function for a best approximation.

To show that R_m^n is boundedly connected suppose f is in $C[0, 1]$, r is greater than 0, p_1/q_1 and p_2/q_2 are in R_m^n and for all x in $[0, 1]$ we have

$$|p_i(x)/q_i(x) - f(x)| < r \quad \text{for } i = 1, 2.$$

For $0 \leqq c \leqq 1$ we have by direct computation that

$$|cp_1(x) - cf(x)q_1(x)| < rcq_1(x),$$

and

$$|(1-c)p_2(x) - (1-c)f(x)q_2(x)| < (1-c)rq_2(x).$$

By adding the last two inequalities and dividing by an appropriate term we find that for every x in $[0, 1]$

$$\left| \frac{cp_1(x) + (1-c)p_2(x)}{cq_2(x) + (1-c)q_2(x)} - f(x) \right| < r.$$

Thus

$$\left\| \frac{cp_1 + (1-c)p_2}{cq_1 + (1-c)q_2} - f \right\| < r.$$

Since this last inequality is valid for every c between 0 and 1, it follows that R_m^n is boundedly connected. We have proved the following.

8. THEOREM. *The metric projection associated with R_m^n is continuous at all f in $C[0, 1]$ which have normal points for best approximations.*

The converse of Theorem 8 is known to be true. For other proofs of Theorem 8 and Corollary 10 and related results see [2], [13], and the references found there.

Let f be in $C[0, 1]$. A sequence $\{h_i\}$ in R_m^n is a *minimizing sequence for f* if $\lim \|h_i - f\| = d(R_m^n, f)$.

Let P denote the metric projection associated with R_m^n.

9. THEOREM. *Let f be in $C[0, 1]$. Every minimizing sequence for f converges in measure to Pf.*

Proof. Let $\{p_i/q_i\}$ be a minimizing sequence for f. Let $\{p_j/q_j\}$ be any subsequence of $\{p_i/q_i\}$. Since $\{p_j\}$ and $\{q_j\}$ are both bounded, we may select a subsequence $\{p_k/q_k\}$ of $\{p_j/q_j\}$ such that $\{p_k\}$ and $\{q_k\}$ both converge to say p and q respectively. Since the norm of $q_k = 1$ for all k, q is not identically 0. Hence $q(x) = 0$ for at most m distinct points. Since the norms of p_k/q_k are bounded, $p(x) = 0$ whenever $q(x) = 0$. If we factor out the common factors in p and q, we are left with a function in R_m^n. Since we started with a minimizing sequence for f, this rational function must be Pf, the best approximation to f.

We must show that p_k/q_k converges in measure to Pf. Let I be a closed subset of $[0, 1]$ which is disjoint from $\{x : q(x) = 0\}$. Since q is continuous and positive on the compact set I, q assumes its minimum there, say $r > 0$. Now for a function on $[0, 1]$ whose restriction to I is continuous, let

$$\|f\|_I = \sup\{|f(x)| : x \text{ in } I\}.$$

For sufficiently large k, $q_k(x) > r/2$ for each x in I. We have

$$\left\| \frac{p_k}{q_k} - Pf \right\|_I = \left\| \frac{p_k q - pq + pq - pq_k}{q_k q} \right\|_I$$
$$\leqq (2/r^2)[\|p\|_I \cdot \|q - q_k\|_I + \|q\|_I \cdot \|p_k - p\|_I].$$

Since q_k converges to q and p_k converges to p, we have that p_k/q_k converges uniformly to Pf on I. Hence p_k/q_k converges in measure to Pf. We started with an arbitrary subsequence of $\{p_i/q_i\}$ and extracted a further subsequence which converged in measure to Pf. Hence the original sequence must itself converge in measure to Pf, and the proof is completed.

10. COROLLARY. *Let f_n be a sequence in $C[0, 1]$ which converges in norm to a function f. Then Pf_n converges in measure to Pf.*

REFERENCES

1. J. Blatter, P. D. Morris, and D. E. Wulbert, *Continuity of the set-valued metric projection*, Math. Ann. (to appear).

2. E. W. Cheney and H. L. Loeb, *Generalized rational approximation*, SIAM J. Numer. Anal. **1** (1964), 11–25.

3. E. W. Cheney and D. E. Wulbert, *Existence and unicity of best approximations*, Math. Scand. (to appear).

4. M. M. Day, *Normed linear spaces*, Springer-Verlag, Berlin, 1958.

5. N. V. Efimov and S. B. Stechkin, *Approximative compactness and Chebychev sets*, Dokl. Akad. Nauk SSSR **140** (1961), 522–524 = Soviet Math. Dokl. **2** (1961), 1226–1228.

6. R. B. Holmes and B. R. Kripke, *Smoothness of approximation*, Abstract 636-138, Notices Amer. Math. Soc. **13** (1966), 618.

7. V. Klee, *Convexity of Chebyshev sets*, Math Ann. **142** (1961), 292–304.

8. ———, *Remarks on nearest points in normed linear spaces*, Proceedings of the Convexity Colloquium, Copenhagen, 1965, pp. 168-176.

9. P. D. Morris, *Metric projections onto subspaces of finite codimension*, Duke Math. J. (to appear).

10. T. Motzkin, *Sur quelques propriétés caractéristiques des ensembles convexes*, Atti. Accad. Lincei Rend. (6) **21** (1935), 563–567.

11. L. P. Vlasov, *Chebyshev sets in Banach spaces*, Dokl. Akad. Nauk SSSR **141** (1961), 19–20 = Soviet Math. Dokl. **2** (1961), 1373–1374.

12. ———, *Chebyshev sets and some of their generalizations*, Mat. Zametki **3** (1968), 59–69. (Russian)

13. H. Werner, *On the local behavior of the rational Tchebycheff operator*, Bull. Amer. Math. Soc. **70** (1964), 554–555.

UNIVERSITY OF WASHINGTON,
SEATTLE, WASHINGTON