ON POLYNOMIALS IN SELF-ADJOINT OPERATORS IN SPACES WITH AN INDEFINITE METRIC(1)

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1. Introduction. Let $H$ be a Hilbert space(2) with the usual inner product $[\cdot, \cdot]$ and norm(3) and with an indefinite inner product $(\cdot, \cdot)$ which, for some orthogonal decomposition $H = H_1 \oplus H_2$ in $H$, is defined by

$$(x, y) = [x_1, y_1] - [x_2, y_2] \quad \text{for all } x, y \in H,$$

where

$$x = x_1 + x_2, \quad y = y_1 + y_2,$$

$$x_1, y_1 \in H_1, \quad x_2, y_2 \in H_2,$$

and $\dim H_1 = \kappa$, a positive integer. Such a space $H$ will be called a space $\Pi_\kappa$ with an indefinite metric. Another, axiomatic definition of the space $\Pi_\kappa$ was given by I. S. Iohvidov and M. G. Krein in [1]; we follow here their terminology, unless otherwise stated, and use the results of their paper.

A linear operator $A$ in $\Pi_\kappa$ is called symmetric if it maps a dense domain $D(A)(4)$ in $\Pi_\kappa$ into $\Pi_\kappa$ and has the property,

$$(Ax, y) = (x, Ay) \quad \text{for all } x, y \in D(A).$$

A linear operator $A^*$ defined in $\Pi_\kappa$ is called the adjoint operator of a linear operator $A$ with a dense domain $D(A)$ in $\Pi_\kappa$ if $A^*$ is the maximum operator such that

$$(Ax, y) = (x, A^*y) \quad \text{for all } x \in D(A) \text{ and all } y \in D(A^*).$$

A symmetric operator is said to be maximal if it has no proper symmetric extension.

A symmetric operator is said to be self-adjoint if $A = A^*$.

It is well known in the theory of operators in Hilbert spaces that any two complex conjugate polynomials in a self-adjoint operator are adjoint to each other. We find that the same property holds for polynomials in a self-adjoint operator in the space $\Pi_\kappa$ with an indefinite metric. Moreover, if there exists a pair of complex conjugate...
polynomials in a symmetric operator one of which is adjoint to the other, then this operator is self-adjoint. We shall prove these assertions in this paper.

2. Closed isometric operators. We shall prove here a theorem on isometric operators for later use. Isometric operators in $\Pi_\kappa$ are, in general, not continuous. However, a closed isometric operator in $\Pi_\kappa$, as we shall show, is continuous.

**Definition 2.1.** A linear operator $V$ is said to be *isometric* if

$$ (Vx, Vx) = (x, x) \quad \text{for all } x \in D(V). $$

**Definition 2.2.** Let $\Pi_\kappa = P \oplus N$, where $P$ is a positive $\kappa$-dimensional subspace and $N$ is the orthogonal complement of $P$. An operator $J$ is called a *metric operator* if it is defined by the relation,

$$ J(x) = x_P - x_N \quad \text{for all } x \in \Pi_\kappa $$

where $x = x_P + x_N$, $x_P \in P$ and $x_N \in N$. The new scalar product $[x, y]_J = (x, Jy)$ is called a $J$-metric and the new norm $|x|_J = ([x, x]_J)^{1/2}$ is called a $J$-norm. By Theorem 1.2 in [1, §2] all the $J$-norms are topologically equivalent.

**Notation.** For any two linear manifolds $L, M$ the notation $L \oplus M$ represents that $(x, y) = 0$ for all $x \in L$ and all $y \in M$. The notation $L \oplus_J M$ represents that $L \oplus M$ and $[x, y]_J = 0$ for all $x \in L$ and all $y \in M$ for some metric operator $J$.

**Theorem 2.3.** If $V$ is a closed isometric operator, then $V$ is a continuous operator with a closed domain $D(V)$ and a closed range $R(V)$.

**Proof.** Let $D_+(V)$ be a positive subspace of $D(V)$ with the greatest possible dimension and $R_+(V) = VD_+(V)$. Since $V$ is an isometric operator, the subspace $R_+(V)$ is a maximal positive subspace of $R(V)$, having the same dimension as $D_+(V)$. We can have the resolutions

$$ D(V) = D_+(V) \oplus D_0(V) \oplus D_-(V) \quad \text{and} \quad R(V) = R_+(V) \oplus R_0(V) \oplus R_-(V) $$

where $D_0(V)$ and $R_0(V)$ are nonpositive orthogonal complements of $D_+(V)$ and $R_+(V)$ in $D(V)$ and $R(V)$ respectively. If the scalar product degenerates on $D_0(V)$, then by Theorem 1.7 in [1, §3] we have

$$ D_0(V) = D_0(V) \oplus D_-(V), $$

where $D_0(V)$ is the isotropic subspace of the linear manifold $D_0(V)$ and $D_-(V)$ is a negative linear manifold. Clearly $R_0(V) = VD_0(V)$ is the isotropic subspace of the linear manifold $R_0(V)$ and $R_-(V) = VD_-(V)$ is a negative linear manifold. Obviously we have

$$ R_0(V) = VD_0(V) \oplus VD_-(V). $$

Thus we have the resolutions,

$$ D(V) = D_+(V) \oplus D_0(V) \oplus D_-(V) \quad \text{(2.1)} $$

and

$$ R(V) = R_+(V) \oplus R_0(V) \oplus R_-(V). \quad \text{(2.2)} $$
If \( \text{cl } (R-(V)) \), the closure of \( R-(V) \), is a negative subspace then the theorem is a direct consequence of Theorem 4.3 in [1, §15]. Thus it remains to prove that \( \text{cl } (R-(V)) \) is a negative subspace.

Assuming that the nonpositive subspace \( \text{cl } (R-(V)) \) is a degenerate subspace, by Theorem 1.7 in [1, §3] we have the decomposition

\[
\text{cl } (R-(V)) = N \oplus R_-
\]

where \( N \) is the isotropic subspace of \( \text{cl } (R-(V)) \) and \( R_- \) is a negative subspace. Similarly we have the decomposition

\[
\text{cl } (D-(V)) = M \oplus D_-
\]

where \( M \) is the isotropic subspace of \( \text{cl } (D-(V)) \) and \( D_- \) is a negative subspace.

Now let \( z_0 \in N \). Then there exists a sequence \( \{x_n\} \in D-(V) \) such that \( \{y_n = Vx_n\} \) is a Cauchy sequence in \( R-(V) \), having \( z_0 \) as its limit. From (2.3) and (2.4) we have

\[
X_n - X^0 + X_n, \quad J_n - J_0 + J_n,
\]

where \( x_0n \in M, \quad x'_n \in D_- \), \( y_0n \in N \) and \( y'_n \in R_-' \) for \( n = 0, 1, 2, \ldots \). Clearly we have

\[
(\langle x'_n, x'_n \rangle, \quad \langle y'_n, y'_n \rangle)
\]

for \( n = 0, 1, 2, \ldots \). Since the scalar product \( \langle , \rangle \) is continuous in both arguments we have

\[
\lim_{n \to \infty} \langle y_n, y_n \rangle = \langle z_0, z_0 \rangle = 0.
\]

It follows from (2.5) that

\[
\lim_{n \to \infty} [\langle x'_n, x'_n \rangle] = - \lim_{n \to \infty} (\langle x'_n, x'_n \rangle) = - \lim_{n \to \infty} (\langle y'_n, y'_n \rangle)
\]

\[
= \lim_{n \to \infty} [\langle y'_n, y'_n \rangle] = - \lim_{n \to \infty} (\langle y_n, y_n \rangle) = 0.
\]

In other words, each of the sequences \( \{x'_n\} \) and \( \{y'_n\} \) converges to the zero vector \( \theta \). Hence the sequence \( \{y_0n\} \) converges to \( z_0 \).

If the sequence \( \{x_0k\} \) has a Cauchy subsequence with a limit \( x \in M \), then \( z_0 = Vx \) and \( x \in D-(V) \) since \( V \) is a closed operator. It follows that \( z_0 \in N \cap R-(V) \), that is \( z_0 = \theta \).

If the sequence \( \{x_0k\} \) had no Cauchy subsequence, then it would have an unbounded subsequence \( \{x_0k\} \) such that \( |x_0k| > k + 1 \) for \( k = 0, 1, 2, \ldots \), since \( M \) is a finite dimensional subspace by Lemma 1.2 in [1, §1].

We define a sequence

\[
w_k = x_0k/d_k = x_0k/d_k + x'_k/d_k \quad \text{for } k = 0, 1, 2, \ldots
\]

The sequence \( \{w_k\} \) is clearly in \( D-(V) \) and the sequence \( \{V(w_k) = y_k/d_k\} \) is clearly a Cauchy sequence in \( R-(V) \) with the limit \( \theta \). The sequence \( \{x'_k/d_k\} \) converges to \( \theta \) and the sequence \( \{x_0k/d_k\} \) is bounded in \( M \) with \( |x_0k/d_k| = 1 \) for \( k = 0, 1, 2, \ldots \).
Let \( \{x_{om}/dn\}_0^\infty \) be a Cauchy subsequence of \( \{x_{0m}/dn\}_0^\infty \), with limit \( w_0 \in M \). It follows that the corresponding subsequence \( \{w_m\}_0^\infty \) is also a Cauchy sequence with limit \( w_0 \). Since \( V \) is a closed operator, we have \( V(w_0) = \theta \) and \( w_0 \in D_-(V) \), that is \( w_0 = \theta \). But \( |w_0|^2 = 1 \) since \( w_0 \) is the limit of the sequence \( \{x_{om}/dn\}_0^\infty \). This contradiction implies that the sequence \( \{x_{om}/dn\}_0^\infty \) is bounded and hence \( z_0 = \theta \). In other words \( N = \{\theta\} \). Now it is easy to show that \( D(V) = F \) and \( R(V) \) are closed. The theorem is proven.

3. Polynomials in self-adjoint operators in the space \( \Pi \). Having proven Theorem 2.3 we are now able to investigate some properties of a symmetric operator by using Cayley-von Neumann transformation. Since every symmetric operator has a closed symmetric extension (see §6 in [1]), we center our attention on closed symmetric operators.

Let \( A \) be a closed symmetric operator with a dense domain \( D(A) \). There exists a nonreal number \( \xi \) which is not a proper value of \( A \) since a symmetric operator in \( \Pi \) can have at most \( 2k \) nonreal proper values (see 1 of §8 in [1]). We define an operator \( V \) by the following formulae:

\[
y = (Ax - \xi x), \quad Vy = (Ax - \xi^* x) \quad \text{for} \quad x \in D(A),
\]

where \( \xi^* \) is the complex conjugate of \( \xi \) or symbolically,

\[
V = (A - \xi I)(A - \xi I)^{-1} \quad \text{and} \quad D(V) = (A - \xi I)D(A).
\]

The operator \( V \) is clearly a closed isometric operator. It follows from Theorem 2.3 that \( V \) is a continuous operator with a closed domain \( D(V) = (A - \xi I)D(A) \). Now it is easy to see that the operator \( (A - \xi I)^{-1} \) is continuous. Thus we have proven the following theorem.

**Theorem 3.1.** Let \( A \) be a closed symmetric operator with a dense domain \( D(A) \). If \( \xi \) is a nonreal number which is not a proper value of \( A \), then the operator \( (A - \xi I)^{-1} \) is continuous with a closed domain \( (A - \xi I)D(A) \).

Before we prove our main theorem, we need to establish a few lemmas for later use.

**Lemma 3.2.** Let \( A \) be a linear operator in a linear space \( \Pi \) and let \( \xi \) be a complex number. If \( (A - \xi I)D(A^m) = D(A^m) \) for some positive integer \( m \), then \( (A - \xi I)D(A^n) = D(A^{n-1}) \) for all natural numbers \( n > m \).

**Proof.** We shall prove this lemma by induction. Let \( n = m + 1 \). It is obvious that \( (A - \xi I)D(A^{m+1}) = D(A^m) \). We need to prove only the reverse inclusion. For any \( x \in D(A^m) \) by assumption there exists \( y \in D(A^m) \) such that \( (A - \xi I)y = x \). It follows that \( Ay \in D(A^m) \), that is \( y \in D(A^{m+1}) \). Hence \( (A - \xi I)D(A^{m+1}) = D(A^m) \) and we have proved our lemma for \( n = m + 1 \). Using the same kind of arguments we can prove the lemma for the case \( n = k + 1 \) by assuming it is true for \( n = k \). The lemma is proven.
**Lemma 3.3.** Let $A$ be a self-adjoint operator in $\Pi_\kappa$ and let $P(\lambda) = \prod_{i=1}^{n} (\lambda - \zeta_i)$ be a polynomial with nonreal roots. If no root of $P(\lambda)$ is a proper value of $A$, then $P(A)D(A^m) = D(A^{m-n})$ for $m > n$, where $m, n$ are natural numbers.

**Proof.** By Theorem 2.9 in [1, §9] we have $(A - \zeta_i I)D(A) = \Pi_\kappa$ for $i = 1, 2, \ldots, n$. Thus this lemma follows Lemma 3.2 immediately.

**Lemma 3.4.** Let $A$ be a maximal symmetric operator in $\Pi_\kappa$. Then $D(A^n)$ is dense in $\Pi_\kappa$ for any natural number $n$.

**Proof.** We shall prove this lemma by induction. For $n = 1$ the lemma is true by the definition of a maximal symmetric operator.

Now we assume this lemma is true for $n = m$. By Theorem 2.9 in [1, §9] we have a pair of complex numbers $(\zeta, \bar{\zeta})$ such that

$$ (A - \zeta)D(A) = \Pi_\kappa \quad \text{and} \quad (A - \bar{\zeta})D(A) = M, $$

where $M$ is a nondegenerate subspace, containing a $\kappa$-dimensional positive subspace. Thus by Theorem 1.5 in [1, §3] we have the resolution

$$ \Pi_\kappa = M \oplus N, $$

where $N$ is the orthogonal complement of $M$. By Lemma 3.2 we have

$$ (A - \zeta)D(A^{m+1}) = D(A^m) \quad (3.3) $$

from relation (3.1).

Now for any $x \in \Pi_\kappa$ we have $y \in D(A)$ such that $x = (A - \zeta)y = (A^* - \bar{\zeta})y$ by relation (3.1). From relation (3.2) we have $y = y_M + y_N$, when $y_M \in M$ and $y_N \in N$. It follows from (3.1) there exists $z \in D(A)$ such that $y_M = (A - \zeta)z$. Since $(A^* - \bar{\zeta})y_N = \theta$, we have

$$ x = (A^* - \bar{\zeta})(A - \zeta)z \quad (3.4) $$

for some $z \in D(A)$. If $x \in \Pi_\kappa$ and $(x, D(A^{m+1})) = 0$ then

$$ 0 = (x, D(A^{m+1})) = ((A^* - \bar{\zeta})(A - \zeta)z, D(A^{m+1})) $$
$$ = ((A - \zeta)z, (A - \bar{\zeta})D(A^{m+1})) $$
$$ = ((A - \zeta)z, (A - \bar{\zeta})D(A^{m+1})). $$

It follows from (3.3) that $0 = ((A - \zeta)z, D(A^m))$. Hence we have $(A - \zeta)z = \theta$ by assumption. Since $\zeta$ is not a proper value of $A$ we must have $z = \theta$. Thus from (3.4) we conclude that $x = \theta$. It thus follows that $D(A^{m+1})$ is dense in $\Pi_\kappa$. The lemma is proved.

**Lemma 3.5.** Let $P(\lambda)$ be a polynomial of degree $n$ and let $F$ be a finite set of $m$ complex numbers. Then we can always find a nonreal number $\zeta_0$ such that all the roots of the polynomial $P(\lambda) - \zeta_0$ are nonreal and these roots are not in the set $F$.

(*) We agree that for any operator $A$, $A^0 = I$ where $I$ is the identity operator.
**Proof.** It is easy to see that if \( \zeta \) and \( \zeta' \) are different numbers, then the polynomials \( P(\lambda) - \zeta \) and \( P(\lambda) - \zeta' \) have no common factors. Hence for only a finite number of complex numbers \( \zeta_i, i = 1, 2, \ldots, m' \) (\( m' \leq m \)) does the corresponding polynomial \( P(\lambda) - \zeta_i \) have roots in the set \( F \). It thus follows that for any complex number \( \zeta \) such that \( \operatorname{Re} \zeta > \operatorname{Re} \zeta_i, i = 1, 2, \ldots, m' \) the polynomial \( P(\lambda) - \zeta \) has no roots in \( F \).

Let \( P(\lambda) = P^{(1)}(\lambda) + iP^{(2)}(\lambda) \), where \( P^{(1)}(\lambda) \) and \( P^{(2)}(\lambda) \) are real polynomials of degree at most \( n \). Let \( \zeta = c + id \), where \( c \neq 0 \) and \( d \) are real numbers such that \( c > \operatorname{Re} \zeta_i \), \( i = 1, 2, \ldots, m' \). If \( \lambda_0 \) is a real root of the polynomial \( P(\lambda) - \zeta \), then we have

\[
(3.5) \quad P^{(1)}(\lambda_0) - c = 0
\]

and

\[
(3.6) \quad P^{(2)}(\lambda_0) - d = 0.
\]

It is clear that for a fixed real number \( c \), there exist at most \( n \lambda_0 \)'s satisfying the relation (3.5). It thus follows that we can find a real number \( d_0 \neq 0 \) such that the polynomial \( P(\lambda) - (c + id_0) \) has no real roots. Hence the number \( \zeta_0 = c + id_0 \) is the desired nonreal number. The lemma is proved.

**Lemma 3.6.** If \( A \) is a closed linear operator in \( \Pi_K \) then the adjoint of the adjoint of \( A \) is \( A \).

**Proof.** Let \( J \) be a metric operator. Clearly \( JA \) is also a closed linear operator since \( J \) is a bicontinuous linear operator by Theorem 1.2 in [1, §2]. Let us denote the adjoint of \( JA \) with respect to the \( J \)-metric by \( (JA)^J \). Since the space \( \Pi_K \) together with a \( J \)-metric is a Hilbert space, we have \( (JA)^{J'} = JA \). It is obvious that for any linear operator \( B \) with a dense domain \( (JB)^J = JB^* \). It thus follows that \( JA = (JA^*)^J \)

\[
= JA^{**}.
\]

Since \( J \) is bijective, we have \( A = A^{**} \). The lemma is proved.

**Theorem 3.7.** Let \( A \) be a symmetric operator in \( \Pi_K \) and let \( P(\lambda) \) and \( \overline{P}(\lambda) \) be complex conjugate polynomials of degree \( n \). Then the operator \( \overline{P}(A) \) is adjoint to \( P(A) \) if and only if \( A \) is a self-adjoint operator.

**Proof.** (1). Let \( A \) be a self-adjoint operator. Since \( A \) can have only a finite number of nonreal proper values, it follows that by Lemma 3.5 we can find a nonreal number \( \xi \) such that the polynomial \( P(\lambda) - \xi \) has no root which is a proper value of \( A \) or its complex conjugate. Hence \( \overline{P}(\lambda) - \xi \) also has no root which is a proper value of \( A \). It follows that \( \xi \) and \( \xi' \) are not proper values of \( P(A) \) and \( \overline{P}(A) \) respectively.

It is clear that \( D(P(A)) = D(A^*) = D(P(A)^*) \). Since \( D(A^*) \) is dense in \( \Pi_K \), by Lemma 3.4, the adjoint operator \( P(A)^* \) of \( P(A) \) exists. Obviously we have \( P(A)^* \supset P(A) \). Therefore it is sufficient to prove \( D(A^*) \supset D(P(A)^*) \) in order to prove \( \overline{P}(A) = P(A)^* \).
By Lemma 3.3 we have

$$ (P(A) - \xi I)D(A^n) = \Pi_\kappa = (\bar{P}(A) - \xi I)D(A^n). $$

For any $x \in D(P(A)^*)$ there exists $z \in D(A^n)$ such that

$$ (P(A)^* - \xi I)x = (\bar{P}(A) - \xi I)z $$

by relation (3.7). In other words, we have $(P(A)^* - \xi I)(x-z) = \theta$. It thus follows that for all $y \in D(A^n)$ we have

$$ 0 = ((P(A)^* - \xi I)(x-z), y) = ((x-z), (P(A) - \xi I)y). $$

Since $(P(A) - \xi I)D(A^n) = \Pi_\kappa$, we have $x - z = \theta$, that is $x = z \in D(A^n)$. Similarly we can prove $P(A)^* = P(A)$. The first part of the theorem is proved.

(2) Now let $P(A)$ and $P(A)^*$ be adjoint to each other. We choose $\xi$ such that the polynomial $P(\lambda) - \xi = \prod_{\iota=1}^n (\lambda - \lambda_\iota)$ has no root which is a proper value or its complex conjugate of the operator $A$, the closed extension of $A$. It thus follows that $\xi$ and $\xi$ are not proper values of $P(A)$ and $P(A)^*$ respectively.

We shall show that $(P(A) - \xi I)D(A^n)$ is dense in $\Pi_\kappa$. Let $x \in \Pi_\kappa$ be such that $(x, (P(A) - \xi I)y) = 0$ for all $y \in D(A^n)$. It follows that for all $y \in D(A^n)$ we have

$$ 0 = ((P(A)^* - \xi I)x, y) = ((P(A) - \xi I)x, y). $$

Since $D(A^n)$ is dense in $\Pi_\kappa$ by Lemma 3.4, we have $(P(A) - \xi I)x = \theta$, the zero vector. As $\xi$ is not a proper value of $P(A)$, $x$ must be the zero vector $\theta$. Therefore $(P(A) - \xi I)D(A^n)$ is dense in $\Pi_\kappa$.

We shall show $(P(A) - \xi I)D(A^n) = \Pi_\kappa$. Let us define an operator $U$ in the $\Pi_\kappa$ by the formulae:

$$ y = (P(A) - \xi I)x, \quad Uy = (P(A) - \xi I)x $$

for all $x \in D(A^n)$. Clearly $U$ is an isometric operator with dense domain in $\Pi_\kappa$. The operator $U$ is bicontinuous by Theorem 4.3 in [1, §15]. Since the operators $(A - \lambda_\iota)^{-1}$, $i = 1, 2, \ldots, n$ are continuous by Theorem 3.1, the operator $(P(A) - \xi I)^{-1} = \prod_{\iota=1}^n (A - \lambda_\iota)^{-1}$ is also continuous. As $\bar{P}(A) = P(A)^*$ is a closed operator, it follows that $U$ and $P(A) - \xi I$ are also closed operators. Applying Theorem 2.3 we conclude that $(P(A) - \xi I)D(A^n) = D(U)$ is a subspace. Since it is dense in $\Pi_\kappa$, it can only be the whole space $\Pi_\kappa$. As $P(A) - \xi I$ is a closed operator, the operator $P(A)$ must be a closed operator. It thus follows from Lemma 3.6 that $P(A) = P(A)^{**}$. Hence by similar arguments we have $(\bar{P}(A) - \xi I)D(A^n) = \Pi_\kappa$.

We shall show that $(A - \xi I)D(A) = \Pi_\kappa = (A - \lambda_\iota)D(A)$. It is sufficient to prove $\prod_{\iota=2}^n (A - \lambda_\iota)D(A^n) = D(A)$. Since $\prod_{\iota=1}^n (A - \lambda_\iota)D(A^n) = \Pi_\kappa$, for any $x \in D(A)$ there exists $x' \in \prod_{\iota=2}^n (A - \lambda_\iota)D(A^n)$ such that $(A - \lambda_\iota)x' = (A - \lambda_\iota)x$, that is $(A - \lambda_\iota)(x' - x) = \theta$. Since $\xi_\iota$ is not a proper value of $A$, we must have $x = x'$.

We shall show that $A = A^*$. Since $D(A) = D(A^*)$ and $\overline{D(A^n)} = \Pi_\kappa$, $A^*$ exists. It is obvious $A^* \supseteq A$; therefore it is sufficient to prove $D(A^*) \subseteq D(A)$. Since
(A - \zeta_i)D(A) = \Pi_k$, for any $x \in D(A^*)$ there exists $y \in D(A)$ such that $(A^* - \zeta_i)x = (A - \zeta_i)y$, that is $(A^* - \zeta_i)(x - y) = \theta$. It follows that $((A^* - \zeta_i)(x - y), z) = 0$ for all $z \in D(A)$. Hence $((x - y), (A - \zeta_i)z) = 0$ for all $z \in D(A)$. As $(A - \zeta_i)D(A) = \Pi_k$, we have $x - y = \theta$, that is $x = y$. So we have $D(A^*) = D(A)$ and $A = A^*$. The theorem is completely proven.

**Theorem 3.8.** Let $A$ be a symmetric operator in $\Pi_k$ and let $P(\lambda)$ be a real polynomial of degree greater than one. Then $P(A)$ is a maximal symmetric operator if and only if $A$ is self-adjoint.

**Proof.** If $A$ is self-adjoint, the operator $P(A)$ must be self-adjoint, that is maximal, by Theorem 3.7. Now let $P(A)$ be maximal and let $\tilde{A}$ be a maximal symmetric extension of the operator $A$. Then $P(A) = P(\tilde{A})$ must hold. If $\tilde{A}$ is self-adjoint, then $P(A)$ is self-adjoint and consequently $A$ is self-adjoint by Theorem 3.7. If $\tilde{A}$ is not self-adjoint, we shall show $P(A) = P(\tilde{A})$ can not be maximal. If $P(\tilde{A})$ were maximal there exists a nonreal number $\zeta$ which is not a proper value of $P(\tilde{A})$ such that

\[(3.8) \quad (P(\tilde{A}) - \zeta)D(\tilde{A}) = \Pi_k = \prod_{i=1}^n (\tilde{A} - \zeta_i)D(\tilde{A}).\]

It follows that the roots of $P(\lambda) - \zeta$ are not proper values of $A$. Since $P(\lambda)$ is a real polynomial of degree at least two and since $\zeta$ is a nonreal number, there exists at least one root of the polynomial $P(\lambda) - \zeta$ in both the upper and the lower half of the complex plane. It thus follows that there exists a root $\zeta_{i0}$ such that $(\tilde{A} - \zeta_{i0})D(\tilde{A}) \neq \Pi_k$. Hence we have

\[\prod_{i=1}^n (\tilde{A} - \zeta_i)D(\tilde{A}) \subset (\tilde{A} - \zeta_{i0})D(\tilde{A}) \neq \Pi_k.\]

This contradiction implies that $A$ must be a self-adjoint operator. The theorem is proven.

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**Reference**


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