QUASICONFORMAL STRUCTURES AND THE METRIZATION OF 2-MANIFOLDS

BY

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1. **Introduction.** There are many conditions that one may impose on a 2-manifold that are equivalent to metrizability. The first definitions of a Riemann surface included as a hypothesis that there was a triangulation of the manifold. In 1925, Rado [15] proved that the presence of a complex-analytic structure on a 2-manifold implies the manifold has a countable basis for its topology, and so is triangulable and metrizable. More recently, several different proofs of the same result have been given [3], [11]. We will show that if a 2-manifold has a K-quasiconformal structure, then it admits an analytic one, and so is metrizable, thus generalizing the classical theorem of Rado.

The converse of Rado's theorem for orientable manifolds was established by M. Heins [9]. Stoilow [17] had shown that a light open mapping is the composition of a homeomorphism and an analytic function. Heins proved that a triangulable orientable 2-manifold has a light open mapping into the 2-sphere. The analytic structure of the 2-sphere may thus be lifted back to the original manifold so that the light open map is analytic. Thus an orientable 2-manifold is metrizable if and only if it admits an analytic structure.

I would like to express my appreciation to Professor G. S. Young, who first suggested this problem to me, and who directed the dissertation on which this paper is based.

2. **Analytic definitions and preliminaries.** There are many approaches to the definition of a quasiconformal mapping, and the geometric ones seem to be the most appealing from an intuitive standpoint. The so-called analytic definition best suits our purposes here however, and we give only it. For other definitions and proofs of equivalence, the reader is referred to the growing literature [1], [2], [6], [8], [14].

If f is a function from the plane into the plane, we define the complex derivatives of f in the usual manner, using the partials f_x and f_y.

$$f_z = (\tfrac{1}{2})(f_x - if_y) \qquad \text{and} \qquad f_{\bar{z}} = (\tfrac{1}{2})(f_x + if_y).$$

DEFINITION 1. A homeomorphism f from one plane region D onto another is said to be K-quasiconformal if

(i) f is ACL in D, and

(ii) $|f_{\bar{z}}| \leqq ((K-1)/(K+1))\,|f_z|$ a.e. in D, $1 \leqq K < \infty$.

Received by the editors August 16, 1967.

(1) The research for this paper was supported by a National Science Graduate Fellowship.

A mapping is called quasiconformal if it is K-quasiconformal for some K. It is well known that (i) implies f_x and f_y exist a.e. in D, so that (ii) is meaningful. Also note that (ii) implies the Jacobian is a.e. positive, so that f is a sense-preserving homeomorphism.

DEFINITION 2. If μ is a function defined in D such that $f_{\bar{z}}=\mu f_z$, then μ is called the complex dilatation of f.

It is a deep result in the theory of quasiconformal functions that if D is an open subset of the plane, and μ is a measurable function defined on D whose L^∞ norm is less than one, then there exists a quasiconformal function f defined on D such that μ is the complex dilatation of f [2, Chapter V] or [10].

In [10], Lehto and Virtanen make use of an equation which allows one to compute the complex dilatation of a composition of two quasiconformal mappings, and we will make extensive use of this equation later. The equation states that if f, g, and h are quasiconformal functions such that $f=g\circ h$, and if μ_1, μ_2, and μ_3 are the complex dilatations of f, g, and h respectively then

$$(*)\qquad \mu_1(z) = \frac{\mu_3(z)+\mu_2(h(z))\exp(-2i\arg h_z)}{1+\mu_3(z)\mu_2(h(z))\exp(-2i\arg h_{\bar{z}})}$$

whenever there is sufficient differentiability. Because Gehring and Lehto [7] have shown that a quasiconformal function is differentiable a.e., and because a quasiconformal function preserves sets of measure zero, routine computation will establish the validity of (*) for computing the complex dilatation of f a.e. In particular, Lehto and Virtanen use (*) to show that if f and h are quasiconformal functions on an open set D which have the same complex dilatations, then there is a conformal mapping g from $h(D)$ to $f(D)$ such that $f=g\circ h$.

3. **Topological definitions and preliminaries.** By a *2-manifold* we mean a connected Hausdorff space such that each point has an open neighborhood homeomorphic to the plane. We use the text of Ahlfors and Sario [3] as a basic reference for the theory of 2-manifolds and Riemann surfaces. We cannot however use the definition of structure found there, for the class of K-quasiconformal mappings is not closed under composition, nor is a homeomorphism which is locally quasiconformal necessarily quasiconformal. Thus we use the more classical definition of a Riemann surface, omitting the requirement of maximality for the structure, as a point of departure for our definition of a quasiconformal manifold.

DEFINITION 3. If M is a 2-manifold, and $\{(U, \varphi)\}$ is a collection of pairs such that:

(1) each U is an open set of M,

(2) the collection of all U covers M,

(3) each φ is a homeomorphism between U and an open set in the plane, and

(4) if (U, φ) and (V, ψ) are two pairs such that $U\cap V\neq\varnothing$, then

$$\psi\circ\varphi^{-1}\colon \varphi(U\cap V)\to\psi(U\cap V)$$

is quasiconformal, then we say $[M, \{(U, \varphi)\}]$ is a quasiconformal manifold, and we call the collection $\{(U, \varphi)\}$ a quasiconformal structure on M. Each U is called a coordinate neighborhood; each φ is called a coordinate map, and the functions $\psi \circ \varphi^{-1}$ are called coordinate transformations. If all the coordinate transformations are K-quasiconformal, then we have a K-quasiconformal manifold. If there is no chance of ambiguity, we refer simply to M as the quasiconformal manifold.

If $K=1$, this is the classical definition of a Riemann surface.

DEFINITION 4. A 2-manifold is simply connected if every simple closed curve is homotopic to a constant map.

REMARK. The Jordan curve theorem, which is well known in the case of the plane, is true for noncompact simply connected 2-manifolds in general. To see this, let H be a homotopy between a given Jordan curve and a constant map. The range of H is a compact set, and so we may cover it with a finite number of open sets each of which is homeomorphic to the plane. The union of these sets is a metrizable 2-manifold in which the curve may be shrunk to a point. We may then use Borsuk's "sweeping" theorem to identify the inside of the curve; its complement is the outside.

DEFINITION 5. Let M and N be connected Hausdorff spaces. Then M is said to be a covering space of N with covering map π if π is a continuous function from M onto N such that each point of N has a neighborhood U with the property that if C is a component of $\pi^{-1}(U)$, then C is open in M, and π restricted to C is a homeomorphism onto U.

From the definition it is immediate that a covering space of a 2-manifold is again a 2-manifold. It is also clear that if M is a covering space for N, and if N has a K-quasiconformal or quasiconformal structure, then the structure may be lifted to M. This can be done because π is a local homeomorphism, and so locally has an inverse. We observe that if M is a metrizable 2-manifold which covers N, then N is also metrizable. This follows because M, being connected, must have a countable basis; the image of this basis under the covering map is a basis for N. Thus N is regular and has a countable basis, and so is metrizable. It is well known [3] that every 2-manifold has a simply connected covering space. These remarks show that to prove every K-quasiconformal 2-manifold is metrizable, it suffices to prove every simply connected one is. Indeed, since the only compact simply connected 2-manifold is the 2-sphere, which is metrizable, we may assume the manifold is not compact.

4. **The theorem.** In the following three theorems we assume M is a simply connected noncompact 2-manifold with a K-quasiconformal structure $\mathscr{S}$. Because there is no maximality requirement in our definition of structure, we also assume, by restricting the coordinate maps if necessary, that the coordinate neighborhoods are connected and simply connected.

PROPOSITION 1. *If* (U, φ) *and* (V, ψ) *are elements of* $\mathscr{S}$ *such that* $U \cup V$ *is simply* *connected, then there is a map ξ from $U \cup V$ into the plane such that:*

(1) *$\mathscr{S} \cup \{(U \cup V, \xi)\}$ is a K-quasiconformal structure on M, and*

(2) *$\xi \circ \varphi^{-1}: \varphi(U) \to \xi(U)$ is conformal.*

Proof. Let μ be defined on the whole plane by setting $\mu(z)$ to be the complex dilatation of $\varphi \circ \psi^{-1}$ at z for $z \in \psi(U \cap V)$, $\mu(z)=0$ otherwise. Let f be a quasiconformal function whose complex dilatation is μ. Since $\varphi \circ \psi^{-1}$ and f have the same dilatation on $\psi(U \cap V)$, there is a conformal mapping g such that $g \circ f = \varphi \circ \psi^{-1}$.

We examine the manifold $U \cup V$, with structure $\{(U, \varphi), (V, f \circ \psi)\}$. Then the coordinate transformation $(f \circ \psi) \circ \varphi^{-1} = f \circ (\psi \circ \varphi^{-1}) = f \circ (\varphi \circ \psi^{-1})^{-1} = f \circ (g \circ f)^{-1} = f \circ f^{-1} \circ g = g$, which is conformal. The other coordinate transformation is g^{-1} which is also conformal. Thus this structure makes $U \cup V$ into a Riemann surface. It is simply connected, and not compact, so by Koebe's uniformization theorem [3, p. 181] there is a conformal mapping ξ from $U \cup V$ into the plane. Thus $\xi \circ \varphi^{-1}$ is conformal.

We must now show $(U \cup V, \xi)$ satisfies (1). We first note that $\psi \circ \xi^{-1} = (f^{-1} \circ f) \circ (\psi \circ \xi^{-1}) = f^{-1} \circ [(f \circ \psi) \circ \xi^{-1}]$, which is a conformal mapping followed by a K-quasiconformal mapping, and so is K-quasiconformal. Moreover, the dilatation of f^{-1} vanishes off $f(\psi[U \cap V])$, and so we may use (*) to show the dilatation of $\psi \circ \xi^{-1}$ vanishes off $\xi(U)$.

Now, let $(W, \tau) \in \mathscr{S}$ be such that $W \cap (U \cap V) \neq \varnothing$. We write $\xi(W \cap [U \cup V])$ as $\xi(W \cap U) \cup \xi(W \cap V)$. On $\xi(W \cap U)$, $\tau \circ \xi^{-1} = \tau \circ \varphi^{-1} \circ \varphi \circ \xi^{-1}$, which is a conformal mapping followed by a K-quasiconformal mapping, and so is K-quasiconformal. On $\xi(W \cap V)$, $\tau \circ \xi^{-1} = \tau \circ \psi^{-1} \circ \psi \circ \xi^{-1}$, which is the composition of two K-quasiconformal mappings, and so is K^2-quasiconformal. Thus $\tau \circ \xi^{-1}$ is locally K^2-quasiconformal, hence is K^2-quasiconformal, and so is ACL.

Now we may use (*) to compute the dilatation of $\tau \circ \xi^{-1}$. We already know that on $\xi(W \cap U)$, the mapping is K-quasiconformal, and have seen that for points in $[\xi(W \cap V) - \xi(W \cap U)]$, the complex dilatation of $\psi \circ \xi^{-1}$ vanishes. Thus for these points, the modulus of the dilatation of $\tau \circ \xi^{-1}$ is the same as $\tau \circ \psi^{-1}$, i.e. is less than or equal to $(K-1)/(K+1)$ a.e. Thus the modulus of the dilatation of $\tau \circ \xi^{-1}$ is less than or equal to $(K-1)/(K+1)$ a.e., and the mapping is K-quasiconformal. This completes the proof of the theorem.

We now remove the hypothesis that $U \cup V$ must be simply connected.

PROPOSITION 2. *If (U, φ) and (V, ψ) are elements of $\mathscr{S}$ such that $U \cap V \neq \varnothing$, then there is a simply connected open set W containing $U \cup V$, and a homeomorphism ξ from W into the plane such that $\mathscr{S} \cup \{(W, \xi)\}$ is a K-quasiconformal structure.*

Proof. If $U \cup V$ is simply connected, apply Proposition 1.

If $U \cup V$ is not simply connected, then we "fill in the holes." Because $U \cap V \neq \varnothing$, their union is connected, and so is a metrizable 2-manifold. Hence its first homology group is countably generated [3, p. 64]. Let $J_1, J_2, \ldots$ be a set of generators. These can be assumed to be carried by Jordan curves, which we again call $J_1, J_2, \ldots$. For each n, let D_n denote the disc bounded by J_n, the existence of which was proven in a previous remark. Finally, let $W = U \cup V \cup D_1 \cup D_2 \cup \cdots$. Then W is a simply connected 2-manifold, and, being the countable union of metrizable 2-manifolds, is itself metrizable. Hence it may be triangulated; moreover, it may be triangulated so that each simplex lies in some coordinate neighborhood [5, p. 419]. Assume W is so triangulated. Let $P_1, P_2, \ldots$ be a canonical exhaustion of W, so that each P_n is a 2-cell [3, p. 61]. D. E. Sanderson [16] has shown that there is a way of removing the elements of P_n that are not in P_{n-1}, one by one, so that at each step the remaining space is still a 2-cell. By reversing this process, we may build up from P_{n-1} to P_n by adding simplexes one at a time so that the result at each stage is simply connected.

Summarizing the above, we have shown that there is an ordering of the elements of the triangulation $\sigma_1, \sigma_2, \ldots$ so that $\bigcup_{n=1}^{\infty} \sigma_n = W$, and $\bigcup_{i=1}^{n} \sigma_i$ is simply connected for each n.

Since the triangulation is subordinate to a covering of coordinate neighborhoods, we may assign to each σ_n a U_n such that $\sigma_n \subset U_n$ and $(U_n, \varphi_n) \in \mathscr{S}$. Let W_n be the interior of $\bigcup_{i=1}^{n} \sigma_i$, and let V_n be the component of $U_n \cap W_n$ that contains the interior of σ_n, so that $W_n = \bigcup_{i=1}^{n} V_i$. Finally, let ψ_n be the restriction of φ_n to V_n. We now use Proposition 1 to define coordinate maps on the W_n.

Let $\xi_1 = \psi_1$, and assume ξ_n has been defined so that $\mathscr{S} \cup \{(W_i, \xi_i), 1 \leqq i \leqq n\}$ is a K-quasiconformal structure on M, with the second set a conformal structure on W_n. We apply Proposition 1 to (W_n, ξ_n) and (V_{n+1}, ψ_{n+1}), letting (W_n, ξ_n) play the role of (U, φ), and letting ξ_{n+1} be the map whose existence is proven.

Then $\{(W_n, \xi_n) : 1 \leqq n < \infty\}$ is a conformal structure for W, and since W is simply connected, we may apply the uniformization theorem, producing a coordinate map ξ defined on all of W, such that $\xi_n \circ \xi^{-1}$ is conformal.

The only thing we now have to show is that $\mathscr{S} \cup \{(W, \xi)\}$ is a K-quasiconformal structure on M. Suppose (U, φ) is any element of $\mathscr{S}$ such that $U \cap W \neq \varnothing$. Let $p \in U \cap W$, and let n be the first integer such that $p \in W_n$. Then there is a neighborhood around p that lies entirely in W_n. On this neighborhood $\varphi \circ \xi^{-1} = \varphi \circ \xi_n^{-1} \circ \xi_n \circ \xi^{-1}$, which is a conformal mapping followed by a K-quasiconformal one, and so is K-quasiconformal. Thus the mapping is a homeomorphism which is locally K-quasiconformal, so is K-quasiconformal, and Proposition 2 is proven.

Convention. In the following, we use α, β, γ, and δ to denote ordinal numbers, and whenever we index a set, we use as indices elements of the smallest ordinal that has the same cardinality as the set to be indexed. If the set is well-ordered, it is ordered by the usual ordinal order on the indexing family.

We present now a lemma in topology, and merely start its proof, the rest being simple verification.

LEMMA 1. *If X is a connected topological space, and $\mathscr{U}$ is a cover for X consisting of open connected sets, then there is a well-ordering of $\mathscr{U}$ such that for each ordinal α, $\bigcup \{U_\beta : \beta \leqq \alpha\}$ is connected.*

Proof. Well-order $\mathscr{U}$, and denote the elements of this ordering by V_α. Set $U_1 = V_1$. Suppose that α is an ordinal, and that for all $\beta < \alpha$, U_β has been chosen to satisfy the condition of the lemma. Let U_α be the first V not already used that meets $\bigcup \{U_\beta : \beta < \alpha\}$. It is not difficult to show this exhausts $\mathscr{U}$.

We are now ready for the main result.

THEOREM. *M is metrizable.*

Proof. Let $\mathscr{U}$ be the set of all coordinate neighborhoods of $\mathscr{S}$, and well-order $\mathscr{U}$ so as to satisfy the condition of Lemma 1. We will use Proposition 2 to construct a tower of manifolds $\{W_\alpha\}$ which will make M into a Riemann surface.

Let $W_1 = U_1$, and $\xi_1 = \varphi_1$. Suppose α is a countable ordinal, and that for all $\beta < \alpha$ we have defined (W_β, ξ_β) so that

(1) W_β is a simply connected submanifold of M,

(2) $U_\beta \subset W_\beta$,

(3) ξ_β is a coordinate map whose domain is W_β,

(4) if $\gamma < \beta$, then W_γ is properly contained in W_β,

(5) $\{(W_\beta, \xi_\beta) : \beta < \alpha\}$ is a conformal structure,

(6) $\mathscr{S} \cup \{(W_\beta, \xi_\beta) : \beta < \alpha\}$ is a K-quasiconformal structure.

Set $V = \bigcup \{W_\beta : \beta < \alpha\}$. Being the union of a tower of simply connected sets, V is simply connected, and with the structure in (5) is a Riemann surface. Using the uniformization theorem again, we have a homeomorphism ψ of V into the plane such that $\xi_\beta \circ \psi^{-1}$ is conformal for all $\beta < \alpha$. We also have that

$$\mathscr{S} \cup \{(W_\beta, \xi_\beta) : \beta < \alpha\} \cup \{(V, \psi)\}$$

is a K-quasiconformal structure, by (6) and the last paragraph in the proof of Proposition 2.

Let U_γ be the first element in the ordering of $\mathscr{U}$ that meets V, but is not contained in V. The connectedness of M implies that if no such U_γ exists, then $V = M$, and we are done. If such a U_γ exists, use Proposition 2 to construct (W_α, ξ_α) so that $V \cup U_\gamma \subset W_\alpha$ and $\xi_\alpha \circ \psi^{-1}$ is conformal. It is because we need V metrizable in order to apply Theorem 2 that we construct this tower only for countable ordinals. But this process must stop, for let ω_1 denote the first uncountable ordinal; set $M_1 = \bigcup \{W_\alpha : \alpha < \omega_1\}$. Then the structure $\{(W_\alpha, \xi_\alpha) : \alpha < \omega_1\}$ makes M_1 into a Riemann surface, and so M_1 has a countable basis for its topology, contradicting the existence of an uncountable tower satisfying (4).

Thus for some countable ordinal α, $M = W_\alpha$, and is metrizable.

REMARK. What we have shown is that if M is a noncompact simply connected 2-manifold with a K-quasiconformal structure $\mathscr{S}$, then there is a K-quasiconformal structure $\mathscr{S}'$ on M which is the union of $\mathscr{S}$ and a conformal structure on M. We can remove the requirement that M not be compact by observing that if M is compact, eventually the uniformizing map will be into the 2-sphere, not the plane. We remove the hypothesis of simple connectivity by noting that if we lift the structure of M to its universal covering space, apply the above there, and then project that structure back down to M, we have the result for an arbitrary K-quasiconformal manifold.

5. **Examples.** We give now two examples of 2-manifolds that have a quasiconformal structure, but are not metrizable. The first example is very far from being simply connected, and gives rise to a family of examples by considering its covering spaces. The first example is a modification of an example of R. L. Moore [12], and is separable, i.e. it has a countable dense subset.

EXAMPLE 1. We call the manifold M; the points of M are of two types. The points of type I are the points in the plane not on the real line. If $p \in M$ is of type I, we give it the same coordinates it has in E^2, $p=(x, y)$, $y \neq 0$. Now for each real number a, define a mapping h_a by $h_a(x, y)=([x-a]/|y|, y)$. This is a homeomorphism of the points of type I onto themselves, leaving the line $y=c$ invariant. Fix a, and for each real number r, $-1<r<1$, let L_r be the inverse image under h_a of the broken line given by $\{(r, y) : (r,y)$ is a point of type I$\}$. Then L_r is a point of type II, and we give it coordinates $[(r, a), 0]$.

To define the coordinate neighborhoods, for each real a, let $U_a=\{[(r, a), 0] : -1 < r < 1\} \cup \{(x, y) : (x, y)$ is of type I, and $h_a[(x, y)]=(r, y)\}$. The corresponding coordinate map φ_a is defined in two steps:

(1) for points of type I, $\varphi_a[(x, y)]=\{(x-a)/|y|\}+iy$,

(2) for points of type II, $\varphi_a([(r, a), 0])=r$.

Each φ_a is 1-1, and maps U_a onto the strip in the plane determined by

$$-1 < \operatorname{Re}(z) < 1.$$

We give M the topology that makes each φ_a a homeomorphism. This is well-defined, and makes M into a 2-manifold. M is separable because the set of all points of type I both of whose coordinates are rational is countable and dense in M. M is not metrizable because $\{[(r, a), 0]$: r is fixed, and a is real$\}$ is a discrete subset of M which has the power of the continuum, and no separable metric space can contain such a subset.

To compute the complex dilatation of a coordinate transformation, we first note that for $a \neq b$, $U_a \cap U_b$ contains no points of type II. Secondly each point of type I in the intersection has a y-coordinate whose absolute value is greater than $(\frac{1}{2})|b-a|$.

Then if $z=x+iy$ is a point in the plane, we have

$$\varphi_a \circ \varphi_b^{-1}(z) = \varphi_a[(|y|x+b, y)] = \{x+(1/|y|)(b-a)\}+iy.$$

Simple computation shows the modulus of the complex dilatation at z is

$|(a-b)/(2y^2-i[a-b])|$, which is bounded away from 1 in $U_a \cap U_b$, and so the transformation is K-quasiconformal for some K depending on a and b.

EXAMPLE 2. Here we show that the cartesian product of the long line with the real line admits a piecewise (real) linear quasiconformal structure. Let L denote the long line; it is the product of the first uncountable ordinal and (0, 1) with the topology induced by lexicographically ordering the elements. That is, $(\alpha, r) < (\beta, s)$ if (1) $\alpha < \beta$, or (2) if $\alpha = \beta$, then $r < s$. Thus the points of the manifold are pairs $[(\alpha, r), s]$ where α is a countable ordinal, $0 \leqq r < 1$, and s real. We let

$$U_\alpha = \{[(\beta, r), s] : \beta < \alpha\}$$

and define the φ_α by transfinite induction.

Let $\varphi_1([(0, r), s]) = r + is$.

Now assume we have defined φ_β for all $\beta < \alpha$, and that $\{(U_\beta, \varphi_\beta)\}$ is a quasiconformal structure. If α has a predecessor, we define φ_α by

$$\varphi_\alpha(p) = (\tfrac{1}{2}) \operatorname{Re} \varphi_{\alpha-1}(p) + i \operatorname{Im} \varphi_{\alpha-1}(p) \quad \text{for } p \in U_{\alpha-1},$$

and extend φ_α linearly so that it maps $U_\alpha - U_{\alpha-1}$ onto the strip $(\frac{1}{2}) \leqq \operatorname{Re} z < 1$, keeping the second coordinate at the same height.

If α is a limit ordinal, pick $\beta_1, \beta_2, \ldots$ so that $\beta_{n+1} > \beta_n$ and the limit of the β_n is α. Pick a corresponding sequence x_n such that $x_{n+1} > x_n$, $0 < x_n < 1$, and the limit of the x_n is 1. We merely repeat countably many times what we did one time when α had a predecessor. Let φ_α map $U_{\beta_n} - U_{\beta_{n-1}}$ onto the strip $x_{n-1} \leqq \operatorname{Re}(z) < x_n$ by "pinching" the real part of φ_{β_n}.

It is routine that this is a quasiconformal structure.

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